SEMINAIRE

Equations aux Dérivées Partielles

1996-1997

Maciej Zworski

Poisson formulæ for resonances.

Séminaire É. D. P. (1996-1997), Exposé n° XIII, 12 p.

<http://sedp.cedram.org/item?id=SEDP_1996-1997____A13_0>

Poisson formulæ for resonances.

Maciej Zworski

Department of Mathematics, University of Toronto
and
Centre de Mathématiques, École Polytechnique

1 Statement of the results.

The purpose of this exposé is to to present a new proof of Poisson formula for resonances. It comes essentially from joint work with Laurent Guillopé [**?**] and the main point is that we avoid the use of Lax-Phillips theory and in particular of the strong Huyghens principle. That was necessary for extending the formula to the case of surfaces with infinite volume hyperbolic ends. It was however the Lax-Phillips theory which provided the original motivation for the formula.

We start by recalling the abstract assumptions of "black box" scattering from [**?**]. Thus we consider a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)) \,, \quad n \ \text{ odd,} \tag{1.1}$$

and an operator

$$\begin{aligned} &P \ : \ \mathcal{H} \longrightarrow \mathcal{H} \,, \quad \text{self-adjoint with a domain } \mathcal{D} \subset \mathcal{H} \\ &\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0,R_0)) \\ &\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} P = -\Delta|_{\mathbb{R}^n \setminus B(0,R_0)} \end{aligned}$$

The last condition can be relaxed to allow operators with coefficients which, with all their derivatives, differ from those of $-\Delta$ super-exponentially, that is, by $\mathcal{O}(\exp(-|x|^{1+\epsilon})$ for some fixed $\epsilon > 0$. (see [**?**] or [**?**] for precise definitions). We also assume that

$$\exists \, k \text{ such that } \ \mathbf{1}_{B(0,R_0)}(P+i)^{-k} \ \text{ is of trace class } \,, \tag{1.2}$$

and that our operator is semi-bounded

$$P \geq -C \,, \quad C \geq 0 \,. \tag{1.3}$$

As simple examples where all the conditions are satisfied we can take

Example 1. $P = -\Delta + V$, $V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, n odd.

Example 2. $P = \Delta_X - 1/4$, $X = \Gamma \backslash \mathbb{H}^2$ is a finite volume, non-compact hyperbolic surface. In this case $n = 1$ and $\mathcal{H}_{R_0}$ is a more complicated Hilbert space – see [**?**] and references given there.

Condition (**??**) implies that

$$\cos t\sqrt{P} - \cos t\sqrt{\Delta}\mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)} \in \mathcal{D}'(\mathbb{R}, \mathcal{L}^1(\mathcal{H},\mathcal{H}))$$

and we can define a distribution on $\mathbb{R}$

$$u(t) = 2\mathrm{tr}\,\left(\cos t\sqrt{P} - \mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)}\cos t\sqrt{\Delta}\mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)}\right). \tag{1.4}$$

The simplest definition of *resonances* comes from the following theorem which in this level of generality was established in [**?**]:

Theorem 1. *The resolvent, $R(\lambda) = (P-\lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$, $\mathrm{Im}\,\lambda > 0$, $\lambda^2 \notin \sigma(P)$, continues meromorphically to $\mathbb{C}$ (when n is odd; to the logarithmic plane when n is even) as an operator*

$$R(\lambda) \; : \; \mathcal{H}_{\mathrm{comp}} \longrightarrow \mathcal{D}_{\mathrm{loc}} \,,$$

with poles of finite rank.

The poles of $R(\lambda)$ are called resonances and the multiplicity is given by

$$m_\lambda(R) = (1+\epsilon(\lambda))\mathrm{dim}\sum_{j=1}^k A_j(\mathcal{H}_{\mathrm{comp}})\,, \quad R(\zeta) = \sum_{j=1}^k \frac{A_j}{(\zeta-\lambda)^j} + A_0(\zeta)\,, \quad \epsilon(\lambda) = \begin{cases} 1 & \lambda = 0 \\ 0 & \lambda \neq 0 \end{cases}\,, \tag{1.5}$$

where A_0 is holomorphic near λ. For $\lambda \neq 0$ this coincides with the more natural definition

$$m_\lambda(R) = \mathrm{rank}\int_{\gamma_{\lambda,\epsilon}} R(\zeta)\zeta d\zeta\,, \quad \lambda \neq 0\,, \quad \theta \mapsto \gamma_{\lambda,\epsilon}(\theta) = \lambda + \epsilon e^{i\theta}\,.$$

Remark. The definition (**??**) is different from the ones used in [**?**],[**?**],[**?**] and hopefully it is now correct. Shmuel Agmon pointed out to me the need to multiply by two at $\lambda = 0$ in a private conversation but the necessity of it became apparent only after the comparison with multiplicities in Theorem 5 below – see the proof of Theorem 2 at the end of Sect.3. In fact, Theorem 5 would look nicer and more natural without the multiplication by two at $\lambda = 0$.

