Alexander I. Shnirelman

Weak solutions of incompressible Euler equations with decreasing energy

WEAK SOLUTIONS OF INCOMPRESSIBLE EULER EQUATIONS WITH DECREASING ENERGY

A. Shnirelman

School of Mathematical Sciences
Tel-Aviv University
and
IHES

1. Consider incompressible Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0; \tag{1}$$

$$\nabla \cdot u = 0. \tag{2}$$

Here $u(x, t)$ is the velocity field of the fluid; $p(x, t)$ is the pressure. For simplicity, we consider the flows on an n-dimensional torus: $x \in M = \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n; t \in [0, T]$.

If $\varphi(x, t) \in C_0^\infty(M \times [0, T])$ is a scalar test-function, and $v(x, t) \in C_0^\infty(M \times [0, T])$ is a vector-function, such that $\nabla \cdot v = 0$, then multiplying both sides of equations (1),(2) by $v(x, t)$ and $\varphi(x, t)$ respectively and integrating by parts, we obtain

$$-\int \int \left[(u, \frac{\partial v}{\partial t}) + (u, \nabla v \cdot u)\right] dxdt = 0; \tag{3}$$

$$-\int \int (u, \nabla\varphi)dxdt = 0. \tag{4}$$

The left hand sides of (1),(2) make sense for arbitrary vector field $u(x, t) \in L^2(M \times [0, T])$. This justifies the following

Definition 1. *Vector field $u(x, t) \in L^2$ is called a weak solution of the Euler equations (1),(2), if $u(x, t)$ satisfies relations (3),(4) for arbitrary test-functions $v(x, t), \varphi(x, t)$.*

The identities (3),(4) are quite general, they express the mass and momentum balance in the fluid. Therefore, when deriving the Euler equations in the fluid mechanics, we obtain first the weak formulation (3),(4), and then, assuming sufficient regularity of the field $u(x, t)$, pass to the differential equations (1),(2).

The properties of generic, non-smooth solutions of (3),(4) are poorly understood. Mention here the result of P. Constantin, W. E and E. Titi [3], and G. Eyink [4], that if $n = 3$, and $u(x, t)$ is a Hölder function of x for every t with the Hölder exponent $\alpha > 1/3$, then the kinetic energy

$$W(t) = \int_M \frac{1}{2} |u(x, t)|^2 dx$$

is constant (this was an old conjecture of L. Onsager [5]). Note that such Hölder solutions of the Euler equations are by no means classical, and nothing is known about their existence and (non)uniqueness under appropriate initial conditions.

On the other hand, V. Scheffer [6] has constructed a strange weak solution $u(x, t) \in L^2(\mathbf{R}^2 \times \mathbf{R})$, such that $u(x, t) = 0$ for all (x, t), satisfying the inequality $|x|^2 + |t|^2 > 1$. This solution is an unbounded, almost everywhere discontinuous vector field, which breaks both uniqueness and energy conservation. Moreover, the energy $W(t)$ is non-monotonous in t, and even unbounded.

Much simpler example of a weak solution $u(x, t)$ on $\mathbf{T}^2 \times \mathbf{R}$, which is identically zero for $|t| > 1$, has been constructed in [9]. This is also a discontinuous, unbounded L^2-field.

2. The physical meaning of weak solutions of the Euler equations is not quite clear. We may assume as a hypothesis, that the velocity field of a slightly viscous and slightly compressible fluid tends to a weak solution of the Euler equations, when viscosity and compressibility tend to zero (here we admit arbitrary rheological model of fluid, whatever nonlinear and nonlocal it is). This is probable because of a very general nature of the Euler equations, which contain nothing but the mass and momentum balance. But the very existence of some L^2-velocity field, describing the turbulent motion at infinite Reynolds number, remains still an unproved hypothesis.

If we want to justify our hypothesis, we have to keep in mind the following fundamental property of the flows at high Reynolds numbers. Every flow of viscous fluid dissipates energy; so, the kinetic energy, in the absence of

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external forces, is always a decreasing function.

The rate of the energy dissipation depends on the viscosity. There is a very well established experimental result, that for sufficiently small viscosities, the rate of energy dissipation does not depend on viscosity and is definitely positive.Thus, if our hypothesis is true, and the turbulent flow at very high Reynolds numbers is described by a weak solution of the Euler equations, then the kinetic energy of this weak solution should decrease. (Thus, we have to reject the overmentioned weak solutions with bounded in time supports as nonphysical.)

