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EQUATIONS AUX DERIVEES PARTIELLES

ON THE PROFILES OF NONLINEAR GEOMETRIC OPTICS

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§1. Profiles and profile equations.

Our goal is to introduce the reader to the nonlinear evolution equations governing the profiles appearing in quasilinear geometric optics.

Studying the profiles corresponds to studying how the original systems propagate weakly nonlinear high frequency waves. As the waves considered have small amplitude, it is not surprising that the governing equations involve only the low order Taylor polynomials of coefficients. An overly optimistic person might even hope to classify the resulting equations into a finite number of canonical classes.

Our point of view is inpired by the very interesting article of Majda, Rosales, and Schonbek [MRS]. Some of their observations are recalled in §4. In the time since that paper was written, the formal aysmptotic expansions of nonlinear geoemetric optics have been rigorously justified in a wide variety of contexts so information about the profiles translates into information about the original hyperbolic systems.

The phenomenon of resonance has some surprising consequences. For simplicity we consider the case of one space variable and the system of equations

$$\partial_t u + A(u)\partial_x u = 0 \ . \tag{1.1}$$

Here $u = u(t,x)$ is a real k-vector valued function and A is a smooth real $k \times k$ matrix valued function of $t, x \in \mathbf{R} \times \mathbf{R}$. *Suppose that the system is strictly hyperbolic in the sense that for all u near 0, $A(u)$ has k distinct real eigenvalues.*

Weakly nonlinear geometric optics expansions for solutions near the background solution $u = 0$, have the form

$$u^\varepsilon(t,x) = \varepsilon U(t,x,\varphi_1(t,x)/\varepsilon,\cdots,\varphi_M(t,x)/\varepsilon) + o(\varepsilon) \tag{1.2}$$

with profile $U(t,x,\theta_1,\cdots,\theta_M)$ almost periodic with repect to the angle variables θ. The phase functions φ_j are real smooth functions with nonvanishing gradients. Solutions of the form 2.1 have amplitudes of order ε and wavelength of order ε. Note that the small parameter does not appear in the equation (1.1), which does not have a natural length scale. The small parameter is introduced by the initial data.

Denote by

$$L \equiv \partial_t + A(0)\partial_x \tag{1.3}$$

the linearized operator at $u = 0$. The equation for u reads

$$0 = Lu + (\partial_u A(0)u)\partial_j u + 0(u^3).$$

For solutions as in 1.2, the second term is $0(\varepsilon)$. As u is $0(\varepsilon)$ it is not surprising that the nonlinear terms are negligeable for times $o(1)$ and one anticipates nonlinear effects to be important for time $0(1)$.

We consider the special case where the profiles are independent of x. This resembles the idea of homogeneous turbulence and the name homogeneous oscillations seems appropriate. Finally we suppose that the phases φ_j are linear functions of t, x. Then 1.2 implies

$$u^\varepsilon \sim \varepsilon U(t, T/\varepsilon, X/\varepsilon)$$

with U almost periodic in the fast variables T, X,

$$U(t,T,X) = \sum_{\alpha \in \mathbf{R}\times\mathbf{R}} U_\alpha(t) e^{i\alpha.(T,X)} \ . \tag{1.4}$$

The most direct derivation of the equations determining U begins by supposing that 1.4 is the first term of an asymptotic expansion

$$u^\varepsilon \sim \varepsilon U(t, t/\varepsilon, x/\varepsilon) + \varepsilon^2 V(t, t/\varepsilon, x/\varepsilon) + \cdots . \tag{1.5}$$

The reader is forwarned that there are examples where the error $u^\varepsilon - \varepsilon U$ is $o(\varepsilon)$ and not much better so that the supposition 1.5 is not correct in those cases [JMR2]. Nevertheless it is our prefered derivation (see [JMR3] for the general case). Substituting 1.5 into 1.1 one finds

$$\partial_t u^\varepsilon + A(u^\varepsilon)\partial_x u^\varepsilon \sim \varepsilon^{-1} W_{-1}(t, T/\varepsilon, X/\varepsilon) + \varepsilon^0 W_0(t, T/\varepsilon, X/\varepsilon) + \cdots \tag{1.6}$$

$$W_0(t,T,X) = \partial_T U + A(0)\partial_X U = L(\partial_T, \partial_X)U \ , \tag{1.7}$$

$$W_1(t,T,X) = L(\partial_T, \partial_X)V + U_t + (\partial_u A(0)U)\partial_X U. \tag{1.8}$$

Here $\partial_u A(0)U$ is the derivative of A applied to the increment U so is a matrix. Equations for the profiles are obtained from the equations $W_j = 0$. The key to unraveling these equations is that the operator $L(\partial_T, \partial_X)$ is neither injective nor surjective.