The Poisson formula for resonances is given in

Theorem 2. *For $t \neq 0$, that is in the sense of $\mathcal{D}'(\mathbb{R}\setminus\{0\})$, we have*

$$u(t) = \sum_{\lambda\in\mathbb{C}} m_\lambda(R) e^{-i\lambda|t|}\,, \tag{1.6}$$

*where u is given by (**??**), $m_\lambda(R)$ by (**??**) and n is odd.*

For scattering by obstacles (that is for $P = -\Delta$, $\mathcal{H} = L^2(\mathbb{R}^2\setminus\mathcal{O})$, $\mathcal{D} = H^2(\mathbb{R}^2\setminus\mathcal{O})\cap H^1_0(\mathbb{R}^2\setminus\mathcal{O})$) and for sufficiently large times, this was obtained by Lax-Phillips [**?**] and by Bardos-Guillot-Ralston [**?**]. Melrose [**?**],[**?**], extended that result to all non-zero times and as stated above the theorem is due to Sjöstrand-Zworski [**?**]. The observation that it can be extended to super-exponentially decaying perturbations was made in [**?**].

The formula (**??**) is clearly equivalent to

$$\widehat{\chi u}(\lambda) = \sum_{\mu \in \mathcal{C}} \hat{\chi}(\lambda + \mu) m_\mu(R) \,, \quad \chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}_+) \,, \quad \lambda \in \mathbb{R} \,.$$

Recently, Sjöstrand [**?**] obtained a local trace formula valid in all dimensions and for more general operators. Specialized to the situation described above his theorem gives

Theorem 3. (Sjöstrand) *For u given by (**??**) and $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}_+)$ but in all dimensions*

$$\widehat{\chi u}(\lambda) = \sum_{\mu^2 \in \lambda^2 \Omega} \hat{\chi}(\lambda + \mu) m_\mu(R) + \mathcal{O}(|\lambda|^{-\infty}) \,, \quad \lambda \in \mathbb{R} \,,$$

where $\Omega = \{ z \in \mathbb{C} : 0 < a < \mathbb{R}{\rm e}\, z < b, -c < {\rm Im}\, z \leq 0 \}$.

For n odd this special consequence of [**?**] follows from (**??**) and from the polynomial bound on the number of resonances[1], [**?**], [**?**].

Despite its severe assumptions Theorem 2 is worth having as it allows the use of the behaviour of $u(t)$ near $t = 0$ and starting with [**?**] that has already had interesting applications, see [**?**],[**?**],[**?**],[**?**],[**?**],[**?**].

We conclude this section with some discussion of the question whether (**??**) could hold through 0. The right hand side of (**??**) does not immediately make sense as a distribution on $\mathbb{R}$, that is, there is no canonical way of extending it through 0. In Example 1 with $n = 1$ and in Example 2 (where $n = 1$ as well) we can use the principal value regularization:

$$\text{p.v.} \sum_{\lambda \in \mathbb{C}} f(\lambda) = \lim_{R \to \infty} \sum_{|\lambda| \leq R} f(\lambda) \,.$$

In Example 1 with $n = 1$ we have [**?**]

$$u(t) = \text{p.v.} \sum_{\lambda \in \mathbb{C}} m_\lambda(R) e^{-i\lambda|t|} + 4|\text{chsupp}\, V| \delta_0(t) \,,$$

and in Example 2, W. Müller (see [**?**],[**?**]) showed that

$$u(t) = \text{p.v.} \sum_{\lambda \in \mathbb{C}} m_\lambda(R) e^{-i\lambda|t|} - \log q \delta_0(t) \,,$$

where q is the factor in the Blaschke product representation of the scattering matrix. Hence something is happening at $t = 0$ and it is related to the distribution of resonances.

Acknowledgments. I am very grateful to Tanya Christiansen, Laurent Guillopé and Richard Melrose for their comments on the first version of this note.

[1]The required polynomial bound on the number of eigenvalues of the reference operator follows from (**??**), see (**??**) and (**??**) below.

2 An application.

As an example of an application of Theorem 2 we present the following result obtained jointly with Antônio Sá Barreto, [?]:

Theorem 4. *If* $V \in \mathcal{C}_c^\infty(\mathbb{R}^n;\mathbb{R})$ *and* n *is odd then* $P = -\Delta + V$ *has infinitely many resonances.*

As mentioned above the compact support condition on V can be relaxed and V can also be taken less regular. Previous results in this direction were obtained by Lax-Phillips [?], Menzala-Schonbek [?], Melrose [?] and Bañuelos-Sá Barreto [?].

