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# Strong unique continuation for second order elliptic differential operators

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**Abstract**

We prove a strong unique continuation result for differential inequalities of the form $|P(x,D)u| \leq C_1|x|^{-2}|u| + C_2|x|^{-1}|\nabla u|$, where $P(x,D) = \sum a_{jk}(x)D_jD_k$ is an elliptic second order differential operator with Lipschitz coefficients such that $a_{jk}(0)$ is real (we suppose $P(0,D) = -\Delta$) . $C_1$ and $C_2$ are positive constants such that $C_2 < \frac{\sqrt{2}}{2}$ . A counterexample du to Alinhac and Baouendi[2] shows that our assumption on the constant $C_2$ is sharp .

**Résumé**

Nous démontrons la propriété du prolongement unique fort pour des inégalités différentielles de la forme $|P(x,D)u| \leq C_1|x|^{-2}|u| + C_2|x|^{-1}|\nabla u|$ , où $P(x,D) = \sum a_{jk}(x)D_jD_k$ est un opérateur elliptique d'ordre deux à coefficients Lipschitz tels que $a_{kj}(0) \in \mathbb{R}$ ( on suppose $P(0,D) = -\Delta$) . $C_1$ et $C_2$ sont deux constantes positives telles que $C_2 < \frac{\sqrt{2}}{2}$ . Cette dernière condition est optimale comme le montre un contrexemple dû à Alinhac et Baouendi[2] .

# 1 Introduction and main results

Let $\Omega$ be a connected open subset of $\mathbb{R}^n (n \geq 2)$ containing 0, and let $P(x,D) = \sum_{j,k=1}^{n} a_{jk}(x)D_jD_k$ be an elliptic differential operator in $\Omega$ such that $a_{jk}(0)$ is real and $a_{jk}$ is Lipschitz continuous in $\Omega$.
In [3], Hörmander proves that if $u \in H^1_{loc}(\Omega)$ satisfying

$$|P(x,D)u| \leq C_1|x|^{-2+\epsilon}|u| + C_2|x|^{-1+\epsilon}|\nabla u| \quad , \ \epsilon > 0$$

and

$$\int_{|x|<R} |u|^2 dx = \mathrm{O}(R^N) \text{ for all } N > 0 \text{ when } R \to 0.$$

Then $u \equiv 0$ in $\Omega$.

It is well known from PliŠ [5] that Hörmander's result fails if we take the functions $a_{jk}$ in any Hölder's class $C^\alpha$ with $\alpha < 1$. Counterexamples due to Alinhac [1] show that it's necessary to assume $a_{jk}$ real at 0 even if it is a smooth function.

In this paper we are interested in the critical case $\epsilon = 0$ in Hörmander's result ; we prove that the same result holds for inequalities of the form

$$|P(x,D)u| \leq C_1|x|^{-2}|u| + C_2|x|^{-1}|\nabla u| \tag{1.1}$$

provided $C_2 < \frac{\sqrt{2}}{2}$ .

**Theorem 1.1.** *Let $P(x,D) = \sum_{j,k=1}^n a_{jk}(x)D_jD_k$ be an elliptic differential operator in a connected open subset $\Omega$ of $I\!R^n$ containing 0, such that $a_{jk}(0)$ is real (we suppose $P(0,D) = -\Delta$ for simplicity) and $a_{jk}$ is Lipschitz continous in$\Omega$ . Let $u \in H^1_{loc}(\Omega)$ be a solution of*

$$|P(x,D)u| \leq C_1|x|^{-2}|u| + C_2|x|^{-1}|\nabla u| \tag{1.1}$$

*with $C_2 < \frac{\sqrt{2}}{2}$ and*

$$\int_{|x|<R} |u|^2 dx = \mathrm{O}(R^N) \ , \ for\ all\ N > 0\ when\ R \to 0\ . \tag{1.2}$$

*Then $u$ is identically zero in $\Omega$.*

**Remark 1.2**

a) In [2] , Alinhac and Baouendi constructed for any $C > 1$ a smooth function $u$ in $\mathbb{R}^2$ flat at 0 , with supp $u = \mathbb{R}^2$ , and satisfying :

$$|\Delta u| \leq C|x|^{-1}|r^{-1}\partial_\theta u|$$

where $x = r(\cos\theta\ ,\ \sin\theta)$ .
But one can easily check that $|r^{-1}\partial_\theta u| \leq (1+\varepsilon)|\partial_r u|$ , where $\varepsilon$ can be taken as small as we want . Then it follows from the identity $|\nabla u|^2 = |\partial_r u|^2 + |r^{-1}\partial_\theta u|^2$ that

$$|\Delta u| \leq (\frac{\sqrt{2}}{2} + \delta)|x|^{-1}|\nabla u|$$

where $\delta$ can be taken arbitrary small . This proves that our assumption on the constant $C_2$ in theorem 1.1 is optimal .
Similar counterexamples are constructed in Wolff[7] for higher dimensions .

b) In theorem 1.1 we have supposed $P(0,D) = -\Delta$ , this can be realised by a linear transform, and then the condition $C_2 < \frac{\sqrt{2}}{2}$ should be replaced by $C_2 < \frac{\sqrt{2}}{2}\lambda_0$ , where $\lambda_0$ is the smallest eigenvalue of the matrix $(a_{jk}(0))$ (we may suppose $(a_{jk}(0))$ positive definite ) .

c) As in Hörmander[3], Theorem 1.1 remains valid if we take the function $a_{jk}$ Lipschitz continous in $\Omega\backslash\{0\}$ and $|\nabla a_{jk}| \leq C|x|^{\delta-1}$ for some $\delta > 0$. As it can be seen in the proof we need only that $|x|^{1-\delta}|\nabla a_{jk}| \to 0$ as $x \to 0$ .