The equation $W_0 = 0$ shows that U satisfies the linearized equations with respect to the fast variables,

$$L(\partial_T, \partial_X)U = 0. \tag{1.9}$$

In particular the values of $U(0,T,X)$ are determined by those of $U(0,0,X)$ by solving a hyperbolic initial value problem. Thus the appropriate initial data for U are $U(0,0,X)$.

Analysing 1.9 in Fourier yields

$$0 = (\partial_T + A(0)\partial_X)U = \sum_\alpha [\alpha_0 I + \alpha_1 A(0)] U_\alpha(t) e^{i\alpha.(T,X)} \ . \tag{1.10}$$

Let λ_j denote the eigenvalues of $A(0)$. Thus for $\alpha \neq 0, U_\alpha$ can be nonzero only for those α such that $\alpha_0 = \lambda_j \alpha_1$ for some j. In that case $U_\alpha(t)$ must belong to the j^{th} eigenspace.

Hyperbolicity implies that $\mathbf{R}^k = \ker(\alpha_0 + \alpha_1 A(0)) \oplus rg(\alpha_0 + \alpha_1 A(0))$. Let E_α denote the corresponding projection of $\mathbf{R}^k$ on the kernel of $\alpha_0 + \alpha_1 A(0)$. In particular E_α vanishes unless $\alpha_0 = \lambda_j \alpha_1$ for some j. If $\{r_j\}$ is a basis of eigenvectors and $\{\ell_j\}$ the associated dual basis of eigenvalues of $A(0)^*$, then in Dirac's notation

$$E_\alpha = |r_j >< \ell_j| \quad \text{when} \quad \alpha_0 = -\lambda_j \alpha_1 \ , \ \alpha_1 \neq 0 \ .$$

The important averaging operator $\mathbf{E}$ is defined on trigonometric series by

$$\mathbf{E}(\sum V_\alpha e^{i\alpha.(T,X)}) \equiv V_0 + \sum_{\alpha \neq 0} E_\alpha V_\alpha e^{i\alpha.(T,X)} \ . \tag{1.11}$$

In the space of formal trigonometric series with smooth coefficients

$$\mathbf{E}^2 = \mathbf{E}, \ \ker(L(\partial_T, \partial_X)) = rg(\mathbf{E}), \ rg(L(\partial_T \ , \partial_X)) = \ker(\mathbf{E}) \tag{1.12}$$

Then 1.9 takes the simple form

$$\mathbf{E}U = U \ , \tag{1.13}$$

and multiplying 1.8 by $\mathbf{E}$ yields

$$\mathbf{E}\left[U_t + (\partial_u A(0)U)\partial_X U\right] = 0 \ . \tag{1.14}$$

The equations 1.13-1.14 have the form of an evolution equation $U_t = \mathcal{F}(U)$ on the set of U satisfying $\mathbf{E}U = U$, and $\mathcal{F}(U) \equiv \mathbf{E}((\partial_u A(0)U)\partial_X U)$.

The next results are very special cases from [JMR1].

Theorem 1.— *Suppose that g and dg/dX are almost periodic $\mathbf{R}^k$ valued functions of $X \in \mathbf{R}$. There is a $t_* \in [0, \infty]$ such that 1.13-1.14 have one and only one solution $U(t,T,X)$ such that $U(0,0,X) = g$, and the functions $U(t,.,.)$ and $\nabla_{t,T,X}U(t,.,.)$ are continous on $[0, t_*[$ with values in the almost periodic functions of T, X equiped with the $L^\infty(R^2_{T,X} : \mathbf{R}^k)$ norm.*

In addition, the spectrum of $U(t)$ is contained in the $\mathbf{Z}$-module generated by spec $(U(0,.,.))$, and, if $t_* < \infty$ then

$$\sup_{|\beta| \leq 1, T, X} |\nabla^\beta_{t,T,X} U(t,T,X)| \to \infty \quad \text{as} \quad t \to t_* \ .$$

Theorem 2.— *Suppose that U and t_* are as in Theorem 1 and $0 < \underline{t} < t_*$. There is an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the initial value problem*

$$\partial_t u^\varepsilon + A(u^\varepsilon)\partial_x u^\varepsilon = 0 \ , \ u^\varepsilon(0,x) = \varepsilon g(X/\varepsilon)$$

has a unique solution $u^\varepsilon \in C^1([0,\underline{t}] \times \mathbf{R})$ *and as* ε *tends to zero*

$$u^\varepsilon(t,x) - \varepsilon U(t,t/\varepsilon,x/\varepsilon) = o(\varepsilon) \quad \textit{in} \quad L^\infty([0,\underline{t}] \times \mathbf{R})$$

and

$$\nabla_{t,x}\{u^\varepsilon(t,x) - \varepsilon U(t,t/\varepsilon,x/\varepsilon)\} = o(1) \quad \textit{in} \quad L^\infty([0,\underline{t}] \times \mathbf{R}) \ .$$