The proof of Theorem 4 is quite simple and can be outlined as follows. Suppose there were *no* resonances. Then by Theorem 2 we would have

$$\text{supp}\, u \subset \{0\}\,. \tag{2.1}$$

The now standard results on the behaviour of the wave trace at $t = 0$ (see [?],[?]) show that

$$u(t) \sim \sum_{j=1}^{\frac{n-1}{2}} d_j(V)\delta_0^{(n-1-2j)}(t) + \sum_{j=\frac{n+1}{2}}^{\infty} d_j(V)|t|^{-n+2j}\,, \tag{2.2}$$

where $d_2(V) = c_2 \int V^2$, $c_2 \neq 0$. If (**??**) holds, that is, there are no resonances, then we only have the first sum and an exact equality. When $n = 3$ then $-3 + 4 > 0$ so in that case we immediately see that $\int V^2 = 0$ and hence $V = 0$ ([?]). Also, as observed in [?] existence of some resonances implies that there must be infinitely many of them: in odd dimensions there is no constant term in the expansion of the singularity at $t = 0$ and if there were only finitely many poles (**??**) would give

$$0 = \lim_{t\to 0+}\left(u(t) - \sum_{j=1}^{\frac{n-1}{2}} d_j(V)\delta_0^{(n-1-2j)}(t)\right) = \text{number of resonances}\ .$$

Hence in dimensions higher than three we would like to extract $d_2(V)$ from (**??**). For that we apply the well known "wave-to-heat" transform to (**??**) and obtain

$$\text{tr}\,(e^{-tP} - e^{t\Delta}) = t^{-\frac{n}{2}} \sum_{j=1}^{\frac{n-1}{2}} \alpha_j d_j(V) t^j\,,\ t > 0\,,\quad \alpha_j \neq 0\,. \tag{2.3}$$

On the other hand the assumption that P has no eigenvalues and zero-resonances implies, by essentially standard heat estimates, that

$$|\text{tr}\,(e^{-tP} - e^{t\Delta})| \leq C_V t^{-\frac{n}{2}+1}\,,\ t > 0\,, \tag{2.4}$$

see [?] and reference given there. Hence $d_2(V) = 0$ which implies $V = 0$. As was pointed out to me by Werner Müller the estimate (**??**) can also be obtained using the Birman-Krein formula for the heat operators (see Sect.3 below for the wave version) and the behaviour of the scattering phase near 0.

When $n = 3$ the result can be extended to cover any super-exponentially decaying self-adjoint elliptic perturbation of the Laplacian, [?]. For $n = 5$ and metric perturbations and for conformal perturbations in higher dimensions the existence of resonances was shown in [?].

3 Proof of Theorem 2.

The new proof of Theorem 2 is based on a trace formula of the Birman-Krein type (already well known in most interesting situations, see [**?**]) and an estimate on the determinant of the scattering matrix in all of the complex plane.

To avoid inessential technical issues we now assume that

$$\sigma_{\mathrm{pp}}(P) \cap (0, \infty) = \emptyset \,, \tag{3.1}$$

that is, that P has no embedded eigenvalues. The contribution of embedded eigenvalues can either be completely separated (as in the case of scattering by compact obstacles with non-connected exteriors) or removed by a generic perturbation (as in the case of finite volume quotients). The continuity of both sides of (**??**) shows then that that identity holds in general (see Sect.4 of [**?**] for a similar argument). A direct argument which would complicate the statement of Theorem 5 below is also possible.

For $\lambda \in \mathbb{R} \setminus \{0\}$ the scattering matrix

$$S(\lambda) \; : \; L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1})$$

can be defined using the asymptotic behaviour of the generalized eigenfunctions of P which in turn can be obtained from the asymptotic behaviour of the resolvent. Thus for $\lambda \in \mathbb{R} \setminus \{0\}$ let $E(\lambda; \omega) \in \mathcal{D}^\infty_{\mathrm{loc}}$, $\omega \in \mathbb{S}^{n-1}$, be the unique solution[2] of

$$(P - \lambda^2) E(\lambda; \omega) = 0 \,, \quad E(\lambda; \omega)|_{\mathbb{R}^n \setminus B(0, R_0)} = e^{i\lambda\langle x, \omega\rangle} + \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} F\left(\lambda; \omega, \frac{x}{|x|}, \frac{1}{|x|}\right) \,, \tag{3.2}$$

where $F \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times [0, R_0^{-1}))$, that is, F has an expansion in $|x|^{-1}$ with a bounded leading term. We then define

$$A(\lambda) \; : \; L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1}) \,, \quad A\phi(\theta) = \int_{\mathbb{S}^{n-1}} A(\theta, \omega) \phi(\omega) d\omega \,, \tag{3.3}$$

with the natural volume measure on $\mathbb{S}^{n-1}$ and the kernel of A is obtained from the scattering amplitude F:

$$A(\theta, \omega) = c_n \lambda^{\frac{n-1}{2}} F(\lambda; \omega, -\theta, 0) \,, \quad c_n = (2\pi)^{-\frac{n-1}{2}} e^{\frac{i\pi}{4}(n-1)} \,.$$

