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On the Ginzburg-Landau and related equations *

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Abstract. We describe qualitative behaviour of solutions of the Gross-Pitaevskii equation in 2D in terms of motion of vortices and radiation. To this end we introduce the notion of the intervortex energy. We develop a rather general adiabatic theory of motion of well separated vortices and present the method of effective action which gives a fairly straightforward justification of this theory. Finally we mention briefly two special situations where we are able to obtain rather detailed picture of the vortex dynamics. Our approach is rather general and is applicable to a wide class of evolution nonlinear equation which exhibit localized, stable static solutions. It yields description of general time-dependent solutions in terms of dynamics of those static solutions "glued" together.

Introduction

In this paper we present our recent results on the Ginzburg-Landau and related Gross-Pitaevskii equations. Our goal is to understand dynamics of vortices. The latter are representatives of *localized*, *particle-like structures* appearing in solutions of many nonlinear evolution equations. We consider the time-dependent Ginzburg-Landau equation of the Schrödinger type:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2 - 1)\psi \ , \qquad \text{(SE)}$$
$$|\psi| \to 1 \quad \text{as} \quad |x| \to \infty \ ,$$

where $\psi : \mathbb{R}^{d+1} \to \mathbb{R}^m$. ($\mathbb{R}^m$ is assumed to possess a complex structure.) This equation comes up in condensed matter physics and nonlinear optics and is also known as the Gross-Pitaevskii or Ginzburg-Pitaevskii equation.

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In this paper we consider the case $m = d = 2$. However many of our arguments are also applicable to other dimensions and to related equations.

Static solutions

The vortices mentioned above are solutions of the corresponding stationary equation, the proper Ginzburg-Landau equation

$$-\Delta\psi + (|\psi|^2 - 1)\psi = 0 \ . \tag{GLE}$$

There are two ways to classify solutions of this equation.

Topological classification

With each $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ we associate the map

$$\hat{\psi} := \frac{\psi}{|\psi|}\Big|_{|x|=R} : S^1 \to S^1 \ .$$

Using a standard definition of the degree (see e.g. A. Schwarz (1993)), we set

$$\deg\psi := \deg\hat{\psi} \in \mathbb{Z} \ .$$

All solutions to the (GLE) are classified according to this topological invariant. Depending on the degree we have e.g.

$$\deg\psi_0 = \pm 1 \Rightarrow \ \text{vortex/antivortex}$$

$$\deg\psi_0 = n \Rightarrow \ n\text{-vortex} \ .$$

The topological classification leads to the topological conservation law for the corresponding time-dependent equations.

Group-theoretical classification

Now we want to isolate symmetric solutions. The symmetry group of (GLE) is

$$G_{\text{sym}} = O(2) \times T(2) \times O(2)$$

(the group of rigid motions of the underlying physical space times the gauge group). Consider the equivariant or "spherically symmetric" solutions ψ_0: $\exists$ homomorphism $\rho : SO(2) \to SO(2)$ s.t.

$$\rho(g)\psi_0(g^{-1}x) = \psi_0(x) \quad \forall g \in O(2) \ .$$

The homotopy class of ρ's determines $\deg \psi$.

Existence and stability

Let L_{ψ_0} be the linearized operator for (GLE) at a solution ψ_0. We use the following definition of the (linearized) stability:

Definition: A solution ψ_0 is said to be *stable* iff

$$\mathrm{spec} L_{\psi_0} \subset \overline{\mathbb{R}^+} \quad \text{and} \quad \mathrm{Null} L_{\psi_0} = \mathfrak{g}_{\mathrm{sym}} \psi_0 \ .$$

Here $\mathfrak{g}_{\mathrm{sym}}$ in the Lie algebra of the group G_{sym}. Note that

$$\mathfrak{g}_{\mathrm{sym}} \psi_0 \subseteq \mathrm{Null} L_{\psi_0} \ .$$

Theorem. $\forall n \in \mathbb{Z}\ \exists$ *a unique (modulo symmetry tranformations) vortex;* $|n| \leq 1$ *vortices are stable and* $|n| > 1$, *unstable.*

References:

Existence: Hervé and Hervé (1994), Chen, Elliott and Qui (1994), Fife and Peletier (1996), Ovchinnikov and Sigal (1997a).