The form 1.13-1.14 of the profile equations is particularly well adapted to multidimensional generalisations. In the special case of one space dimension, there is a simpler presentation, which does not extend cleanly to the multidimensional case. Our approach was to expand in a Fourier series and then apply equation 1.9. Operating in the reverse order, the general solution of 1.9 is of the form

$$U(t,T,X) = \sum \sigma_j(t, X - \lambda_j T) r_j \tag{1.15}$$

with scalar valued $\sigma_j(t,\theta)$. Almost periodicity in T, X holds if the σ_j are almost periodic in θ. Introduce the phase functions

$$\varphi_j(t,x) \equiv x - \lambda_j t \tag{1.16}$$

which satisfy the eikonal equation $\det L(\partial_t\varphi, \partial_x\varphi) = 0$. Then

$$U = \sum (\sigma_j(t, \varphi_j(T,X)) r_j \ . \tag{1.17}$$

Equations for the σ_j are derived by plugging 1.17 into 1.13-1.14. Introduce $\mathbf{E}_j$ the projector which extracts the λ_j part of U, namely

$$\mathbf{E}_j V \equiv \sum_{\alpha_0 = -\lambda_j \alpha_1} E_\alpha U_\alpha e^{i\alpha(T,X)} \ . \tag{1.18}$$

Then

$$\mathbf{E} = \Sigma \mathbf{E}_j \ , \ \mathbf{E}_j = |r_j >< \ell_j| \mathbf{E} \ , \ \mathbf{E}_j \mathbf{E} = \mathbf{E}\mathbf{E}_j = \mathbf{E}_j \ . \tag{1.19}$$

Equation 1.14 is then equivalent to

$$\mathbf{E}_j \left[U_t + (\partial_u A(0) U) \partial_X U \right] = 0 \quad \text{for} \quad 1 \le j \le k \ . \tag{1.20}$$

Define $k \times k$ matrices B_j by

$$B_j \equiv (\partial_u A(0)) r_j \ . \tag{1.21}$$

Equation 1.20 holds if and only if for $1 \le j \le k$,

$$\partial_t \sigma_j(t, \varphi_j(T,X)) + \sum_{\mu,\nu} < \ell_j | B_\mu | r_\nu > \Gamma_j \left[\sigma_\mu(t, \varphi_\mu(T,X)) \partial_\theta \sigma_\nu(t, \varphi_\nu(T,X)) \right] = 0 \tag{1.22}$$

where Γ_j is the operator, acting on scalar valued trigonmetric series, which extracts the part with spectrum in $\mathbf{R}(1, -\lambda_j)$,

$$\Gamma_j(\Sigma v_\alpha e^{i\alpha \cdot (T,X)}) \equiv \sum_{\alpha_0 = -\lambda_j \alpha_1} v_\alpha e^{i\alpha \cdot (T,X)} . \tag{1.23}$$

§2. One mode solutions of the profile equations.

To analyse the equations 1.22 for the amplitudes σ_j, one must compute

$$\Gamma_j\left[\sigma_\mu(t,\varphi_\mu(T,X))\partial_\theta\sigma_\nu(t,\varphi_\nu(T,X))\right] \tag{2.1}$$

for all j,μ,ν. Abreviate 2.1 as $\Gamma_j(\sigma_\mu\partial_\theta\sigma_\nu)$ and evaluate for $j=1$, the other values of j being similar. Two easy consequences of the definitions are

$$\Gamma_1(\sigma_1\partial_\theta\sigma_1)=\sigma_1\partial_\theta\sigma_1, \quad \text{and} \quad \Gamma_j(\sigma_1\partial_\theta\sigma_1)=0 \quad \text{for} \quad j\neq 1\ . \tag{2.2}$$

Proof. Expand

$$\sigma_1\partial_\theta\sigma_1=\Sigma\hat\sigma_1(n)\ im\ \hat\sigma_1(m)\ \exp\ (i(n+m)\varphi_1(T,X))\ .$$

The spectrum, in T,X is contained entirely in the set $\alpha_0=-\lambda_1\alpha_1,\alpha_1\neq 0$. ////

It follows that one generates a subclass of solutions by choosing $\sigma_j\equiv 0$ for $j\geq 2$, and $\sigma_1(t,\theta)$ a solution of Burgers' equation

$$\partial_t\sigma_1+<\ell_1|B_1|r_1>\partial_\theta(\sigma_1^2/2)=0\ , \tag{2.3}$$

The equation is genuinely nonlinear if and only if the selfinteraction coefficient $<\ell_1|B_1|r_1>$ is nonzero. Recall the following facts about Burgers' equation for initial data periodic in θ.