Then the scattering matrix is

$$S(\lambda) = I + A(\lambda) \,. \tag{3.4}$$

A more natural but less explicit and perhaps somewhat mysterious definition of $S(\lambda)$ is given as the intertwining operator between the incoming and outgoing generalized eigenfunctions: $S(\lambda) E(-\lambda; \bullet) = E(\lambda; \bullet)$ where we consider $E(\lambda; \bullet)$ as a function on $\mathbb{S}^{n-1}$ with values in $\mathcal{D}^\infty_{\mathrm{loc}}$. One could also use the standard dynamic definition (see e.g. [**?**], Chapter 14) where $S(\lambda)$ comes

[2] Uniqueness follows from Rellich's uniqueness theorem and (**??**).

from the spectral decomposition of the scattering operator along the generalized eigenspaces of the free Laplacian with which it commutes or the general stationary definition (see [?] for a formulation based on the structure at infinity).

The kernel of the operator $A(\lambda)$ is analytic in (θ, ω) and in particular $A(\lambda)$ is a trace class operator. Hence we can define

$$s(\lambda) = \det S(\lambda) \,, \quad \sigma(\lambda) = \frac{i}{2\pi} \log \det S(\lambda) \,, \tag{3.5}$$

where $\sigma(\lambda)$ is the scattering phase. Heuristically it measures the averaged phase shift of a wave passing through the "black box" perturbation. The resonances close to the continuous spectrum correspond to the peaks in its derivative and that roughly explains the physical origin of the notion of resonance. In fact, the Poisson formula of Theorem 2 and its proof below below give some rigorous meaning to that connection.

The first trace formula relates the wave group and the scattering phase, $\sigma(\lambda)$:

Theorem 5. *For P satisfying the assumptions of Sect.1 and (??) and for $u(t)$ and $\sigma(\lambda)$ defined by (??) and (??) respectively we have $\sigma' \in \mathcal{S}'(\mathbb{R})$ and*

$$u(t) = \widehat{\frac{d\sigma}{d\lambda}}(t) + 2 \sum_{\mathrm{Im}\,\lambda > 0} m_\lambda(R) \cos(t\lambda) + m_0(R) \,, \quad t \neq 0$$

This is motivated by classical results of Birman-Krein and can be proved using the Maaß-Selberg type relations which in Euclidean scattering appeared already in the work of Buslaev (see [?] and [?]). By adding an explicit term $C_{R_0} D_t^{n-1} \delta_0(t)$ to the left hand side we could make this formula valid for all t but that is not relevant to our discussion (this is more subtle in even dimensions).

As an operator on $L^2(\mathbb{S}^{n-1})$ the scattering matrix $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ and it extends to a meromorphic family of operators in $\mathbb{C}$ (when n is odd, and in the logarithmic plane if n is even). The continuation still posesses the structure (??) with $A(\lambda)$ meromorphic with values in smoothing (analytic) operators. The simplest way of defining the multiplicity of the poles of $S(\lambda)$ is through $s(\lambda)$:

$$m_\lambda(S) = k \,, \quad s(\mu) = (\lambda - \mu)^{-k} f(\mu) \,, \quad f(\lambda) \neq 0 \,, \ f \text{ is holomorphic near } 0 \,. \tag{3.6}$$

We then have

$$m_\lambda(S) = m_\lambda(R) - m_{-\lambda}(R) \,, \tag{3.7}$$

that is the multiplicities of the poles of $S(\lambda)$ and of $R(\lambda)$ essentially coincide. The relation (??) can be proved either by the methods of Sect.2 of [?] or by establishing first the simpler case of simple poles and then using the generic simplicity of resonances proved in [?] or the perturbation methods of [?].

To estimate the number of resonances of P we follow [?],[?],[?] and introduce a reference operator $P^\sharp$. Roughly speaking, we take $R > R_0$ and compactify $\{x : |x_i| \leq R, i = 1, \cdots, n\}$ to obtain a torus $\mathbb{T}^n_R$. Then $P^\sharp$ is a self-adjoint operator on $\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}^n_R \setminus B(0, R_0))$ given as P on $\mathcal{H}_{R_0}$ and as the positive Laplacian on the torus on $L^2(\mathbb{T}^n_R \setminus B(0,R_0))$ (see [?] for a more precise description). Condition (??) implies that

$$\sharp\{\lambda \; : \; \lambda^2 \in \sigma(P^\sharp)\,, \; \lambda^2 \leq r^2\} \leq C_\epsilon r^{m+\epsilon}\,, \quad n \leq m \leq 2k\,, \tag{3.8}$$

for any $\epsilon > 0$ and we take m to be the smallest real number for which this estimate holds. Of course, in many situations we can simply take $\epsilon = 0$ and for elliptic perturbations $m = n$, $\epsilon = 0$.

