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EQUATIONS AUX DERIVEES PARTIELLES

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The correct form of radiation conditions is found in scattering problem for N-particle quantum systems. The estimates obtained allow us to give an elementary proof of asymptotic completeness for such systems in the framework of the theory of smooth perturbations.

1. One of the main problems of scattering theory is a description of asymptotic behaviour of N interacting quantum particles for large times. The complete classification of all possible asymptotics (channels of scattering) is called asymptotic completeness. The final result can easily be formulated in physics terms. Two particles can either form a bound state or are asymptotically free. In case $N \geq 3$ a system of N particles can also be decomposed asymptotically into its subsystems (clusters). Particles of the same cluster form a bound state and different clusters do not interact with each other.

There are two essentially different approaches to a proof of asymptotic completeness for multiparticle ($N \geq 3$) quantum systems. The first of them, started by L. D. Faddeev [1], relies on the detailed study of a set of equations derived by him for the resolvent of the corresponding Hamiltonian. This approach was developped in [1] for the case of three particles and was further elaborated in [2, 3]. The attempts [4, 5] towards a straightforward generalization of Faddeev's method to an arbitrary number of particles meet with numerous difficulties. However, the results of [6] for weak interactions are quite elementary.

Another approach relies on the commutator method [7] of T. Kato. In the theory of N-particle scattering it was introduced by R. Lavine [8, 9] for repulsive potentials. A proof of asymptotic completeness in the general case is much more complicated and is due to I. Sigal and A. Soffer [10]. In the recent paper [11] G. M. Graf gave an accurate proof of asymptotic completeness in the time-dependent framework. The distinguishing feature of [11] is that all

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intermediary results are also purely time-dependent and most of them have a direct classical interpretation. Papers [10, 11] were to a large extent inspired by V. Enss (see e.g. [12]) who was the first to apply a time-dependent technique for the proof of asymptotic completeness.

The aim of this lecture is to outline an elementary proof of asymptotic completeness (Theorem 7) for N-particle Hamiltonians with short-range potentials which fits into the theory of smooth perturbations [7, 13]. Our approach hinges on new estimates which establish some kind of radiation conditions (for the precise statement, see Theorem 8) for N-particle systems. We omit some details. However, basic intermediary results are formulated and their proofs are sketched.

2. Let us briefly recall some basic definitions of the scattering theory. For a self-adjoint operator H in a Hilbert space $\mathcal{H}$ we introduce the following standard notation: $\mathcal{D}(H)$ is its domain; $\sigma(H)$ is its spectrum; $E(\Omega; H)$ is the spectral projection of H corresponding to a Borel set $\Omega \subset \mathbb{R}$; $\mathcal{H}^{(ac)}(H)$ is the absolutely continuous subspace of H; $P^{(ac)}(H)$ is the orthogonal projection on $\mathcal{H}^{(ac)}(H)$; $\mathcal{H}^{(p)}(H)$ is the subspace spanned by all eigenvectors of the operator H; $\sigma^{(p)}(H)$ is the spectrum of the restriction of H on $\mathcal{H}^{(p)}(H)$, i.e. $\sigma^{(p)}(H)$ is the closure of the set of all eigenvalues of H. Norms of vectors and operators in different spaces are denoted by the same symbol $\|\cdot\|$; I is always the identity operator; C and c are positive constants whose precise values are of no importance.

Let K be H-bounded operator. It is called H-smooth (in the sense of T. Kato) on a Borel set $\Omega \subset \mathbb{R}$ if for every $f = E(\Omega; H)f \in \mathcal{D}(H)$

$$\int_{-\infty}^{\infty} \|K \exp(-iHt) f\|^2 \, dt \leq C\|f\|^2.$$

Let now H_j, $j = 1, 2$, be a couple of self-adjoint operators and let J be a bounded operator in a Hilbert space $\mathcal{H}$. The wave operator for the pair H_1, H_2 and the "identification" J is defined by the relation

$$W^{\pm}(H_2, H_1; J) = s - \lim_{t \to \pm\infty} \exp(iH_2 t) J \exp(-iH_1 t) P^{(ac)}(H_1) \qquad (1)$$