. The solutions have $L^2(\theta)$ norm independent of time so long as they remain Lipshitzean in θ.

. Nonconstant solutions do not remain Lipshitzean but have unique extensions to global weak entropic solutions on $[0,\infty[\times S^1$ for which the $L^2(S^1)$ norm is nonincreasing.

. The solution operators $S(t)$ so defined are $L^1(S^1)$ contractions.

. The total variation of solutions is a decreasing function of time.

. The term $\partial_\theta\sigma^2$ leads to interaction among the Fourier coefficients σ. In a sense Burgers' equation likes the coefficients to decay like $1/n$. If one starts with smooth data, that is rapid decrease, shocks tend to form and shocks have this regularity. If one starts with very singular data, say L^∞, there is a regularizing effect which forces the solution to be BV, functions whose coefficients are $0(1/|n|)$. These vague comments constitute a sort of Kolmogorov's law, and it would be nice to have more precise versions. Using the Hopf-Lax explicit solution [L] it is not hard to show that jump discontinuities form. For times when $\sigma(t)$ has such a discontinuity,

$$\lim_{|n|\to\infty}\sup\ |\hat\sigma(t,n)n|>0 \tag{2.4}$$

Conjecture. For nonconstant entropic θ-periodic solutions of Burgers' equation, 2.4 holds for a strictly positive fraction of times $t \in [0, \infty[$.

Summarizing, the system 1.1 has solutions

$$u^\varepsilon(t, x) = \varepsilon\sigma_j(t, (x - \lambda_j t)/\varepsilon)r_j + o(\varepsilon) \tag{2.5}$$

whenever σ_j is a smooth periodic solution of Burgers'equation on $[0, \underline{t}] \times S^1$. Here $o(\varepsilon)$ is measured in $L^\infty([0, \underline{t}] \times S^1)$. In this sense Burgers behavior is always present in 1.1. The main feature is the breaking of waves which is caused by the generation of high frequencies thanks to the fact that $\partial_\theta \sigma^2$ is an operator which is nonlocal in the frequency space.

§3. Equations for 3×3 resonant interactons.

Our main goal is to study the interaction of high frequency wave trains like those in §2. To do that we simplify to the case of 3×3 systems, that is $k = 3$. The key fact is that for $k \geq 3$ there is always a resonance relation since three linear functions of two variables are always linearly dependent. Thus there are constants $\gamma_j \neq 0$ such that $\Sigma \gamma_j \varphi_j = 0$. Phases $\varphi_j(t, x) = f_j(x - \lambda_j t)$ with nonlinear function f_j satisfy the eikonal equation and one gets fine single mode waves. However, generically there would not be any resonance. *The choice of linear phases was not at all innocent.*

Replacing the phases φ_j by $\tilde{\varphi} \equiv \gamma_j \varphi_j$ and the amplitudes $\sigma_j(t, \theta)$ by $\tilde{\sigma}_j \equiv \sigma_j(t, \theta/\gamma_j)$ preserves the linearity of the phases and the almost periodicity of the amplitudes but simplifies the resonance relation to

$$\tilde{\varphi}_1 + \tilde{\varphi}_2 + \tilde{\varphi}_3 = 0 \tag{3.1}$$

and 1.22 becomes

$$\partial_t \tilde{\sigma}_j(t, \tilde{\varphi}_j(T, X)) + \tag{3.2}$$

$$\sum_{\mu,\nu} < \ell_j | B_\mu | r_\nu > \gamma_\nu \Gamma_j(\tilde{\sigma}_\mu(t, \tilde{\varphi}(T, X)) \partial_\theta \tilde{\sigma}_\nu(t, \tilde{\varphi}_\nu(T, X)) = 0$$

the only change being the insertion of the factor γ_ν. We drop the tilda's remembering that our phases are no longer given by 1.16.

We seek solutions with σ_j 2π-periodic in θ.