The consequent bound on the number of resonances is

$$\sum_{|\lambda| \leq r} m_\lambda(R) \leq C_\epsilon r^{m+\epsilon}\,. \tag{3.9}$$

and we can from a Weierstrass product over the poles of $R(\lambda)$:

$$P(\lambda) = \prod_{\mu \in \mathbb{C}} E\left(\frac{\lambda}{\mu}, [m]\right)^{m_\mu(R)}\,, \quad E(z,p) = (1-z)\exp\left(z + \cdots + \frac{z^p}{p}\right)\,, \tag{3.10}$$

which defines an entire function of order m, more precisely, $|P(\lambda)| \leq C\exp(C|\lambda|^{m+\epsilon})$. From (??) and the meromorphy of $s(\lambda)$ we immediately get *for n odd*

$$s(\lambda) = e^{g(\lambda)} \frac{P(-\lambda)}{P(\lambda)}\,, \tag{3.11}$$

where g is an entire function. Theorem 2 follows from Theorem 5 and

Proposition 6. *The entire function g given in* (??) *is a polynomial. That is there exists l such that*

$$|g(\lambda)| \leq C + C|\lambda|^l\,.$$

This is proved using the techniques developed for estimating the number of scattering poles [?],[?],[?],[?],[?] and is the only component of the argument that has not been available in the last 25 years. The connection between Theorems 2 and 5 was already exploited before by Melrose [?] to obtain scattering asymptotics and his argument was later used in [?],[?],[?].

In Sect.4 we give a complete argument in the case of $P = -\Delta + V$, $V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, n odd and indicate the method for the extension to the general case (which is already essentially treated in Sect.3 of [?]).

To see how Proposition 6 implies Theorem 2 we write

$$\begin{aligned}\left(\frac{d}{d\lambda}\right)^{2M+1} \log s(\lambda) &= \left(\frac{d}{d\lambda}\right)^{2M+1} g(\lambda) \\ &+ \sum_{\zeta \in \mathbb{C}} m_\zeta(R) \left[\left(\frac{d}{d\lambda}\right)^{2M+1} \log E\left(-\frac{\lambda}{\zeta}, [m]\right) - \left(\frac{d}{d\lambda}\right)^{2M+1} \log E\left(\frac{\lambda}{\zeta}, [m]\right)\right]\end{aligned}$$

which for $2M \geq \max\{l, [m]\}$ is equal to

$$\sum_{\zeta \in \mathbb{C}} m_\zeta(R) \left(\frac{(2M)!}{(\lambda+\zeta)^{2M+1}} - \frac{(2M)!}{(\lambda-\zeta)^{2M+1}}\right)$$

that is we obtain

$$\left(\frac{1}{i}\frac{d}{d\lambda}\right)^{2M}\frac{d\sigma}{d\lambda}(\lambda)=\frac{i}{2\pi}\sum_{\zeta\in\mathbb{R}}m_\zeta(R)\left(\frac{1}{i}\frac{d}{d\lambda}\right)^{2M}\left(\frac{1}{(\lambda-\zeta)}-\frac{1}{(\lambda+\zeta)}\right).$$

Denoting by $\mathcal{F}$ the Fourier transform on $\mathbb{R}$ we observe that for $\operatorname{Im}\zeta<0$

$$\mathcal{F}^{-1}\left(t^{2M}e^{-i\zeta|t|}\right)(\lambda)=\frac{i}{2\pi}\left(\frac{1}{i}\frac{d}{d\lambda}\right)^{2M}\left(\frac{1}{(\lambda-\zeta)}-\frac{1}{(\lambda+\zeta)}\right),$$

which with ζ replaced by $-\zeta$ is also valid for $\operatorname{Im}\zeta>0$. Hence we obtain

$$\left(\frac{1}{i}\frac{d}{d\lambda}\right)^{2M}\frac{d\sigma}{d\lambda}(\lambda)=\sum_{\operatorname{Im}\zeta<0}m_\zeta(R)\mathcal{F}^{-1}\left(t^{2M}e^{-i\zeta|t|}\right)(\lambda)-\sum_{\operatorname{Im}\zeta>0}m_\zeta(R)\mathcal{F}^{-1}\left(t^{2M}e^{i\zeta|t|}\right)(\lambda).$$

On the other hand, by Theorem 5

$$\mathcal{F}\left(\left(\frac{1}{i}\frac{d}{d\lambda}\right)^{2M}\frac{d\sigma}{d\lambda}\right)(t)=$$
$$t^{2M}u(t)-\sum_{\operatorname{Im}\zeta>0}m_\zeta(R)t^{2M}e^{-i\zeta|t|}-\sum_{\operatorname{Im}\zeta>0}m_\zeta(R)t^{2M}e^{i\zeta|t|}-t^{2M}m_0(R),$$

and hence

$$t^{2M}u(t)=t^{2M}\sum_{\zeta\in\mathbb{C}}m_\zeta(R)e^{-i\zeta|t|},\quad\text{in }\mathcal{D}'(\mathbb{R}),\tag{3.12}$$

which implies (**??**).