The proof of theorem 1.1 is based on Carleman's method. First we show that any function satisfying (1.1) and (1.2) should satisfy for all $|\alpha| \leq 2$ :

$$\int_{|x|<R} |D^\alpha u|^2 dx \;=\; \mathrm{O}(e^{-CR^{-1}}) \;,\; C > 0.$$

This allows us to use strictly convex weights like $\exp(\frac{\gamma}{2}(\log|x|)^2), \gamma > 0$ , rather than the usual weights $|x|^{-\gamma}$.

Let's introduce the following notations :
We shall denote by $\langle .,. \rangle_2$ the inner product of the Hilbert space $L^2(\mathbb{R}^n\backslash\{0\})$ with respect to the measure $|x|^{-n}dx$, and by $\|.\|_2$ the corresponding norm.We set

$$\varphi_\gamma(x) = \exp(\frac{\gamma}{2}(\log|x|)^2) \;,\; \gamma > 0.$$

**Theorem 1.2.** *For any $\gamma > 0$ (large enough), and for any $u \in C_0^\infty(X\backslash\{0\})$ with $X$ a sufficiently small neighborhood of $0$, we have the estimate*

$$C\| \; |x|^2\varphi_\gamma P(x,D)u\|_2 \geq \gamma^{3/2}\|\varphi_\gamma u\|_2 + \gamma^{1/2}\| \; |x|\varphi_\gamma \nabla u\|_2 \tag{1.3}$$

*where $C$ is a positive constant depending only on $P(x,D)$.*

# 2 proof of the results

After a linear transform, we may assume that $P(0,D) = \Delta$ the Laplace operator in $\mathbb{R}^n$. As in Hörmander [3], let's introduce polar coordinates in $\mathbb{R}^n\backslash\{0\}$ by seting $x = e^t\omega$, with $t \in \mathbb{R}$ and $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. We have then

$$\frac{\partial}{\partial x_j} = e^{-t}(\omega_j\partial_t + \Omega_j)$$

where $\Omega_j$ is a vector field in $S^{n-1}$. Then the operator $P(x,D)$ takes the form

$$P(x,D) = -e^{-2t}\sum_{j,k=1}^{n} a_{jk}(e^t\omega)(\omega_j\partial_t - 1 + \Omega_j)(\omega_k\partial_t + \Omega_k).$$

While the Laplacian becomes

$$e^{2t}\Delta = \partial_t^2 + (n-2)\partial_t + \Delta_\omega \tag{2.1}$$

where $\Delta_\omega = \sum_{j=1}^n \Omega_j^2$ is the Laplace-Beltrami operator in $S^{n-1}$.

The vector fields $\Omega_j$ have the properties

$$\sum_{j=1}^{n} \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} \Omega_j \omega_j = n-1.$$

The adjoint of $\Omega_j$ as an operator in $L^2(S^{n-1})$ is

$$\Omega_j^* = (n-1)\omega_j - \Omega_j. \tag{2.2}$$

Since the functions $a_{jk}$ are Lipschits continous, we have

$$-a_{jk}(e^t\omega) = \delta_{jk} + \mathrm{O}(e^t) \text{ as } \ t \ \to -\infty.$$

The operator $P(x, D)$ can then be written in the form :

$$e^{2t}P(x,D) = \partial_t^2 + (n-2)\partial_t + \Delta_\omega + \sum_{j+|\alpha|\leq 2} C_{j\alpha}(\partial_t)^j \Omega^\alpha \tag{2.3}$$

where $\Omega^\alpha$ denotes the product $\Omega_1^{\alpha_n} \cdots \Omega_n^{\alpha_n}$ , $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $C_{j\alpha}$ are functions satisfying

$$C_{j\alpha}(t,\omega) = \mathrm{O}(e^t) \text{ and } dC_{j\alpha}(t,\omega) = \mathrm{O}(e^t) \text{ as } t \to -\infty, \text{ for any } d \in \{\partial_t, \Omega_1, \cdots, \Omega_n\}.$$

**Lemma 2.1.** *There exists a positive constant $C$ such that for any $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ , for any $\tau \in \{k + \frac{1}{2}\ ,\ k \in \mathbb{N}\}$ , and for any $\delta > 0$ , we have the estimate :*

$$(1+\delta)\int |x|^{-2\tau+4}|\Delta u|^2|x|^{-n}dx \geq C\delta \sum_{|\alpha|=2} \tau^{-2} \int |x|^{-2\tau+4}|D^\alpha u|^2|x|^{-n}dx$$
$$+ (\frac{1}{2} - \delta)\int |x|^{-2\tau+2}|\nabla u|^2|x|^{-n}dx + C\delta\tau^2 \int |x|^{-2\tau}|u|^2|x|^{-n}dx \tag{2.4}$$

**Proof.** Let $v = e^{-\tau t}u$ and $\Delta_\tau v = e^{-\tau t}\Delta(e^{\tau t}v)$. Then it suffices to prove ( with a new constant $C$) :

$$\int\int |e^{2t}\Delta_\tau v|^2 dt d\omega \ \geq \ C\delta \sum_{j+|\alpha|=2} \tau^{-2} \int\int |(\partial_t)^j \Omega^\alpha v|^2 dt d\omega$$
$$+ (\frac{1}{2} - \delta)\int\int |(\partial_t + \tau)v|^2 dt d\omega \ + \ (\frac{1}{2} - \delta)\sum_{j=1}^{n}\int\int |\Omega_j v|^2 dt d\omega$$
$$+ C\delta\tau^2 \int\int |v|^2 dt d\omega \tag{2.5}$$