To write the equation for σ_1, return to 2.1 for $j = 1$. The cases $\mu = \nu$ have already been calculated. Expand

$$\Gamma_1(\sigma_\mu \partial_\theta \sigma_\nu) = \Gamma_1 \sum_{n,m} \hat{\sigma}_\mu(n) im \hat{\sigma}_\nu(m) \exp(i(n\varphi_\mu(T, X) + m\varphi_\nu(T, X)) \ . \tag{3.3}$$

The operator Γ_1 annihilates all terms for which $n\varphi_\mu + m\varphi_\nu$ is not a multiple of φ_1. For example if $\mu = 1$ and $m \neq 0$ the term is annihilated. For $m = 0$, the term vanishes thanks to the im factor so

$$\Gamma_1(\sigma_1 \partial_\theta \sigma_\nu) = 0 \quad \text{for} \quad \nu \neq 1 \ . \tag{3.4}$$

Similarly if $\nu = 1$ all terms with $\mu \neq 0$ are killed and

$$\Gamma_1(\sigma_\mu \partial_\theta \sigma_1) = \hat{\sigma}_\mu(0)\partial_\theta \sigma_1 \quad \text{for} \quad \mu \neq 1 \tag{3.5}$$

The remaining terms are crucial. Consider $\mu = 2$, $\nu = 3$, the reverse choice being similar. In 3.3 only terms with $n = m$ survive. For them use the relation $\varphi_2 + \varphi_3 = -\varphi_1$ to get

$$\begin{aligned} \Gamma_1 \sigma_2 \partial_\theta \sigma_3 &= \Sigma \hat{\sigma}_\mu(n) in \hat{\sigma}(n) \exp(-in\varphi_1(T,X)) \\ &= \Sigma[\partial_\theta(\sigma_2 * \sigma_3)]^\wedge(n)\exp(-in\varphi_1) \\ &= R[\partial_\theta(\sigma_2 * \sigma_3)](\varphi_1(T,X)) \end{aligned} \tag{3.6}$$

where R is the reflection operator and $*$ is convolution

$$Rg(\theta) \equiv g(-\theta) \tag{3.7}$$

$$\sigma_2 * \sigma_3 \equiv \int_{S^1} \sigma_2(\theta - \psi)\sigma_3(\psi)d\psi/2\pi \ . \tag{3.8}$$

The equation for σ_1 then takes the form

$$\partial_t \sigma_1 = (c^{st}\bar{\sigma}_2 + c^{st}\bar{\sigma}_3)\partial_\theta \sigma_1 + c^{st}\partial_\theta(\sigma_1^2) + c^{st}\partial_\theta(R(\sigma_2 * \sigma_3)) \ .$$

The right hand side is a θ derivative so the mean value of σ_1 is independent of time and similarly the other means. We restrict attention to the case

$$\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}_3 = 0. \tag{3.9}$$

In that case the profile equations take the simple form

$$\begin{array}{ccccccc} \partial_t \sigma_1 & + & c_1 \partial_\theta \sigma_1^2 & + & b_1 R \partial_\theta(\sigma_2 * \sigma_3) & = & 0 \\ \partial_t \sigma_2 & + & c_2 \partial_\theta \sigma_2^2 & + & b_2 R \partial_\theta(\sigma_1 * \sigma_3) & = & 0 \\ \partial_t \sigma_3 & + & c_3 \partial_\theta \sigma_3^2 & + & b_3 R \partial_\theta(\sigma_1 * \sigma_2) & = & 0. \end{array} \tag{3.10}$$

The system is integrodifferential.

The uniqueness part Theorem 1 implies that if $\sigma_k(t,\theta)$ are odd functions of θ at $t = 0$, they remain so and the system becomes

$$\begin{array}{ccccccc} \partial_t \sigma_1 & + & c_1 \partial_\theta \sigma_1^2 & - & b_1 \partial_\theta(\sigma_2 * \sigma_3) & = & 0 \\ \partial_t \sigma_2 & + & c_2 \partial_\theta \sigma_2^2 & - & b_2 \partial_\theta(\sigma_1 * \sigma_3) & = & 0 \\ \partial_t \sigma_3 & + & c_3 \partial_\theta \sigma_3^2 & - & b_3 \partial_\theta(\sigma_1 * \sigma_2) & = & 0. \end{array} \tag{3.11}$$

An even simpler case arises if for $1 \le j \le 3$, $c_j = c$ and $b_j = b$. Then there are solutions $\sigma = (v, v, v)$ provided

$$v_t + c(v^2)_\theta - b(v * v)_\theta = 0, \qquad v(t,\theta) = -v(t,-\theta). \tag{3.12}$$

Thus 3.12 is a model equation for resonant interaction. In 3.11, the c_j terms are of Burgers type and the the b_j terms represent the effect of the resonant interaction of the waves of the other families on the given family. They would be absent if there were no resonance.