The knowledge of the order of the singularity of $u(t)$ at $t=0$ and the proof of the Poisson formula presented above give an improvement in the estimate of the order of the scattering matrix which from (**??**) and from Proposition 6 can so far be estimated by $\max\{l,m\}$[3]:

Theorem 7. *If m is the order of growth of eigenvalues of the reference operator $P^\sharp$ (see (**??**)) then the order of the determinant of the scattering matrix, $s(\lambda)$, as a meromorphic function is m, that is we can take $l=[m]$ in Proposition 6. In particular for any elliptic compactly supported perturbation of $-\Delta$ in $\mathbb{R}^n$, n, odd, the order of the determinant of the scattering matrix is n.*

This means in particular that in (**??**) we can replace $2M$ by $[m]$.

4 Proof of Proposition 6.

We will now prove Proposition 6 in the case of $P=-\Delta+V$, $V\in L^\infty_{\text{comp}}$, n odd, and then indicate the way the argument can be adapted to the general case. By Hadamard's factorization

[3] This observation is a direct consequence of some recent e-mail exchanges with Richard Melrose.

theorem (see for instance [?], 8.24) it is enough to show that there exists an entire function, $h(\lambda)$, such that for some N

$$|h(\lambda)s(\lambda)| \leq Ce^{C|\lambda|^N}, \quad |h(\lambda)| \leq Ce^{C|\lambda|^N}.$$

That will in turn follow from an estimate

$$|s(\lambda)| \leq Ce^{C|\lambda|^N}, \quad \lambda \notin \bigcup_{m_\zeta(S)>0} D(\zeta, \langle\zeta\rangle^{-m-\epsilon}) \tag{4.1}$$

where m is equal to the exponent in the bound on the number resonances (**??**)) and $\epsilon > 0$ is fixed. Once we have (**??**) we can for instance take $P(\lambda)$ and use the maximum principle.

To estimate $s(\lambda)$ we shall use the Weyl inequalities following the method introduced by Melrose [?],[?] and then refined in [?],[?],[?]. For that we recall that if A is a trace class operator with eigenvalues $\{\lambda_j(A)\}_{j=1}^\infty$, $|\lambda_1(A) \geq \cdots \geq |\lambda_k(A)| \to 0$, then $\det(I+A) = \prod_{j=1}^\infty (1+\lambda_j(A))$. The characteristic values of A are defined as eigenvalues of $|A| = (AA^*)^{\frac{1}{2}}$, $\mu_1(A) \geq \cdots \geq \mu_k(A) \to 0$. Then one form of Weyl's inequality says that

$$|\det(I+A)| \leq \det(I+|A|). \tag{4.2}$$

Hence, using (**??**)

$$|s(\lambda)| \leq \prod_{j=1}^{\infty}(1+\mu_j(A)) \tag{4.3}$$

and to estimate $\mu_j(A)$ we need a good representation for A. That is particularly simple in the potential case (see for instance [?]):

$$A(\lambda)(\omega,\theta) = C_n\lambda^{n-2}\int e^{-i\lambda\langle x,\omega,\rangle}V(x)u_-(\lambda,x,\theta)dx\,, \; u_-(\lambda,\bullet,\theta) = (I-R_0(\lambda)V)^{-1}(e^{i\lambda\langle\bullet,\theta\rangle})\,, \tag{4.4}$$

where $R_0(\lambda)$ is the resolvent for $P = -\Delta$, $(\lambda^2 - \Delta)R_0(\lambda) = I$, and $R_0(\lambda)$ is bounded on L^2 for $\operatorname{Im}\lambda > 0$. If $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to one on supp V then we check easily that

$$\chi u_-(\lambda,\bullet,\theta) = (I-\chi R_0(\lambda)V)^{-1}(\chi e^{i\lambda\langle\bullet,\theta\rangle})\,,$$

Thus (**??**) can be rewritten as

$$A(\lambda) = C_n\lambda^{n-2}\mathbb{E}^\chi(-\lambda)V(I-H_V(\lambda))^{-1t}\mathbb{E}^\chi(\lambda)\,, \tag{4.5}$$

where $H_V(\lambda) = \chi R_0(\lambda)V$ and

$$\mathbb{E}^\chi(\lambda) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{S}^{n-1})\,, \quad \mathbb{E}^\chi(\lambda)(\theta,x) = \chi(x)e^{i\lambda\langle x,\theta\rangle}\,.$$

The factor $(I-H_V(\lambda))^{-1}$ is now meromorphic and we will have to avoid its poles in our estimate. We have (using $\mu_j(AB) \leq \mu_j(A)\|B\|$)

$$\mu_j(A(\lambda)) \leq |C_n||\lambda|^{n-2}\mu_j(\mathbb{E}^\chi(-\lambda))\|V(I-H_V(\lambda))^{-1}\|\|\mathbb{E}^\chi(\lambda)\|\,. \tag{4.6}$$