By (2.1) we have

$$e^{2t}\Delta_\tau = \partial_t^2 + (2\tau + n - 2)\partial_t + \tau(\tau + n - 2) + \Delta_\omega \ ,$$

hence

$$\int\int |e^{2t}\Delta_\tau v|^2 dtd\omega \ = \ \int\int |\partial_t^2 v|^2 dtd\omega \ + \ \int\int |\Delta_\omega v|^2 dtd\omega$$

$$+ \ 2\sum_{j=1}^{n}\int\int |\partial_t\Omega_j v|^2 dtd\omega \ + \ (2\tau^2 + 2(n-2)\tau + (n-2)^2)\int\int |\partial_t v|^2 dtd\omega$$

$$+ \ \tau^2(\tau+n-2)^2\int\int |v|^2 dtd\omega \ - \ 2\tau(\tau+n-2)\sum_{j=1}^{n}\int\int |\Omega_j v|^2 dtd\omega. \tag{2.6}$$

We shall give a lower bound of

$$I(\tau,v) = \tau^2(\tau+n-2)^2\int\int |v|^2 dtd\omega + \int\int |\Delta_\omega v|^2 dtd\omega - 2\tau(\tau+n-2)\sum_{j=1}^{n}\int\int |\Omega_j v|^2 dtd\omega.$$

We recall that the spectrum of $-\Delta_\omega$ as an operator in $L^2(S^{n-1})$ is $\{k(k+n-2), k\in\mathbb{N}\}$, and each eigenspace may be identified with $E_k$ the space of spherical harmonics of degree $k$. It follows that

$$\int\int |\Delta_\omega v|^2 dtd\omega = \sum_{k\geq 0} k^2(k+n-2)^2\int\int |v_k|^2 dtd\omega$$

and

$$\sum_{j=1}^{n}\int\int |\Omega_j v|^2 dtd\omega = \sum_{k\geq 0} k(k+n-2)\int\int |v_k|^2 dtd\omega \ ,$$

where $v_k$ is the projection of $v$ on $E_k$.
After replacing in $I(\tau,v)$, we obtain :

$$I(\tau,v) = \sum_{k\geq 0}\Big(\tau(\tau+n-2) - k(k+n-2)\Big)^2\int\int |v_k|^2 dtd\omega.$$

We have $\Big(\tau(\tau+n-2) - k(k+n-2)\Big)^2 = (\tau-k)^2(\tau+k+n-2)^2$, and since $\tau\in\{k+\frac{1}{2} \ , \ k\in\mathbb{N}\}$ , we get

$$(\tau-k)^2(\tau+k+n-2)^2 \geq \frac{1}{2}\tau(\tau+n-2) + \frac{1}{2}k(k+n-2) \ ,$$

which gives

$$I(\tau,v) \geq \frac{1}{2}\tau(\tau+n-2)\int\int |v|^2 dtd\omega + \frac{1}{2}\sum_{j=1}^{n}\int\int |\Omega_j v|^2 dtd\omega.$$

If we replace in (2.6) we obtain

$$\int\int |e^{2t}\Delta_\tau v|^2 dtd\omega \geq \int\int |\partial_t^2 v|^2 dtd\omega \ + \ 2\sum_{j=1}^{n}\int\int |\partial_t\Omega_j v|^2 dtd\omega$$

$$+\frac{1}{2}\sum_{j=1}^{n}\int\int|\Omega_j v|^2 dtd\omega \ + \ \frac{1}{2}\tau(\tau+n-2)\int\int|v|^2 dtd\omega$$

$$+\left(2\tau^2+2(n-2)\tau+(n-2)^2\right)\int\int|\partial_t v|^2 dtd\omega \tag{2.7}$$

If we multiply (2.6) by $\frac{\delta}{2\tau(\tau+n-2)}$ and add to (2.7) , and using the inequality

$$\int\int|\Delta_\omega v|^2 dtd\omega \geq C_n\sum_{|\alpha|=2}\int\int|\Omega^\alpha v|^2 dtd\omega$$

we obtain the desired result (2.5) . Thus lemma 2.1 is proved .

**Remark.2.1.** The estimate (2.4) in lemma.2.1 remains valid if we suppose $u \in H^2_{loc}(\Omega)$ with compact support and satisfying for all $|\alpha| \leq 2$ and all $N > 0$ , $\int_{|x|<R}|D^\alpha u|^2 dx = \mathrm{O}(R^N)$ as $R \to 0$. We can easily see this by cutting $u$ off for small $|x|$ and regularising .

**Lemma 2.2.** *Let $u$ be as in theorem 1.1. Then $u \in H^2_{loc}(\Omega)$ and satisfies for all $|\alpha| \leq 2$*

$$\int_{|x|<R}|D^\alpha u|^2 dx = \mathrm{O}(e^{-CR^{-1}}) \text{ as } R \to 0 \tag{2.8}$$

*where $C$ is a positive constant .*

**Proof.** First we shall prove that $u \in H^2_{loc}(\Omega)$ , and satisfies for all $|\alpha| \leq 2$ :

$$\int_{|x|<R}|D^\alpha u|^2 dx = \mathrm{O}(R^N) \ , \text{ for all } N > 0 \text{ as } R \to 0. \tag{2.9}$$

Let $u \in H^1_{loc}(\Omega)$ be a solution of (1.1) satisfying (1.2) . From (1.1) we have immediately $P(x,D)u \in L^2_{loc}(\Omega\backslash\{0\})$ . By regularising and using Friedrichs' lemma and ellipticity of $P(x,D)$ , we get without difficulties $u \in H^2_{loc}(\Omega\backslash\{0\})$ .
Following Hörmander[4](Corollary17.1.4. , p.8) we obtain for all $|\alpha| \leq 2$ :

$$\int_{R<|x|<2R}|D^\alpha u|^2 dx = \mathrm{O}(R^N) \ , \text{ for all } N > 0 \text{ as } R \to 0. \tag{2.10}$$

Hence $u$ is the sum of a function in $H^2_{loc}(\Omega)$ and a distribution with support at 0. But no distribution with support at 0 is in $L^2_{loc}$ . It follows that $u \in H^2_{loc}(\Omega)$ .
Since $u \in H^2_{loc}(\Omega)$ it is clear that from (2.10) we have also :

$$\int_{|x|<R}|D^\alpha u|^2 dx = \mathrm{O}(R^N) \ , \text{ for all } N > 0 \text{ as } R \to 0.$$