A first remark is that the interaction terms are less singular than the Burgers terms. This is illustrated by the following estimates

$$\begin{aligned}
||\partial_\theta(\sigma * \sigma)||_{L^\infty} &\le ||\sigma||_{L^\infty} \quad ||\partial_\theta\sigma||_{L^1} \\
||(\sigma\partial_\theta\sigma)||_{L^\infty} &\le ||\sigma||_{L^\infty} \quad ||\partial_\theta\sigma||_{L^\infty} \\
||\partial_\theta(\sigma * \sigma)||_{L^1} &\le ||\sigma||_{L^1} \quad ||\partial_\theta\sigma||_{L^1} \\
||(\sigma\partial_\theta\sigma)||_{L^1} &\le ||\sigma||_{L^\infty} \quad ||\partial_\theta\sigma||_{L^1}.
\end{aligned}$$

A consequence is that the local existence of profiles given by Theorem 1, follows arguments familiar from the local existence theory for quasilinear hyperbolic systems.

§4. Profiles for compressible inviscid 1-d Euler.

For this system, 3.11 simplifies even further. If one takes $\lambda_1 < \lambda_2 < \lambda_3$, then σ_1 and σ_3 represent genuinely nonlinear waves called acoustic waves. However, σ_2 in not only linearly degenerate, which means $c_2 = 0$ but the interaction coefficient $b_2 = 0$ also. Thus σ_2 is independent of time. In addition, one has $c_1 = c_3$ and $b_1 = -b_3$. Thus with

$$k(\theta) \equiv -b_1\partial_\theta\sigma_2(\theta), \tag{4.1}$$

the profile equations take the elegant form

$$\partial_t\sigma_1 + c\partial_\theta\sigma_1^2 + k * \sigma_3 = 0 \tag{4.2$_1$}$$

$$\partial_t\sigma_3 + c\partial_\theta\sigma_3^2 - k * \sigma_1 = 0. \tag{4.2$_2$}$$

Since σ_2 is constant the interaction term appears as a linear perturbation of the Burgers system. In addition the linearized equation

$$\partial_t\sigma_1 + k * \sigma_3 = 0, \quad \partial_t\sigma_3 - k * \sigma_1 = 0$$

has evolution operator which is particularly simple, namely

$$\hat{\sigma}_1(t,n) + i\hat{\sigma}_3(t,n) = e^{ik(n)t}(\hat{\sigma}_1(0,n) + i\hat{\sigma}_3(0,n)). \tag{4.3}$$

In particular the $L^2(S^1)$ norm is preserved. It follows that the $L^2(S^1)$ norm of σ_1, σ_3 is conserved (resp. nonincreasing) for smooth (resp. entropy satisfying weak) solutions of 4.2.

If $\sigma_2 \in BV$, then k is a bounded measure so the linear operator $\sigma \to (k * \sigma_3, -k * \sigma_1)$ is bounded on all $L^p(S^1)$ spaces. It follows (see [MRS]) that for arbitrary $BV(S^1)$ initial data , there are unique global entropy satisfying solutions of 4.2 satisfying

$$||\sigma_1(t), \sigma_3(t)||_{BV(S^1)} \leq \exp (Ct)||\sigma_1(0), \sigma_3(0)||_{BV(S^1)}, 0 \leq t < \infty . \tag{4.4}$$

In the same paper, one finds interesting exact solutions when k is constant plus a sum of delta functions. In addition they performed numerical simulations for $\sigma_2 = \sin(\theta)$. These revealed the following qualitative features.

. There is a tendency to avoid wave breaking. As the σ_1 component became steep, instead of breaking it would back off, and the σ_3 wave would steepen and so on.

. The waves seemed to be recurrent, and the authors suggested that this might correspond to solutions almost or quasi periodic in time. The growth allowed by 4.4 was not observed.

. Traveling cusp shaped waves were often present. The system 4.2 seems to like cusps. Thus motivated, Pego proved the following result.

Theorem 3.— *([P]). For $\sigma_2 = \sin(\theta)$, there is an explicit one parameter family of smooth solutions of 4.2 each of which is periodic in time. The limiting value of the parameter yields a time periodic solution with a traveling cusp.*

Applying Theorem 2, one sees that for arbitrarily large times $\underline{t}$, the Euler equations have x-periodic solutions $u^\varepsilon \in C^\infty([0,T] \times \mathbf{R})$ of the form

$$\varepsilon[\sigma_1(t,(x - \lambda_1 t)/\varepsilon)r_1 + \sin(x/\varepsilon)r_2 + \sigma_3(t,(x + \lambda_1 t)/\varepsilon)r_3] + o(\varepsilon).$$

Similar initial data for Burgers equation would lead to shock formation in finite time independent of ε, so these solutions avoid the formation of shocks for arbitrarily long periods of time.