The natural question is, whether there exist weak solutions whose energy monotonically decreases? This work gives an affirmative answer:

Theorem. *If $n = 3$, then there exists a weak solution $u(x, t) \in L^2(\mathbf{T}^3 \times [0, T])$, such that its kinetic energy $W(t)$ monotonically decreases.*

We do not assert that this solution is already physically meaningful. The "true" definition of weak solution should include relations (3),(4), and some other requirements. Monotonicity of energy is one of them, but possibly not the most fundamental.

3. The simplest mechanical system without explicit friction, but with decaying energy, is the system of two particles, moving freely along the line, which *stick* upon collision, and form one new particle. If m_1, m_2 are the masses, and v_1, v_2 are the velocities of the particles before collision, then the new particle will have the mass $m = m_1 + m_2$, and the velocity v, such that its momentum $mv = m_1 v_1 + m_2 v_2$. It is easy to see that its kinetic energy $mv^2/2$ is strictly less than $m_1 v_1{}^2/2 + m_2 v_2{}^2/2$.

The idea of our construction is to organize a flow in such a manner, that the fluid particles collide and stick; this sticking is just the sink of the energy. These collisions should include essential part of the fluid particles; each particle may meet and stick other ones many and even infinitely many times.

It is clear that such flow should be sufficiently non-regular, for in the smooth velocity fields different particles cannot meet each other; if the velocity field is not sufficiently irregular, then the rate of collisions is not enough to absorb positive energy (this is the possible explanation of the results of [3], [4].)

4. Our main tools are the Generalized Flow, introduced by Y. Brenier [1]. Remind that in the "classical" fluid dynamics the fluid configurations are identified with smooth, volume-preserving diffeomorphisms: all particles

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are labeled by the points of M where they were at $t = 0$, and every other configuration of fluid particles is obtained from the initial one by a smooth permutation, i.e. smooth mapping $\xi : M \to M$. But such description of the fluid motion is not appropriate for irregular flows.

The idea of Brenier is to *decouple* the fluid particles and the points of the flow domain M. Brenier has proposed to introduce a separate space Ω of fluid particles. This is a measurable space with a probabilistic measure $\mu(d\omega)$.

Generalized Flow (GF) is a measurable map $x : \Omega \times [0, T] \to M$, satisfying the following two conditions:

(i) For every $\varphi(x, t) \in C_0^\infty(M \times [0, T])$,

$$\int_\Omega \mu(d\omega) \int_0^T \varphi(x(t, \omega), t) dt = \int_M \int_0^T \varphi(x, t) dx dt;$$

(ii)

$$J = \int_\Omega \mu(d\omega) \int_0^T \frac{1}{2} |\dot{x}(t, \omega)|^2 dt < \infty.$$

The first condition may be called incompressibility, and the second one expresses finiteness of the action (i.e. the mean energy).

Generalized Flows were introduced by Y. Brenier for solving the following problem. Given two fluid configurations, i.e. smooth volume preserving diffeomorphisms, ξ_0 and ξ_1; find an incompressible flow $\xi_t, 0 \leq t \leq 1$, which connects ξ_0 with ξ_1 and minimizes the action J. In a good case, when the solution of this problem exists, it is a classical solution of the Euler equations (1),(2). But generally the solution does not exist (see [7]), and the minimum of the action J is achieved for some Generalized Flow (see [1],[2]) for details).

GFs proved to be a powerful and flexible tool for different problems, related to the geometry of the group of volume preserving diffeomorphisms (see [8]). In the present work we see a new application of GFs. Note that many of our constructions here are similar to the ones described in [8].

5. We start from introducing some classes of GFs.

Definition 1. *Generalized Flow $x(\omega, t)$ is called a GF with definite velocity (GFDV), if there exists a vector field $u(x, t)$, such that*

(i) $u \in L^2(M \times [0, T])$;

(ii) $\nabla \cdot u = 0$ in the weak sense;

(iii) For almost all (ω, t),

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$$\dot{x}(\omega, t) = u(x(\omega, t), t).$$

The field $u(x, t)$ is called a field, associated with the GFDV $x(\omega, t)$. It is clear that this field is essentially unique.

The second important notion is *Generalized Flow with Local Interaction* (GFLI).

Definition 2. *GF $x(\omega, t)$ is called a Generalized Flow with Local Interaction, if for every test-field $v(x, t) \in C_0^\infty$, such that $\nabla \cdot v = 0$,*

$$\int_\Omega \mu(d\omega) \int_0^T (\ddot{x}(\omega, t), v(x(\omega, t), t)) dt = 0. \tag{5}$$

The next is an important class of the GFLI.