Stability: Lieb and Loss (1994) and Mironescu (1994) for disc and $|n| \leq 1$. Ovchinnikov and Sigal (1997a) for $\mathbb{R}^2$ and all n.

Related results: Chanillo and Kiesling (1995), Mironescu (1996) and Shafrir (1994).

Remarks.

(a) S. Gustafson (1997b) extended the result above to the non-commutative case. He showed that for $d = m \geq 3$ (monopoles, etc.) spherically symmetric solutions exist only for $n = \pm 1$, and they are unique and stable.

(b) Gustafson and Sigal (1998) generalized the above theorem to the order parameter ψ coupled to a magnetic field (i.e. to magnetic or Abrikosov vortices).

Idea of the proof of the stability result

We follow Ovchinnikov and Sigal (1997a). The outline below, though formally incorrect, gives a fairly good impression of our approach. Let ψ_0 be the 1-vortex. Since it breaks translation symmetry ($\psi_0(x) \neq \psi_0(x+h) \forall h \neq 0$),

$$\partial_{x_j}\psi_0 \text{ are zero modes of } L_{\psi_0} \ ,$$

the linearized operator. (In fact $\partial_{x_1}\psi_0$ and $\partial_{x_2}\psi_0$ are "proportional" to each other, so we can consider just $\partial_{x_1}\psi_0$.)

We find a positivity (open) cone $\Gamma \subset L^2(\mathbb{R}^2, \mathbb{R}^2)$ s.t.

(i) $\partial_{x_1}\psi_0 \in \Gamma$,

(ii) $\exp(-tL_{\psi_0}) : \overline{\Gamma} \to \Gamma$ (i.e. it is positively improving w.r.t. Γ).

Then the Perron-Frobenius theory implies that $\sigma(L_{\psi_0}) \subset [0, \infty)$ and 0 is a non-degenerate eigenvalue. □

The reason that the argument above is incorrect is that the property (ii) does not quite hold. For $|n| \leq 1$, this hole can be patched up problem can be circumvented, while for $|n| > 1$, not. In the latter case we construct a test function ξ s.t.

$$\langle \xi, L_{\psi_0}(\xi) \rangle < 0 \ ,$$

which shows that L_{ψ_0} has a negative eigenvalue. □

Renormalized Energy

(GLE) is the equation for critical points of the celebrated Ginzburg-Landau functional

$$\mathcal{E}(\psi) = \frac{1}{2}\int \{|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2\} d^2x \ .$$

There is one problem with this functional though

Theorem. *Let ψ be a C^1 vector field on $\mathbb{R}^2$ s.t. $|\psi| \to 1$ as $|x| \to \infty$. If $\deg \psi \neq 0$, then $\mathcal{E}(\psi) = \infty$.*

Thus if we want to use energetic arguments for vortices we have to modify $\mathcal{E}(\psi)$. We introduce the renormalized energy functional as follows (see Ovchinnikov and Sigal (1997a))

$$\mathcal{E}_{\text{ren}}(\psi) = \frac{1}{2}\int \{|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + \frac{1}{2}(|\psi|^2 - 1)^2\} d^2x \ ,$$

where $r = |x|$ and χ is a smooth cut-off function, $= 0$ for $r \leq 1$ and $= 1$ for $r \geq 2$. Critical points of this functional are still given by (GLE) and it is a constant of motion for (SE).

In order to introduce our next key notion, we need the following notation and definition. Let $\mathbf{c} = (\mathbf{z}, \mathbf{n})$, where

$$\mathbf{z} = (z_1, \ldots, z_k) \ , \ z_j \in \mathbb{R}^2 \ ,$$
$$\mathbf{n} = (n_1, \ldots, n_k) \ , \ n_j \in \mathbb{Z} \ .$$

Definition: We say that ψ has a *configuration* $\mathbf{c}$, conf $\psi = \mathbf{c}$, iff ψ has zeros only at $z_1 \ldots z_k$ with local indices $n_1, \ldots, n_k$.