Let's now prove (2.8) . By assumption we have for all $v \in H^2_{loc}(\Omega)$ :

$$|P(x,D)v(x) - \Delta v(x)|^2 \leq C_0|x|^2\sum_{|\alpha|=2}|D^\alpha v(x)|^2 \tag{2.11}$$

where $C_0$ is a positive constant depending only on $P(x,D)$.
Let $\delta > 0$ to be chosen later . Let $v \in H^2_{comp}(\Omega)$ , whith $supp(v) \subset \{x, |x| < \delta\tau^{-1}\}$ , and satisfying for all $|\alpha| \leq 2$ :

$$\int_{|x|<R} |D^\alpha v|^2 dx = \mathrm{O}(R^N) \ , \text{ for all } N > 0 \text{ as } R \to 0.$$

By remark.2.1 at the end of the proof of Lemma.2.1 , we can apply (2.4) to $v$. If we combine it with (2.11) we get :

$$\begin{aligned}(1+\delta) \int |x|^{-2\tau-n+4} |P(x,D)v|^2 dx \ &\geq \ C\delta\tau^2 \int |x|^{-2\tau-n|\alpha|} |v|^2 dx \\ &+ (\frac{1}{2} - \delta) \int |x|^{-2\tau-n+2} |\nabla v|^2 dx \\ &+ \left(C\delta - 2C_0(1+\delta)\delta^2\right) \tau^{-2} \sum_{|\alpha|=2} \int |x|^{-2\tau-n+4} |D^\alpha v|^2 dx \qquad (2.12)\end{aligned}$$

where $C$ is as in lemma 2.1.

Let $u \in H^1_{loc}(\Omega)$ be a solution of (1.1) satisfying (1.2) . Thus we can apply (2.12) to $v = \chi_\tau u$ , where $\chi_\tau \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_\tau = 1$ for $|x| \leq \frac{1}{2}\delta\tau^{-1}$ , and $\chi_\tau = 0$ for $|x| \geq \delta\tau^{-1}$ . Then for $\tau$ sufficiently large we have, with $R = \frac{1}{2}\delta\tau^{-1}$ :

$$\begin{aligned}(1+\delta) \int |x|^{-2\tau-n+4} |P(x,D)(\chi_\tau u)|^2 dx \ &\geq \ C\delta\tau^2 \int_{|x|<R} |x|^{-2\tau-n|\alpha|} |u|^2 dx \\ &+ (\frac{1}{2} - \delta) \int_{|x|<R} |x|^{-2\tau-n+2} |\nabla u|^2 dx \\ &+ \left(C\delta - 2C_0(1+\delta)\delta^2\right) \tau^{-2} \sum_{|\alpha|=2} \int_{|x|<R} |x|^{-2\tau-n+4} |D^\alpha u|^2 dx \ . \qquad (2.13)\end{aligned}$$

But

$$\begin{aligned}\int |x|^{-2\tau-n+4} |P(x,D)(\chi_\tau u)|^2 dx \ &= \ \int_{|x|<R} |x|^{-2\tau-n+4} |P(x,D)u|^2 dx \\ &+ \int_{|x|>R} |x|^{-2\tau-n+4} |P(x,D)(\chi_\tau u)|^2 dx \ ,\end{aligned}$$

and since $u$ is a solution of (1.1) it follows that

$$\int |x|^{-2\tau-n+4} |P(x,D)(\chi_\tau u)|^2 dx \ \leq \ 2C_1^2 \int_{|x|<R} |x|^{-2\tau-n} |u|^2 dx$$

$$+ \ 2C_2^2 \sum_{|\alpha|=1} \int_{|x|<R} |x|^{-2\tau-n+2} |D^\alpha u|^2 dx \ + \ \int_{|x|>R} |x|^{-2\tau-n+4} |P(x,D)(\chi_\tau u)|^2 dx.$$

Now if we replace in (2.13) we obtain

$$(1+\delta)\int_{|x|>R}|x|^{-2\tau-n+4}|P(x,D)(\chi_\tau u)|^2dx \;\ge\; (\delta\tau^2-2(1+\delta)C_1^2)\int_{|x|<R}|x|^{-2\tau-n}|u|^2dx$$
$$+\left(\frac{1}{2}-\delta-2(1+\delta)C_2^2\right)\int_{|x|<R}|x|^{-2\tau-n+2}|\nabla u|^2dx$$
$$+\;\left(C\delta-2C_0(1+\delta)\delta^2\right)\tau^{-2}\sum_{|\alpha|=2}\int_{|x|<R}|x|^{-2\tau-n+4}|D^\alpha u|^2dx\;.$$

We have by hypothesis $C_2<\frac{\sqrt{2}}{2}$ , hence if we choose $\delta$ sufficiently small we have $\left(\frac{1}{2}-\delta-2(1+\delta)C_2^2\right)>0$ , and $(C\delta-2C_0(1+\delta)\delta^2)>0$ . Thus for $\tau$ sufficiently lagre we get

$$C\int_{|x|>R}|x|^{-2\tau-n+4}|P(x,D)(\chi_\tau u)|^2dx \ge \sum_{|\alpha|\le 2}\tau^{2-2|\alpha|}\int_{|x|<R}|x|^{-2\tau-n+2|\alpha|}|D^\alpha u|^2dx \quad (2.14)$$

where $C$ is a new positive constant .
By construction of $\chi_\tau$ we have $|D^\alpha\chi_\tau|\le C'R^{-|\alpha|}$ , where $C'$ is a positive constant . It follows then

$$\int_{|x|>R}|x|^{-2\tau-n+4}|P(x,D)(\chi_\tau u)|^2dx \;\le\; C''R^{-2\tau-n}\|u\|_{H^2}^2 \quad (2.15)$$

where $\|u\|_{H^2}$ is the $H^2$norm of $u$ in the ball $B(0,2R)$ , and $C''$ a positive constant . On the other hand we have

$$\sum_{|\alpha|\le 2}\tau^{2-2|\alpha|}\int_{|x|<R}|x|^{-2\tau-n+2|\alpha|}|D^\alpha u|^2dx \;\ge\; \sum_{|\alpha|\le 2}\tau^{2-2|\alpha|}\int_{|x|<R/2}|x|^{-2\tau-n+2|\alpha|}|D^\alpha u|^2dx$$
$$\ge \sum_{|\alpha|\le 2}\tau^{2-2|\alpha|}(R/2)^{-2\tau-n+2|\alpha|}\int_{|x|<R/2}|D^\alpha u|^2dx$$