Problems. Are there nonconstant x-periodic solutions of the Euler equations which are smooth for all positive times? Time periodic?

§5. The sawtooth miracle.

In contrast to the case of 1-d gas dynamics, the general system 3.11 may have solutions which explode in finite time. An elegant construction of such solutions is given by Hunter [H].

Let $S(\theta)$ be the 2π-periodic sawtooth function such that

$$S(\theta) = \theta \text{ for } -\pi < \theta < \pi. \tag{5.1}$$

Then in the sense of distributions one has (! 3.13)

$$\partial_\theta(S^2) = 2S, \text{ and, } \partial_\theta(S * S) = S. \tag{5.2}$$

Thus one generates discontinuous solutions of 3.11 of the form

$$\sigma_j(t,\theta) = a_j(t)S(\theta) \tag{5.3}$$

provided

$$\begin{array}{ccccccc} \partial_t a_1 & + & 2c_1 a_1^2 & - & b_1 a_2 a_3 & = & 0 \\ \partial_t a_2 & + & 2c_2 a_2^2 & - & b_2 a_1 a_3 & = & 0 \\ \partial_t a_3 & + & 2c_3 a_3^2 & - & b_3 a_1 a_2 & = & 0. \end{array} \tag{5.4}$$

The solution is entropy statisfying if and only if

$$a_j c_1 \geq 0 \text{ for } j = 1,2,3. \tag{5.5}$$

Thanks to the sign condition 5.5, the selfinteraction terms $c_j a_j^2$ always act to decrease the size of $|a_j|$. For definiteness suppose that the $c_j \geq 0$ so the entropy condition is simply $a_j \geq 0$.

Then if the b are large and positive they tend to make nonegative solutions of 5.4 grow. If the b are positive and large compared to the c_j all strictly positive solutions will diverge to infinity in finite time. In this way one sees that there are explosive discontinuous solutions of the profile equations. A recent result of Schochet [S] justifies nonlinear geometric optics with discontinuous profiles. His L^1 bounds on the error are not proved on time intervals which approach the blow up times of the profiles. To use Theorem 2 we need to construct smooth explosive profiles.

Conjecture. If the c_j are nonnegative and there is a solution a of 5.4 with strictly positive a_j such that $|a(t)|$ diverges to infinity as $t \to t_* < \infty$, then for any $\delta > 0$ there is a smooth 2π periodic solution σ of 3.11 which explodes at time $\underline{t} < t_* + \delta$.

Conjecture. Suppose that 1.1 is a system of conservation laws and there is an entropic sawtooth solution of the profile equations which explodes at time $t_* < \infty$ and that $\underline{t} > t_*$. Then there are BV periodic initial data of arbitrarily small sup norm and uniformly bounded BV norm for which the initial value problem does not have an entropy solution in $BV([0,\underline{t}] \times S^1)$.

The argument of [JMR4] yields the following.

Theorem 4.— *Suppose that the b_j are strictly positive. Then there is a $\delta > 0$ and a $t_1 \in [0, \infty[$ such that if $|c_j| < \delta$ for $1 \leq j \leq 3$, there is a $t_2 \in]0, t_1]$ and an odd solution $\sigma \in C^\infty([0, t_2[\times S^1 : \mathbf{R}^3)$ of 5.4 such that*

$$\int_{S^1} \sigma_j(\theta) \sin(\theta) d\theta \to \infty \text{ as } t \to t_2. \tag{5.6}$$

Idea of the proof. When $c_j = 0$, the equations 5.4 are local in Fourier. If the intial data are positive multiples of $\sin(\theta)$, then the solution is of the form $\sigma_j = \beta_j(t) \sin(\theta)$ with $\beta_j - b_j\beta_j^2 = 0$, so β diverges in finite time.

If the c_j are sufficiently small one can control the spreading effect in Fourier of the Burgers term to show that the solution of 5.4 with the same initial data explodes with first Fourier coefficients dominant. /////

Theorem 5.— *Suppose that $b_j, \delta, |c_j| < \delta$, and $t_2(c)$ are as in the previous theorem. Then there is a $C > 0$ such that for all $N > 0$ and $\eta > 0$ there is a $\underline{t} \in]0, t_2]$ and a solution $u \in C^\infty([0, \underline{t}] \times S^1)$ of 1.1 such that*

$$|u(t, \theta)| \leq \eta \text{ for } t, \theta \in [0, \underline{t}] \times S^1, \int_{S^1} |\partial_\theta u(0, \theta)| d_\theta < C \ , \tag{5.7}$$

$$\int_{S^1} |\partial_\theta u(\underline{t}, \theta)| d\theta \geq N \int_{S^1} |\partial_\theta u(0, \theta)| d\theta. \tag{5.8}$$

Proof. Take initial data $u = \Sigma \varepsilon \sigma_j(0, \theta/\varepsilon) r_j$ with σ as in Theorem 4 and $\varepsilon = 1/m$. Let m tend to infinity and apply Theorem 2. /////

Remarks.