Now we introduce *intervortex energy* as

$$E(\mathbf{c}) := \inf\{\mathcal{E}_{\text{ren}}(\psi) | \text{conf}\psi = \mathbf{c}\} \ . \qquad (*)$$

(For variational problems with topological constraints see Fröhlich and Struwe (1990).) Ovchinnikov and Sigal (1997b) argue that (actually the $\to$ direction is proven)

$$(*) \text{ has a minimizer}$$
$$\longleftrightarrow \nabla_{\mathbf{z}} E(\mathbf{c}) = 0$$

and show that in the cases of interest and for intervortex distances $\gg 1$,

$$\nabla_{\mathbf{z}} E(\mathbf{c}) \neq 0 \ .$$

Hence for large intervortex distances we expect that there are *no stationary vortex configurations.*

For intervortex distances of order $O(1)$ stationary configurations do exist, e.g. (see Ovchinnikov and Sigal (1998b)); they correspond to various discrete subgroups of $O(2)$:

Fig. 1. Static vortex configurations

<u>Pinning</u>

Introduce impurities in order to nail the vortices down:

$$\mathcal{E}_\lambda(\psi) = \mathcal{E}_{\text{ren}}(\psi) + \sum_{j=1}^{K} \frac{\lambda_j}{2} \int \delta_{b_j} |\psi|^2 \ ,$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\delta_b(x) = \frac{1}{2\pi r_0}\delta(|x-b| - r_0)$,

Ovchinnikov and Sigal (1997b) argue that if $\lambda_j \geq \text{const}|\nabla_{z_j} E(\mathbf{c})| \forall j$, then $\mathcal{E}_\lambda(\psi)$ has a minimizer in the class $\{\text{conf}\psi = \mathbf{c}\}$.

<u>Asymptotics of $E(\mathbf{c})$</u>

Let $R(\mathbf{c})$ be the intervortex distance. Ovchinnikov and Sigal (1997b) show that as $R(\mathbf{c}) \to \infty$,

$$E(\mathbf{c}) = \sum_{i=1}^{K} E_{n_j} + H(\mathbf{c}) + O(R(\mathbf{c})^{-1}) \ , \tag{AS}$$

where E_n is the (proper) energy of the n-vortex and $H(\mathbf{c})$ is the Kirchhoff-Onsager Hamiltonian:

$$H(\mathbf{c}) = -\pi \sum_{i \neq j} n_i n_j \ln|z_i - z_j| \ .$$

The idea of a demonstration of (AS) is as follows. The upper bound, $E(\mathbf{c}) \leq$ r.h.s.(AS), is obtained by choosing an appropriate test function and performing a rather delicate many-body geometrical analysis. To prove the lower bound, $E(\mathbf{c}) \geq$ r.h.s.(AS), we use the pinning energy functional with $\lambda = O(R(\mathbf{c})^{-1})$. This gives

$$E(\mathbf{c}) \geq \inf\{\mathcal{E}_\lambda(\psi) | \mathrm{conf}\psi = \mathbf{c}\} - CR(\mathbf{c})^{-1} \ .$$

For $\lambda_j \geq CR(\mathbf{c})^{-1}$, the minimization problem on the r.h.s. has a minimizer. The latter satisfies the Euler-Lagrange equation

$$-\Delta\psi + (|\psi|^2 - 1)\psi = -\sum \delta_{b_j}\psi \ .$$

This equation allows us to produce estimates on the minimizer in question which show that it is of the same form as the aforementioned test function and therefore

$$\inf\{\mathcal{E}_\lambda(\psi) | \mathrm{conf}\psi = \mathbf{c}\} = \text{ r.h.s. (AS)} \ .$$

The last two relations produce the desired lower bound which, together with the upper bound mentioned above, yields (AS).

An expansion related to (AS) is derived in Bethuel, Brezis and Hélein (1994).