If we combine this estimate with (2.14) and (2.15) we get for sufficiently small $R$ :

$$\sum_{|\alpha|\le 2}\int_{|x|<R/2}|D^\alpha u|^2dx \;\le\; C\|u\|_{H^2}^2R^{-2}2^{-\frac{\delta}{R}}\;,$$

that's

$$\sum_{|\alpha|\le 2}\int_{|x|<R/2}|D^\alpha u|^2dx \;=\; \mathrm{O}(e^{-aR^{-1}}) \quad (2.16)$$

where $a$ is a positive constant (we can take $a=\frac{\delta}{2}\log 2$ ).
We recall that $\tau\in\{k+\frac{1}{2}\;,k\in\mathbb{N}\}$ and $R=\frac{\delta}{2}\tau^{-1}$ . It follows that $R$ must be in the set $\{\;R_k\;,\;k\in\mathbb{N}\}$ , where $R_k=\frac{\delta}{2}(k+\frac{1}{2})^{-1}$ . But since $R_k\le R_{k+1}\le 2R_k$ and $R_k\to 0$ as $k\to\infty$ , one can easily see that (2.16) holds for all small positive $R$ with $a$ replaced by $\frac{a}{2}$ . This achieves the proof of the Lemma .

To prove theorem 1.2 we need a lemma that we take from Hörmander [4] (p. 12). Let's introduce the following notations :

For $k=1,\cdots,n$, we set $D_k=\frac{1}{i}\Omega_k$ and $D_0=\frac{1}{i}\partial_t$. We denote by $D^\alpha$ any product of the form $D_0^{\alpha_0}\cdots D_n^{\alpha_n}$ , $\alpha=(\alpha_0,\cdots,\alpha_n)\in\mathbb{N}^{n+1}$. If we set $\omega_0=0$ it follows from (2.2) that $D_k^*=D_k+i(n-1)\omega_k$ for any $k\in\{0,\cdots,n\}$.

**Lemma .2.3.** *Let $I$ be an open interval of $\mathbb{R}$ , and $A(t,\omega)\in C^0(I\times S^{n-1})\cap L^\infty(I\times S^{n-1})$ such that $D_kA\in L^\infty(I\times S^{n-1})$ for $k=0,\cdots,n$. Then There exists a positive constant $M$ such that for any $u,v\in C_0^\infty(I\times S^{n-1})$ , and for any $\alpha,\beta\in\mathbb{N}^{n+1}$ , with $|\alpha|,|\beta|\leq 2$ , we have :*

$$|\langle AD^\alpha u,D^\beta v\rangle_2-\langle AD^\beta u,D^\alpha v\rangle_2|\leq M\sum_{\alpha',\beta'}\|L\ D^{\alpha'}u\|_2\|L\ D^{\beta'}v\|_2 \tag{2.17}$$

*where the sum is taken over all $\alpha',\beta'$ such that $\max(|\alpha'|,|\beta'|)\leq\max(|\alpha|,|\beta|)$ and $|\alpha'|+|\beta'|$ $\leq|\alpha|+|\beta|-1$ , and where $L(t,\omega)=\max\left(|A(t,\omega)|^{1/2},|D_0A(t,\omega)|^{1/2},\cdots,|D_nA(t,\omega)|^{1/2}\right)$.*

**Proof.** First we note that when $|\alpha|=|\beta|=0$, the left hand side of (2.17) is zero and the statement is abvious. When $|\alpha|=1$ and $|\beta|=0$ we have

$\langle AD_ku,v\rangle_2-\langle Au,D_kv\rangle_2=\langle AD_ku,v\rangle_2-\langle D_k^*(Au),v\rangle_2$. But $D_k^*=D_k+i(n-1)\omega_k$ for any $k\in\{0,\cdots,n\}$. Hence

$$\langle AD_ku,v\rangle_2-\langle Au,D_kv\rangle_2=-\langle(i(n-1)\omega_kA+D_kA)u,v\rangle_2\ ,$$

and by Schwarz inequality we get :

$$|\langle AD_ku,v\rangle_2-\langle Au,D_kv\rangle_2|\ \leq\ M\|Lu\|_2\|Lv\|_2 \tag{2.18}$$

which proves the lemma when $|\alpha|=1$ and $|\beta|=0$.

When $|\alpha|=|\beta|=1$ we have

$$\langle AD_ku,D_jv\rangle_2-\langle AD_ju,D_kv\rangle_2=\langle D_j^*(AD_ku),v\rangle_2-\langle D_k^*(AD_ju),v\rangle_2$$

$=\langle(A[D_j,D_k]+i(n-1)\omega_jAD_k-i(n-1)\omega_kAD_j)u,v\rangle_2\ +\ \langle(D_j(A)D_k-D_k(A)D_j)u,v\rangle_2.$
An easy computation shows that $[D_k,D_j]=\omega_kD_j-\omega_jD_k$ if $k,j\in\{1,\cdots,n\}$ and $[D_k,D_j]=0$ if $j=0$ or $k=0$. Thus if we replace in the last identity we get

$$|\langle AD_ku,D_jv\rangle_2-(AD_ju,D_kv\rangle_2|\ \leq\ M(\|LD_ju\|_2+\|LD_ku\|_2)\|Lv\|_2. \tag{2.19}$$

This proves the lemma when $|\alpha|=|\beta|=1$ .