Definition 3. *GF $x(\omega, t)$ is called a pressureless GF, if (5) is valid for every test-field $v(x, t) \in C_0^\infty$, without restriction $\nabla \cdot v = 0$.*

Roughly speaking, this means that if we find the "sum" $f(x, t)$ of accelerations of all particles, passing the point x at the time moment t, then we obtain a potential field (in the case of GFLI), or zero (in the case of a pressureless GFLI).

The field f cannot be defined so simply. But the weak definition (5) makes sense, since $x(\omega, t) \in H^1$ for almost all ω; hence, $v(x(\omega, t), t) \in H^1$, while $\ddot{x}(\omega, t) \in H^{-1}$, and the integral (1.4) is defined.

Theorem 1. *If a GF $x(\omega, t)$ is both a GFLI and a GFDV, and $u(x, t)$ is an associated velocity field, then $u(x, t)$ is a weak solution of the Euler equations.*

Weak solution $u(x, t)$ is called a *pressureless* weak solution, if (3) is valid for arbitrary test-field $v(x, t) \in C_0^\infty$, without restriction $\nabla \cdot v = 0$. If $u(x, t)$ is sufficiently regular, then it satisfies (1),(2) with $p \equiv 0$; all such smooth solutions are easy to describe. But, as we shall see, there exist many nontrivial nonsmooth pressureless solutions.

Theorem 2. *If the GF $x(\omega, t)$ is both a pressureless GFLI and a GFDV, and $u(x, t)$ is the associated velocity field, then $u(x, t)$ is a pressureless weak solution of the Euler equations.*

The next important class of GFs are *Sticking* GFs (SGFs). Their definition requires some preliminary constructions.

Let Σ_t be a family of measurable partitions of Ω, $0 \le t \le T$; this means that for every $t \in [0, T]$ there is defined a measurable space Σ_t, and a measurable mapping $\pi_t : \Omega \to \Sigma_t$. We identify elements $\sigma \in \Sigma_t$ and the sets $\pi_t^{-1}(\sigma)$; just these sets are the elements of the partition Σ_t.

Let $\nu_t(d\sigma)$ be a direct image of measure μ in Σ_t. For ν_t-almost all $\sigma \in \Sigma_t$ there is defined a conditional measure $\chi_{t,\sigma}$, i.e. a measure in Ω, such that it is concentrated on σ, and for every μ-integrable function $\Psi(\omega)$,

$$\int_\Omega \Psi(\omega)\mu(d\omega) = \int_{\Sigma_t} \nu_t(d\sigma) \cdot \int_\sigma \Psi(\omega)\chi_{t,\sigma}(d\omega).$$

Suppose that the family Σ_t satisfies the following

Condition 1. *(i) Σ_t is a coarsening family; this means that if $t_1 < t_2$, then for every $\sigma \in \Sigma_{t_1}$ there exists some $\sigma' \in \Sigma_{t_2}$, such that $\sigma \subset \sigma'$;*

(ii) The family Σ_t is continuous from the right:

$$\Sigma_t = \bigwedge_{t' > t} \Sigma_{t'}.$$

Now we are able to define the Sticking Generalized Flow (SGF).

Definition 3. *GF $x(\omega, t)$ is called a SGF, associated with the family Σ_t of partitions, satisfying Condition 1, if*

(i) For every $t \in [0, T]$ and every $\sigma \in \Sigma_t$, if $\omega_1, \omega_2 \in \sigma$, then $x(\omega_1, t) = x(\omega_2, t)$;

(ii) For every $t \in [0, T]$, and every $\sigma \in \Sigma_t$,

$$\int_\sigma \dot{x}(\omega, t)\chi_{\sigma,t}(d\omega) = \int_\sigma \dot{x}(\omega, 0)\chi_{\sigma,t}(d\omega).$$

This definition requires some comments. Condition (i) means that all the particles, belonging to $\sigma \in \Sigma_t$, have stucked and formed one particle by the moment t (we may identify this "large" particle with the class σ). The coarsening condition (Condition 1, (i)) implies that these particles keep moving together for all $t' > t$. This implies, in its turn, that the (right) velocity $\dot{x}(\omega, t)$ is the same for all $\omega \in \sigma \in \Sigma_t$. Hence, for almost all t, and ν_t-almost all $\sigma \in \Sigma_t$, the (true) velocity is defined and constant on σ.

Part (ii) of Definition 3 says that the momentum of the composed particle σ is equal to the sum of momenta of "small" particles ω, which constitute σ.

We must distinguish between the *trivial* and *non-trivial* SGFs.