Multivortex dynamics

Problem: Consider (SE) with an initial condition corresponding to several vortices at large distances from each other. The goal is to show that the corresponding solution can be described in terms of moving vortices and find the dynamic law *for the vortex centers.*

Nonlinear adiabatic theory

Let ψ be a solution of (SE) with an initial condition of a configuration $\mathbf{c}$ and low energy, say of order $E(\mathbf{c}) + O(1)$. To describe this solution, we proceed as follows (in what follows $\mathbf{n} = (n_1, \ldots, n_k)$ is fixed and is not displayed in the notation):

(i) Pick a "minimizer", $\psi_{\mathbf{z}}$ of $\mathcal{E}_{\mathrm{ren}}(\psi)$ in $\{\mathrm{conf}\psi = \mathbf{c}\}$, $\mathbf{c} = (\mathbf{z}, \mathbf{n})$,

(ii) Define the intervortex energy $E(\mathbf{z}) := \mathcal{E}(\psi_{\mathbf{z}})$ and write the Hamiltonian equation

$$\dot{\mathbf{z}} = J\nabla E(\mathbf{z}), \tag{$*$}$$

where J is a "symplectic" matrix on $\bigoplus_{i=1}^{k} \mathbb{R}^2$:

$$J = \operatorname{diag}\left(\frac{1}{\pi n_j} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \ ,$$

(iii) Insert the solution, $\mathbf{z}(t)$, of the Hamiltonian system above with an appropriate initial condition into $\psi_{\mathbf{z}}$. This gives the adiabatic order parameter as $\psi_{\mathbf{z}(t)}$,

(iv) We expect that the solution ψ is of the form

$$\psi = e^{i\alpha(t)} \psi_{\mathbf{z}(t)} + \psi_{\text{disp}} \ ,$$

where $\alpha(t)$ is some slowly varying real function of t and ψ_{disp} is a radiation to ∞, the latter of the order $O(R(\mathbf{z})^{-2})$.

Effective Action Method

Now we explain the origin of the nonlinear adiabatic theory (see Ovchinnikov and Sigal (1998a) for more details, some of the general ideas originate with Whitham (1974), Manton (1981) and Stuart (1994)). Let $S(\psi)$ be the action functional for Eqn (1.1):

$$S(\psi) \ = \ \int \Big\{ - \int \frac{1}{2} \text{Im}(\psi \dot{\overline{\psi}}) d^2 x + \mathcal{E}_{\text{ren}}(\psi) \Big\} dt \, ,$$

where $\mathcal{E}_{\text{ren}}(\psi)$ is the renormalized Ginzburg-Landau functional introduced above. First we find an approximate minimizer, $\psi_{\mathbf{z}}$, of $\mathcal{E}_{\text{ren}}(\psi)$ under the constraint that the vortices are fixed at positions $z_1, \ldots, z_k$, $(z_1, \ldots, z_k) = \mathbf{z}$. Next, we allow $\mathbf{z}$ to depend on time and plug $\psi_{\mathbf{z}(t)}$ into $S(\psi)$. The resulting action functional,

$$S_{\text{eff}}(\mathbf{z}) \ \equiv \ S(\psi_{\mathbf{z}}) \, ,$$

describes the dynamics of the vortex centers in the leading approximation; it is equal modulo $\int O(\ln R(\mathbf{z}) \cdot R(\mathbf{z})^{-2}) dt$ to the action functional

$$S_{\text{vort}}(\mathbf{z}) \ = \ \int \Big\{ - \frac{\pi}{2} \sum_{j=1}^{k} z_j \wedge \dot{z}_j - E(\mathbf{z}) \Big\} dt \, ,$$

whose critical points satisfy Eqn $(*)$.

To go beyond this approximation we write $\psi = \psi_{\mathbf{z}} + \alpha$, where α is supposed to be a small fluctuation field around $\psi_{\mathbf{z}}$ and expand $S(\psi)$ in α up to the second order. Critical points of the resulting functional satisfy the system of coupled equations

$$\partial_{\mathbf{z}} S_{\text{eff}}(\mathbf{z}) \;=\; -\nabla_{\mathbf{z}} \text{Re} \int \overline{\alpha} \partial_{\overline{\psi}} S(\psi_{\mathbf{z}}) \tag{CEa}$$

$$S''(\psi_{\mathbf{z}})\alpha \;=\; -\partial_{\overline{\psi}} S(\psi_{\mathbf{z}})\,, \tag{CEb}$$