When $|\alpha|$ or $|\beta|=2$ it suffices to set $u'=D_ju$ or $v'=D_jv$ and apply (2.18) and (2.19) to these functions .

**Proof of theorem.1.2.**

We use the same notations as in Lemma 2.3 :
for $k=1,\cdots,n$, we set $D_k=\dfrac{1}{i}\Omega_k$ and $D_0=\dfrac{1}{i}\partial t$. We denote by $D^\alpha$ any product of the form $D_0^{\alpha_0}\cdots D_n^{\alpha_n}, \alpha=(\alpha_0,\cdots,\alpha_n)\in \mathbb{N}^{n+1}$.

Set $u=e^{-\frac{1}{2}\gamma t^2}v$ and $P_\gamma v=e^{\frac{1}{2}\gamma t^2}P(e^{-\frac{1}{2}\gamma t^2}v)$ , $\gamma>0$ , $v\in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. (We recall that we work in polar coordinates $x=e^t\omega$). Thus by (2.3) the operator $P_\gamma$ can be written

$$e^{2t}P_\gamma=(\partial_t-\gamma t)^2+(n-2)(\partial_t-\gamma t)+\Delta_\omega+\sum_{j+|\alpha|\le 2}C_{j\alpha}(t,\omega)(\partial_t-\gamma t)^j\Omega^\alpha,$$

where the functions $C_{j\alpha}$ satisfy $C_{j\alpha}=\mathrm{O}(e^t)$ as $t\to-\infty$, and $D_k(C_{j\alpha})=\mathrm{O}(e^t)$ as $t\to-\infty$, for any $k\in\{0,\cdots,n\}$.
The estimate (1.3) in theorem.1.2 is then equivalent to

$$C\int\int|e^{2t}P_\gamma v|^2dtd\omega\ge\gamma^3\int\int|tv|^2dtd\omega+\gamma\int\int|\partial_t v|^2dtd\omega+\gamma\sum_{j=1}^n\int\int|\Omega_j v|^2dtd\omega \quad (2.20)$$

($C$ a positive constant).

We shall prove (2.20).
Let $P_\gamma^-$ be the operator obtained from $P_\gamma$ when $\partial_t,\Omega_j$ and $C_{j\alpha}$ are replaced by $-\partial_t,-\Omega_j$ and $\bar{C}_{j\alpha}$ respectively. We shall give a lower bound of the difference :

$$D(\gamma,v)=\|e^{2t}P_\gamma v\|_2^2-\|e^{2t}P_\gamma^- v\|_2^2\ ,$$

and the sum

$$S(\gamma,v)=\|e^{2t}t^{-1}P_\gamma v\|_2^2+\|e^{2t}t^{-1}P_\gamma^- v\|_2^2.$$

We have

$$D(\gamma,v)=4Re\langle(\partial_t^2+\gamma^2t^2-(n-2)\gamma t-\gamma+\Delta_\omega)v,(-2\gamma t+n-2)\partial_t v\rangle_2+R(\gamma,v),$$

where $R(\gamma,v)$ is a sum of terms of the form :

$$\gamma^{4-|\alpha|-|\beta|}Re\Big(\langle AD^\alpha v,D^\beta v\rangle_2-\langle AD^\beta v,D^\alpha v\rangle_2\Big)$$

with $|\alpha|\le 2,|\beta|\le 2$ and $A$ is a function satisfying for $|\alpha|\le 1,|D^\alpha A|=\mathrm{O}(t^4e^t)$ as $t\to-\infty$. (In fact the function $A$ is obtained from products of the functions $C_{j\alpha}$ or products of such functions and the function $t^k$ for $k\le 4$).
Let $T_0<0$ such that $|T_0|$ is large enough to be chosen later . If $v\in C_0^\infty(]-\infty,T_0[\times S^{n-1})$ we have by lemma 2.3 :

$$|\langle AD^\alpha v,D^\beta v\rangle_2-\langle AD^\beta v,D^\alpha v\rangle_2|\le\sum_{\alpha',\beta'}\|L\ D^{\alpha'}v\|_2\|L\ D^{\beta'}v\|_2$$

where the sum is taken over all $\alpha',\beta'$ such that $\max(|\alpha'|,|\beta'|)\le\max(|\alpha|,|\beta|)$ and $|\alpha'|+|\beta'|\le|\alpha|+|\beta|-1$ , and where $L$ satisfies $L(t,\omega)=\mathrm{O}(t^2e^{t/2})$ as $t\to-\infty$ . It follows then :

$$|R(\gamma,v)| \ \le \sum_{|\alpha|\le 2} \gamma^{3-|\alpha|-|\beta|}\|LD^\alpha v\|_2^2 \ . \tag{2.21}$$

Integration by parts gives, with $v \in C_0^\infty(]-\infty, T_0[\times S^{n-1})$ ,

$$4Re\langle(\partial_t^2 + \gamma^2 t^2 - (n-2)\gamma t - \gamma + \Delta_\omega)v, (-2\gamma t + n - 2)\partial_t v\rangle_2 \ =$$

$$= \ 4\gamma\|\partial_t v\|_2^2 + \|f(t)v\|_2^2 - 4\gamma\sum_{j=1}^n \|\Omega_j v\|_2^2$$

where $f^2(t) = 12\gamma^3 t^2 - 12(n-2)\gamma^2 t - 2\gamma^2 + (n-2)^2\gamma$.