Definition 4. *SGF $x(\omega, t)$ is called trivial, if for every t, for ν_t-almost all $\sigma \in \Sigma_t$, $\dot{x}(\omega, t) = \mathrm{const}$ for $\chi_{t,\sigma}$-almost all $\omega \in \sigma$.*

Otherwise the SGF $x(\omega, t)$is called non-trivial.

Trivial SGFs are easy to describe. Every composite particle σ consists of small particles ω, moving from the beginning with equal velocities; this means that their trajectories coincide, and they stick at the moment t only formally; the sticking does not affect their motion. Since there is no interaction between the particles, the trivial SGF is a free motion of noninteracting particles with a constant density. If the SGF is not trivial, it is called a *nontrivial* SGF.

Theorem 3. *Suppose that the following conditions are satisfied:*

(i) $x(\omega, t)$ is a nontrivial SGF;

(ii) $x(\omega, t)$ is an L^3-GF;

(iii) $x(\omega, t)$ is a GFDV, with associated velocity field $u(x, t)$.

Then

(i) $u(x, t)$ is a weak solution of the Euler equations;

(ii)Moreover, $u(x, t)$ is a pressureless weak solution;

(iii)Kinetic energy $W(t) = \frac{1}{2} \int_M |u(x, t)|^2 dx$ is a truly decreasing function of t (i.e., $W'(t) \leq 0$, and $W(T) < W(0)$) .

The main result of this work is the following

Theorem 4. *If $n = 3$, then there exists a GF $x(\omega, t)$, which satisfies all conditions of Theorem 4.*

Thus, we obtain an example of a weak solution of the Euler equations with decreasing energy.

6. We prove Theorem 4 by explicit construction; its idea is the following. Consider first arbitrary smooth flow of incompressible fluid (call this fluid the main phase) in M, whose velocity field $u_0(x, t)$ is *not* a solution of the Euler equations. This means that it is necessary to apply some external force $f_0(x, t)$ to the fluid, so that it moved in prescribed manner.

Suppose now, that in the same volume there is a finite number of other fluids, having constant densities, and moving with constant and different velocities (call these fluids controlling phases). This is a simple GF, called *multiflow*. Now suppose that the particles of controlling phases are being absorbed by the particles of the main phase; the rates of absorption are, generally, some functions of (x, t), which we may define arbitrarily. This GF may be called a *multiflow with mass exchange*. The absorbed particles add their momentum to the momentum of the absorbing ones. This is equivalent to the action of some force $g(x, t)$, depending on the rates of absorption,

and we may define these rates (which are in our hands) in such a way, that $g(x, t) = f(x, t)$ for all (x, t). Thus, the main phase will move in a given way without external forces; but the densities of the phases are not constant any more, and the overall density becomes, generally, variable. Now we may introduce a few new phases (call them *compensating* phases), having constant velocities, but non-constant densities, and moving freely, without interaction with previously defined phases, so that the overall density is sufficiently close to constant.

Thus we obtain an SGF, which, however, is not a GFDV, and whose density is only approximately constant. Next is the crucial step of our construction. For every phase, we define a perturbation of its motion (so that changes of coordinates, velocities, and accelerations of all particles are arbitrarily small), such that this perturbation nearly separates all phases. This means that at every point (x, t) outside some set in $M \times [0, T]$ of arbitrarily small measure the densities of all phases but one become arbitrarily small. This construction is technically the most difficult part of the work, and may be done only if $n = 3$ (there is not enough freedom if $n = 2$).

But the GF just constructed is no more a SGF, because we have slightly, but arbitrarily changed the trajectories of particles. In order to compensate the new accelerations, we introduce new controlling phases, and define the absorption rates by all the previously defined phases, so that this absorption exactly compensates the extra accelerations. Then we introduce new compensating phases, which move without interaction with all other phases, and whose role is to make the overall density close to constant (much closer than on the previous step).

Again, we have an SGF, which is not perfectly incompressible, and not perfectly a GFDV. We construct a new perturbation of the motion of every phase, which separates phases much better than it was on the previous step; then compensate the small accelerations by absorption of particles of new controlling phases, compensate the variations of density by new compensating phases, etc. It is possible to perform all the steps so that we obtain *a convergent process*. This means that the sequence of GFs, regarded as random processes, converges in a weak sense to some random process, which satisfies all hypotheses of Theorem 1.2.

The weak solution we have constructed is still far from the real fluid flows. Its only "realistic" feature is the decay of kinetic energy. But two other features may also have some relation to the reality: the direct interaction of

particles with different velocities, without any cascade, and the absence of the pressure. Possible role of these phenomena in the "true" weak solutions is still unclear.

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