where ∂_ε stands for the variational derivative w.r. to ε and $S''(\psi)$ is the Hessian of S at ψ, and where we dropped the higher order term $\nabla_{\mathbf{z}} \frac{1}{2} \text{Re} \int \int \overline{\alpha} S''(\psi_{\mathbf{z}})\alpha$. We demonstrate that provided $\mathbf{z}$ satisfies $(*)$, one can perturb $\psi_{\mathbf{z}}$ slightly in such a way that Eqn (CEb) has a solution of the order $\alpha = O(R(\mathbf{z})^{-1})$, provided $t \le R(\mathbf{z})^p$ for some $p \ge 0$. To this end we decompose the space $\mathbb{R}^2$ into several regions determined by the configurations $\mathbf{z}$ and estimate Eqn (CEb) separately in each region. We call this method, the method of geometric solvability.

Finally, we observe that Eqns (CE) stripped of inessential terms read

$$\dot{\mathbf{z}} = J\nabla_{\mathbf{z}} E(\mathbf{z}) - \int \ddot{\chi} \nabla_{\mathbf{z}} \varphi_0 d^2x \ ,$$

$$(\partial_t^2 - 2\Delta)\chi = -\ddot{\varphi}_0 \ ,$$

where $\varphi_0(x) = \sum_{j=1}^k n_j \theta(x - z_j)$ and χ = phase of α. Here $\theta(x)$ is the polar angle of $x \in \mathbb{R}^2$. This systems represents finite dimensional Hamiltonian equations for the vortex centers $\mathbf{z}$ coupled to a wave equation for the phase fluctuation χ.

Special case: Two simple vortices

Take an initial condition for (SE) describing two simple vortices at the distance R from each other.

Two vortices of the same charge: the vortices rotate around each other with the frequency $\omega = \frac{1}{R^2}$

Fig. 2. Motion of two 1-vortices

and radiate at the same time, so that the distance between them grows as

$$R(t) = (3\pi t)^{\frac{1}{6}}$$

modulo lower order terms.

Two vortices of opposite charges: there is a critical distance R_{cr} s.t. for $R > R_{\text{cr}}$ there exists a travelling wave solution corresponding to the vortices moving parallel to each other

Fig. 3. Motion of a vortex-antivortex pair

while for $R < R_{\text{cr}}$, the vortices, as they move parallel to each other, emit a shock wave (Cherenkov radiation) and eventually collapse onto each other.

$\arcsin\left(\frac{\sqrt{2}}{v}\right)$

Fig. 4. Shock wave produced by a vortex pair

References. The Hamiltonian dynamics of vortices was first suggested by Onsager (1949) and then elaborated by Gross (1966) and Cheswick and Morrison (1980) on the basis of analogy with the motion of an incompressible fluid. It was derived using multiscale expansion by Neu (1990) and using the nonlinear adiabatic theory by Ovchinnikov and Sigal (1998a). The rigorous proof that the vortices indeed are well defined for "low energy" solutions ψ and that their centers are governed by the Hamiltonian system mentioned was given by Lin and Xin (1998) and Colliander and Jerrard (1998) (these authors considered (SE) in a bounded domain and with ε^{-2} in front of $(|\psi|^2 - 1)^2$).

The radiation phenomena was found in Ovchinnikov and Sigal (1988a), where the coupled equations for the vortex motion and radiation, Eqns (CE), were derived.

The special case of two vortices of the same charge was analyzed by Ovchinnikov and Sigal (1998c). The existence of a solitary wave for two vortices of opposite charge at a large distance from each other was predicted by Jones and Roberts (1982) (see also Iordanskii and Smirnov (1978), Jones, Putterman and Roberts (1986), Kuznetzov and Rasmunssen (1995) and Pismen and Nepomnyashchy (1993) and references therein) and was rigorously proven by Bethuel and Sout (1998). The appearance of the shock wave at small distances was suggested by Ovchinnikov and Sigal (1998c).

Conclusion

In this paper we presented a definition of a renormalized Ginzburg-Landau energy and a conclusive result on stability of vortices of the Ginzburg-Landau equation. In order to describe the dynamics of several vortices, we introduced the notion of the intravortex energy and developed a general adiabatic theory. This theory is justified by the method of effective action functional. Finally, we considered two examples where the general theory is applied.

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