It we combine this with (2.21) we get

$$D(\gamma,v) \ge 4\gamma\|\partial_t v\|_2^2 + \|f(t)v\|_2^2 - 4\gamma\sum_{j=1}^n \|\Omega_j v\|_2^2 - \sum_{|\alpha|\le 2}\gamma^{3-2|\alpha|}\|LD^\alpha v\|_2^2. \tag{2.22}$$

We have directly from the definition of $P_\gamma$ and $P_\gamma^-$, with $v \in C_0^\infty(]-\infty, T_0[\times S^{n-1})$ :

$$S(\gamma,v) \ \ \ge \ \ \frac{1}{2}\|t^{-1}((\partial_t - \gamma t)^2 + (n-2)(\partial_t - \gamma t) + \Delta_\omega)v\|_2^2$$

$$+ \ \frac{1}{2}\|t^{-1}((\partial_t + \gamma t)^2 - (n-2)(\partial_t + \gamma t) + \Delta_\omega)v\|_2^2 \ - \sum_{|\alpha|\le 2}\gamma^{4-2|\alpha|}\|C_\alpha D^\alpha v\|_2^2 \tag{2.23}$$

where $C_\alpha$ are functions satisfying $C_\alpha = \mathrm{O}(te^t)$ as $t \to -\infty$.

We have

$$\begin{aligned}
&\frac{1}{2}\|t^{-1}((\partial_t - \gamma t)^2 + (n-2)(\partial_t - \gamma t) + \Delta_\omega)v\|_2^2 \\
&+ \ \frac{1}{2}\|t^{-1}((\partial_t + \gamma t)^2 - (n-2)(\partial_t + \gamma t) + \Delta_\omega)v\|_2^2 \\
&= \ \|t^{-1}\partial_t^2 v\|_2^2 + \|t^{-1}\Delta_\omega v\|_2^2 + 2\sum_{j=1}^n \|t^{-1}\partial_t\Omega_j v\|_2^2 \\
&+ \ \|g(t)\partial_t v\|_2^2 - \sum_{j=1}^n \|\ell(t)\Omega_j v\|_2^2 + \|h(t)v\|_2^2
\end{aligned}$$

where

$$\begin{aligned}
g^2(t) &= (-2\gamma + (n-2)t^{-1})^2 - 2\gamma^2 + 2(n-2)\gamma t^{-1} + 2\gamma t^{-2}, \\
h^2(t) &= (\gamma^2 t - (n-2)\gamma - \gamma t^{-1})^2 - 2(n-2)\gamma t^{-3} - 6\gamma t^{-4}, \\
\ell^2(t) &= 2(\gamma^2 - (n-2)\gamma t^{-1} - \gamma t^{-2}) + 6t^{-4}.
\end{aligned}$$

If we replace in (2.23) we obtain :

$$S(\gamma,v)\geq \|t^{-1}\partial_t^2 v\|_2^2+\|t^{-1}\Delta_\omega v\|_2^2+2\sum_{j=1}^n\|t^{-1}\partial_t\Omega_j v\|_2^2$$

$$+\|g(t)\partial_t v\|_2^2+\|h(t)v\|_2^2-\sum_{j=1}^n\|\ell(t)\Omega_j v\|_2^2-\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\|C_\alpha D^\alpha v\|_2^2. \tag{2.24}$$

Multiplying (2.22) by $\gamma$ and adding to (2.24) we obtain

$$\begin{aligned}\gamma D(\gamma,v)+S(\gamma,v) \ &\geq\ \|t^{-1}\partial_t^2 v\|_2^2+\|t^{-1}\Delta_\omega v\|_2^2+2\sum_{j=1}^n\|t^{-1}\partial_t\Omega_j v\|_2^2\\ &+\ 4\gamma^2\|\partial_t v\|_2^2\ +\|g(t)\partial_t v\|_2^2\ +\ \gamma\|f(t)v\|_2^2+\|h(t)v\|_2^2\\ &-\ \sum_{j=1}^n\|\ell(t)\Omega_j v\|_2^2-4\gamma^2\sum_{j=1}^n\|\Omega_j v\|_2^2-\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\|L'D^\alpha v\|_2^2,\end{aligned} \tag{2.25}$$

where $L'=O(t^2e^{t/2})$ as $t\to-\infty$ .

We have for all $\epsilon>0$ ,

$$\begin{aligned}&\sum_{j=1}^n\|\ell(t)\Omega_j v\|_2^2+4\gamma^2\sum_{j=1}^n\|\Omega_j v\|_2^2=2((\frac{1}{2}\ell^2+2\gamma^2)v,\Delta_\omega v)_2\\ &\leq\ \epsilon^{-1}\|(\frac{1}{2}\ell^2+2\gamma^2)tv\|_2^2+\epsilon\|t^{-1}\Delta_\omega v\|_2^2.\end{aligned} \tag{2.26}$$

If $|T_0|$ and $\gamma$ are large enough we have $\gamma f^2+h^2-\epsilon^{-1}(\frac{1}{2}\ell^2+2\gamma^2)^2t^2\geq(12-9\epsilon^{-1})\gamma^4t^2$ for all $t\in]-\infty,T_0[$ . by choosing $0<\epsilon<1$ such that $12-9\epsilon^{-1}>0$ , we get from (2.25) and (2.26) :

$$\gamma D(\gamma,v)+S(\gamma,v)\ \geq\ \|t^{-1}\partial_t^2 v\|_2^2+(1-\epsilon)\|t^{-1}\Delta_\omega v\|_2^2+2\sum_{j=1}^n\|t^{-1}\partial_t\Omega_j v\|_2^2$$

$$+\|g(t)\partial_t v\|_2^2+2\gamma^2\|\partial_t v\|_2^2+(12-9\epsilon^{-1})\gamma^4\|tv\|_2-\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\|L'D^\alpha v\|_2^2 \tag{2.27}$$