1. The unbounded variation amplification in finite time shows that solutions of 3×3 systems which are small in amplitude and of moderate size in total variation can display behavior radically different from Burgers'equation thanks to resonant interaction of small amplitude oscillations.

2. In the 2×2 case, the analysis of Glimm and Lax [GL] shows that for L^∞ small solutions and fixed positive finite time, the variation per period is bounded by a fixed positive constant. Thus 5.7-5.8 is impossible in that case.

3. Glimm [G] constructed solutions of small initial variation whose variation at time $\underline{t}$ is bounded by a fixed multiple of the initial variation. Thus his solutions too cannot satisfy 5.7-5.8. This also indicates (it does not *prove* since we lack a uniqueness theorem for weak solutions) that though our solutions are as small as we like in sup norm their initial variations must be bounded below.

4. Theorem 5 complements and contrasts with the unbounded amplification of the BV norm in dimensions $d > 1$ [R]. The latter depends on the phenomenon of focussing which is a linear phenomenon which can take place in an arbitrarily small neighborhood of a single point. The current result depends on resonance which is nonlinear and requires initial oscillations over a finite range in x so that the resonant interaction takes place over a finite interval of time. In particular the phenomenon does not take place for solutions of arbitrarily small initial variation.

References

[G] J. Glimm, Solutions in the large for nonlinear systems of conservation laws, Comm. Pure Appl. Math. 18(1965), 685-715.

[GL] J. Glimm and P.D. Lax, Decay of solutions of systems of nonlinear hyperbolic hyperbolic conservation laws, Memoirs A.M.S. 101, 1970.

[Je] A. Jeffrey, Breakdown of the solution to a completely exceptional system of hyperbolic equations, J. Math. Anal. Appl. 45(1974) 375-381.

[Jo] F. John, Formation of singularities in one-dimensional nonlinear wave propagation, Comm. Pure Appl. Math. 37(1974) 377-405.

[JMR1] J-L. Joly, G. Metivier, and J. Rauch, Resonant one dimensional nonlinear geometric optics, Journal of Functional Analysis, to appear.

[JMR2] J-L. Joly, G. Metivier, and J. Rauch, Formal and rigorous nonlinear high frequency hyperbolic waves, pp. 121-143 in Proceedings of Varenna Conference on Nonlinear Hyperbolic Equations and Field Theory, M.K. Murthy and S. Spagnolo eds., Pitman Research Notes in Math, 1992

[JMR3] J-L. Joly, G. Metivier, and J. Rauch, Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves, Duke Math. J., to appear

[JMR4] J-L. Joly, G. Metivier, and J. Rauch, Unbounded variation amplification for 3x3 systems of conservation laws, preprint.

[H] J. Hunter, Strongly nonlinear hyperbolic waves, in Notes on Numerical Fluid Dynamics Vol.24, Nonlinear Hyperbolic Equations, eds. J. Ballman and R. Jeltsch, Vieweg, Braunschweig, 1989.

[L] P.D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math. 10(1957), 537-566.

[MR] A. Majda and R. Rosales, Resonantly interacting weakly nonlinear hyperbolic waves I, a single space variable. Stud.Appl.Math. 71(1984)149-179.

[P] R. Pego, Some explicit resonating waves in weakly nonlinear gas dynamics, Stud. Appl. Math. 71(1984)263-270.

[R] J. Rauch, BV estimates fail for most quasilinear hyperbolic systems in dimension greater than one, Comm.Math.Phys. 106(1986)484-489.

[S] S. Schochet, Resonant nonlinear geometric optics for weak solutions of conservation laws, preprint.

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Page/Line	
I-2/5	$\varepsilon U(t, t/\varepsilon, x/\varepsilon)$ in place of $\varepsilon U(t, T/\varepsilon, X/\varepsilon)$
I-2/6	$U(t, T, X)$ in place of U
I-3/-1	$\varepsilon g(x/\varepsilon)$ in place of $\varepsilon g(X/\varepsilon)$
I-4/formula (1.18)	V_α in place of U_α

[MRS] A. Majda, R. Rosales, and M. Schonbek, A canonical system of integro differential equations arising in resonant nonlinear acoustics, Stud. Appl. Math. 79 (1988) 205-262.