The estimate on $\mu_j(\mathbb{E}^\chi(-\lambda))$ follows from Melrose's method and was given in Proposition 2 of [?] (with a slight improvement from [?] used here):

$$\begin{aligned}\mu_j(\mathbb{E}^\chi(\lambda)) &= \mu_j((I-\Delta_{\mathbb{S}^{n-1}})^{-k}(I-\Delta_{\mathbb{S}^{n-1}})^k\mathbb{E}^\chi(\lambda)) \\ &\leq \mu_j((I-\Delta_{\mathbb{S}^{n-1}})^{-k})\|(I-\Delta_{\mathbb{S}^{n-1}})^k\mathbb{E}^\chi(\lambda)\| \leq j^{-\frac{2k}{n-1}}C^{2k}(2k)!e^{C|\lambda|} \\ &\leq C_1 e^{C|\lambda|-j^{\frac{1}{n-1}}/C}, \quad k = \left[\frac{j^{\frac{1}{n-1}}}{2C}\right]+1\,.\end{aligned} \tag{4.7}$$

To estimate $\|(I-H_V(\lambda))^{-1}\|$ we use Theorem 5.1 of Chapter V of [?][4]

$$\|(I-H_V(\lambda))^{-1}\| \leq |\det(I+H_V(\lambda)^{n+1})|^{-1}|\det(I+|H_V(\lambda)|^{n+1})\,, \tag{4.8}$$

where the power $n+1$ makes the operator be of trace class ($(n+1)/2$ would of course suffice in this case). The second term can be optimally estimated as in [?] using arguments similar to (??). If we are not concerned with optimal bounds, as we do not have to be here, that is quite easy. For optimal bounds, in this and in the general case, Vodev developed a method for estimating determinants of this type, again based on (??) – see [?] and [?]. Here we get

$$\det(I+|H_V(\lambda)|^{n+1}) \leq Ce^{C|\lambda|^n}\,,$$

and using (??) we conclude that $h(\lambda) = \det(I + H_V(\lambda)^{n+1})$ is an entire function of order at most n; let $\mathcal{L}$ denote the set of its zeros. The minimum modulus theorem for entire functions (see for instance [?], 8.71) now shows that for any $\epsilon > 0$ and any $\delta > 0$

$$|h(\lambda)| \geq \frac{1}{C_\epsilon}e^{-|\lambda|^{n+\epsilon}}, \quad \lambda \notin \bigcup_{\zeta\in\mathcal{L}} D(\zeta, \langle\zeta\rangle^{-n-\delta})\,, \tag{4.9}$$

so that, going back to (??)

$$\|(I-H_V(\lambda))^{-1}\| \leq \frac{1}{C_\epsilon}e^{|\lambda|^{n+\epsilon}}, \quad \lambda \notin \bigcup_{\zeta\in\mathcal{L}} D(\zeta, \langle\zeta\rangle^{-n-\delta})\,.$$

Combining this and (??) in (??) we obtain

$$\mu_j(A) \leq C_\epsilon e^{|\lambda|^{n+\epsilon}-j^{\frac{1}{n-1}}/C} \quad \lambda \notin \bigcup_{\zeta\in\mathcal{L}} D(\zeta, \langle\zeta\rangle^{-n-\delta})\,,$$

and consequently, by (??) and the maximum principle away from the poles of s:

$$|s(\lambda)| \leq Ce^{C|\lambda|^{2n-1+\epsilon}}, \quad \lambda \notin \bigcup_{m_\zeta(S)>0} D(\zeta, \langle\zeta\rangle^{-m-\epsilon})\,,$$

which is (??) with $N = 2n-1+\epsilon$. Even with the use of the methods which give optimal bounds on the number of scattering poles (that corresponds to the estimate on $h(\lambda)$) this estimate is

[4]That gives an elegant argument; one can also obtain a similar estimate using a direct argument based on Crammer's rule.

highly inaccurate. However, as observed in Theorem 7 above, the use of the trace formula and of the order of the singularity of $u(t)$ at $t = 0$ gives the correct order n.

In the general case we procceed similarly but we use a more general, but less intuitive, representation of A:

$$A(\lambda) = C_n \lambda^{n-2} \mathbb{E}^{\phi_1}(-\lambda)(I + K(\lambda, \lambda_0))^{-1}[\Delta, \chi]^t \mathbb{E}^{\phi_2}(\lambda)\,, \tag{4.10}$$

where $K(\lambda, \lambda_0)$ is the compact operator appearing in the proof of Theorem 1 given in [**?**], $\phi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus B(0, R_0); [0, 1])$, $i = 1, 2$, $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n; [0, 1])$, $\chi \equiv 1$ near $B(0, R_0)$, are suitably chosen cut-off functions. To estimate $\|(I + K(\lambda, \lambda_0))^{-1}\|$ away from its poles we use the methods of [**?**],[**?**].

References

[1] Sh. Agmon. *A perturbation theory for resonances*, Journées "Équations aux Dérivées partielles", Saint-Jean de Monts, 1996.