By ellipticity of $\Delta_\omega$ we have

$$\|t^{-1}\Delta_\omega v\|_2^2\ \geq\ C\sum_{|\alpha|=2}\|t^{-1}\Omega^\alpha v\|_2^2$$

and since

$$\gamma^2\sum_{j=1}^n\|\Omega_j v\|_2^2=-\gamma^2(v,\Delta_\omega v)_2\leq\frac{1}{2}\gamma^4\|tv\|_2^2+\frac{1}{2}\|t^{-1}\Delta_\omega v\|_2^2$$

we have

$$(1-\epsilon)\|t^{-1}\Delta_\omega v\|_2^2+(12-9\epsilon^{-1})\gamma^4\|tv\|_2^2\geq C\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\|t^{1-|\alpha|}\Omega^\alpha v\|_2^2.$$

where $C$ is a positive constant . If we replace in (2.27) we obtain

$$\gamma D(\gamma, v) + S(\gamma, v) \;\geq C \sum_{|\alpha|\leq 2} \gamma^{4-2|\alpha|} \|t^{1-|\alpha|} D^\alpha v\|_2^2 - \sum_{|\alpha|\leq 2} \gamma^{4-2|\alpha|} \|L' D^\alpha\|_2^2$$

We recall that $L'(t, \omega) = \mathrm{O}(t^2 e^{t/2})$ as $t \to -\infty$. Hence if $|T_0|$ is sufficiently large we get for $v \in C_0^\infty(]-\infty, T_0[\times S^{n-1})$ :

$$\gamma D(\gamma, v) + S(\gamma, v) \geq C' \sum_{|\alpha|\leq 2} \gamma^{4-2|\alpha|} \|t^{1-|\alpha|} D^\alpha v\|_2^2 \;\;, \;\; C' \text{ a positive constant.}$$

But

$$\begin{aligned} \gamma D(\gamma, v) + S(\gamma, v) \;&=\; \gamma \|e^{2t} P_\gamma v\|_2^2 - \gamma \|e^{2t} P_\gamma^- v\|_2^2 \\ &+\; \|e^{2t} t^{-1} P_\gamma v\|_2^2 + \|e^{2t} t^{-1} P_\gamma^- v\|_2^2 \\ &\leq\; (\gamma + 1) \|e^{2t} P_\gamma v\|_2^2 \;, \end{aligned}$$

that's

$$(\gamma + 1) \|e^{2t} P_\gamma v\|_2^2 \geq C' \sum_{|\alpha|\leq 2} \gamma^{4-2|\alpha|} \|t^{1-|\alpha|} D^\alpha v\|_2^2$$

which is better than the desired result.

**Remark.2.2.** By using a sequence of cut-off functions for small $|x|$ and regularising we can see that theorem.1.2 remains valid if $u \in H^2_{loc}(X)$ with compact support and satisfying for all $|\alpha| \leq 2$ , $\int_{|x|<R} |D^\alpha u|^2 dx = \mathrm{O}(e^{-CR^{-1}})$ as $R \to 0$ , $C > 0$ .

**Proof of theorem 1.1**

Following Hörmander [4] (theorem 17.2.1) it suffices to prove that $u = 0$ in a neighborhood of 0.
Let $u \in H^1_{loc}(\Omega)$ be a solution of (1.1) satisfying (1.2) . By lemma 2.3 $u$ is in $H^2_{loc}$ and satisfies (2.8). Thus by remark.2.2 above we can apply (1.3) to the function $\xi u$ where $\xi \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \leq R_0$ and $\xi(x) = 0$ for $|x| \geq 2R_0$, ($R_0 > 0$ small enough). Then we have, with $C$ a positive constant,

$$\begin{aligned} C \int \varphi_\gamma^2 |x|^{-n+4} |P(x, D)(\xi u)|^2 dx \;&\geq\; \gamma^3 \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n} |u|^2 dx \\ &+ \gamma \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n+2} |\nabla u|^2 dx \end{aligned}$$

On the other hand we have

$$\int \varphi_\gamma^2 |x|^{-n+4} |P(x, D)(\xi u)|^2 dx \;=\; \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n+4} |P(x, D) u|^2 dx$$

$$+ \int_{|x|>R_0} \varphi_\gamma^2 |x|^{-n+4} |P(x,D)(\xi u)|^2 dx \ ,$$

and since $u$ is a solution of (1.1) we get

$$\begin{aligned} \int \varphi_\gamma^2 |x|^{-n+4} |P(x,D)(\xi u)|^2 dx \ \leq \ & 2C_1^2 \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n} |u|^2 dx \\ + \ & 2C_2^2 \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n+2} |\nabla u|^2 dx \\ + \ & \int_{|x|>R_0} \varphi_\gamma^2 |x|^{-n+4} |P(x,D)(\xi u)|^2 dx. \end{aligned}$$

We obtain then

$$\int_{|x|>R_0} \varphi_\gamma^2 |x|^{-n+4} |P(x,D)(\xi u)|^2 dx \ \geq \ (\gamma^3 - 2CC_1^2) \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n} |u|^2 dx$$

$$+ (\gamma - 2CC_2^2) \int_{|x|<R_0} \varphi_\gamma^2 |x|^{-n+2} |\nabla u|^2 dx.$$

We recall that $\varphi_\gamma(x) = \exp(\frac{\gamma}{2}(\mathrm{Log}|x|)^2)$. Hence for $|x| > R_0$ we have $\varphi_\gamma^2(x) < \exp(\frac{\gamma}{2}(\mathrm{Log} R_0)^2)$ and $\varphi_\gamma^2(x) > \exp(\frac{\gamma}{2}(\mathrm{Log} R_0)^2)$ for $|x| < R_0$. Then for $\gamma$ sufficiently large we get

$$\int_{|x|>R_0} |x|^{-n+4} |P(x,D)(\xi u)|^2 dx \ \geq \ (\gamma^3 - 2CC_1^2) \int_{|x|<R_0} |x|^{-n} |u|^2 dx$$

$$+ (\gamma - 2CC_2^2) \int_{|x|<R_0} |x|^{-n+2} |\nabla u|^2 dx.$$

Letting $\gamma \to \infty$, we get $u = 0$ in $B(0, R_0)$.

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