[2] R. Bañuelos and A. Sá Barreto. *On the heat trace of Schrödinger operators*, Comm. Partial Differ. Equations **20** (1995), 2153-2164.

[3] C. Bardos, J.-C. Guillot and J.V. Ralston. *La relation de Poisson pour l'équation des ondes dans un ouvert non borné*, Commun. Partial Differ. Equations **7** (1982), 905–958.

[4] V. Buslaev. *Scattering plane waves, spectral asymptotics and trace formulas in exterior problems*, Dokl. Akad. Nauk SSSR, **197** (1971), 999–1002.

[5] I. C. Gohberg and M. G. Krein. *Introduction to the theory of linear nonselfadjoint operators*, Translations of mathematical monographs **18**, American Mathematical Society, Providence, 1969.

[6] L. Guillopé. *Asymptotique de la phase de diffusion pour l'opérateur de Schrödinger avec potentiel*, C.R. Acad. Sci., Paris, Ser.I **293**(1981), 601-603.

[7] L. Guillopé and M. Zworski. *Upper bounds on the number of resonances for non-compact Riemann surfaces*, J. Funct. Anal. **129** (1995), 364–389.

[8] L. Guillopé and M. Zworski. *Scattering asymptotics for Riemann surfaces*, to appear in Ann. of Math.

[9] L. Hörmander. *The analysis of linear partial differential operators II*, Springer Verlag, Berlin, 1983.

[10] F. Klopp and M. Zworski. *Generic simplicity of resonances*, Helv.Phys. Acta. **68**(1995), 531–538.

[11] P. Lax and R. Phillips. *Decaying modes for the wave equation in the exterior of an obstacle.* Comm. Pure App. Math. **22** (1969), 737–787.

[12] P. Lax and R. Phillips. *The time delay operator and a related trace formula.* in *Topics in Functional Analysis.* Advances in Math. Suppl. Studies **3** (1978), 197–295.

[13] R.B. Melrose. *Scattering theory and the trace of the wave group*, J. Func. Anal. **45** (1982), 429–440.

[14] R.B. Melrose. *Polynomial bounds on the number of scattering poles*, J. Funct. Anal. **53** (1983), 287–303.

[15] R.B. Melrose. *Polynomial bounds on the distribution of poles in scattering by an obstacle*, Journées "Équations aux Dérivées partielles", Saint-Jean de Monts, 1984.

[16] R.B. Melrose. *Weyl asymptotics for the phase in obstacle scattering*, Commun. Partial Diff. Equations **13**(1988), 1431-1439.

[17] R.B. Melrose. *Geometric scattering theory*, Cambridge University Press, Cambridge, New York, Melbourne, 1995.

[18] R.B. Melrose and M. Zworski. *Scattering metrics and geodesic flow at infinity*, Invent. Math. **124**(1996), 389–436.

[19] G. Perla-Menzala and T. Schonbek. *Scattering frequencies for the wave equation with potential term*, J. Funct. Anal. **55**(1984), 297-322.

[20] W. Müller. *Spectral geometry and scattering theory for certain complete surfaces of finite volume*, Invent. Math. **109** (1992), 265–305.

[21] A. Sá Barreto and Siu-Hung Tang. *Existence of resonances in metric scattering,* in preparation.

[22] A. Sá Barreto and M. Zworski. *Existence of resonances in three dimensions*, Comm. Math. Phys. **173**(2) (1995), 401–415.

[23] A. Sá Barreto and M. Zworski. *Existence of resonances in potential scattering*, Comm. Pure Appl. Math. **49**(1996), 1271-1280.

[24] J. Sjöstrand. *A trace formula and review of some estimates for resonances*, Publications du Centre de Mathématiques de l'École Polytechnique No.1153, Octobre, 1996.

[25] J. Sjöstrand. *A trace formula for resonances and application to semi-classical Schrödinger operators*, Séminaire EDP, École Polytechnique, Novembre, 1996.

[26] J. Sjöstrand and M. Zworski. *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4** (1991), 729–769.

[27] J. Sjöstrand and M. Zworski. *Lower bounds on the number of scattering poles II*, J. Funct. Anal. **123** (1994), 336–367.

[28] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, 1939

[29] G. Vodev. *Sharp polynomial bounds on the number of scattering poles for perturbations of the Laplacian*, Comm. Math. Phys. **146** (1992), 39–49.

[30] G. Vodev. *Asymptotics of scattering poles for degenerate perturbations of the Laplacian*, J. Funct. Anal. **138** (1996), 295-310.

[31] M. Zworski. *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), 311–323.

[32] M. Zworski. *A remark on isopolar potentials*, unpublished note.

[33] M. Zworski. *Counting scattering poles.*, Proceedings of the Taniguchi International Workshop *Spectral and scattering theory*, M. Ikawa Ed., Marcel Dekker, New York, Basel, Hong Kong, 1994.