under the assumption that this strong limit exists. We emphasize that all definitions and considerations for " $+$ " and " $-$ " are independent of each other. Clearly, for every $f_2 = W^{\pm}(H_2, H_1; J) f_1$

$$\exp(-iH_2 t) f_2 \sim J \exp(-iH_1 t) f_1, \quad t \to \pm\infty,$$

where " $\sim$ " means that the difference between left and right sides tends to zero. If the wave operator (1) exists, then the intertwining property

$$E_2(\Omega) W^{\pm}(H_2, H_1; J) = W^{\pm}(H_2, H_1; J) E_1(\Omega)$$

($\Omega \subset \mathbf{R}$ is any Borel set and $E_i(\Omega) = E(\Omega; H_i)$) holds. It follows that the range $R(W^{\pm}(H_2, H_1; J))$ of the operator (1) is contained in $\mathcal{H}^{(ac)}(H_2)$ and its closure is an unvariant subspace of H_2. Moreover, if the wave operator is isometric on some subspace $\mathcal{H}_1$, then the restrictions of H_1 and H_2 on the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2 = W^{\pm}(H_2, H_1; J)\mathcal{H}_1$ respectively are unitarily equivalent. This equivalence is realized by the wave operator.

We need the following sufficient condition of existence of wave operators (see e.g. [14]).

Proposition 1 *Let an operator $\mathcal{J}$ be H_1-bounded and let its adjoint $\mathcal{J}^*$ be H_2-bounded. Suppose that*

$$H_2\mathcal{J} - \mathcal{J}H_1 = \sum_n K_{2,n}^* K_{1,n}, \quad n = 1, \ldots, N < \infty,$$

where the operators $K_{j,n}$ are H_j-bounded and are H_j-smooth on some bounded interval Λ. Then the wave operators

$$W^{\pm}(H_2, H_1; \mathcal{J}E_1(\Lambda)), \quad W^{\pm}(H_1, H_2; \mathcal{J}^*E_2(\Lambda))$$

exist.

3. We consider the Schrödinger operator $H = T + V$ where $T = -\Delta$ and V is multiplication by a real function $V(x)$ defined as follows. Suppose that some finite number of subspaces X^α of $X := \mathbf{R}^d$ is given and let x^α be the orthogonal projection of $x \in X$ on X^α. Set

$$V(x) = \sum_\alpha V^\alpha(x^\alpha), \tag{2}$$

where V^α are decreasing functions of variables x^α. Note that in the two-particle case the sum (2) consists of a single term V^α with $X^\alpha = X$. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any neighbourhood (neighbourhoods of all subspaces are supposed to be conical) of $X_\alpha = X \ominus X^\alpha$ and $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case. Operators H studied here were introduced in [15] and are natural generalizations of N-particle Hamiltonians. Consideration of more general class of operators allows to unravel better the geometry of the problem.

We assume that each V^α is a sum of short-range V_s^α and long-range V_l^α terms:

$$V^\alpha = V_s^\alpha + V_l^\alpha. \tag{3}$$

We say that a potential V^α is short-range if $V_l^\alpha = 0$. It is convenient to split all conditions on V^α into two parts (the first has a preliminary nature). To formulate them we need to introduce the operator $T^\alpha = -\Delta_{x^\alpha}$ in the space $\mathcal{H}^\alpha = L_2(X^\alpha)$.

Assumption 2 *Operators*

$$V^\alpha(T^\alpha+I)^{-1}, \quad (|x^\alpha|+1)V_s^\alpha(T^\alpha+I)^{-1}, \quad (|x^\alpha|+1)|\nabla V_l^\alpha|(T^\alpha+I)^{-1}$$

are compact in the space $\mathcal{H}^\alpha$.

Assumption 3 *For some $\rho>1$ operators*

$$(|x^\alpha|+1)^\rho V_s^\alpha(T^\alpha+I)^{-1}, \quad (|x^\alpha|+1)^\rho|\nabla V_l^\alpha|(T^\alpha+I)^{-1}$$

are bounded in the space $\mathcal{H}^\alpha$.

Compactness of $V^\alpha(T^\alpha+I)^{-1}$ ensures that the operator H is self-adjoint on the domain $\mathcal{D}(H)=\mathcal{D}(T)=:\mathcal{D}$ in the Hilbert space $\mathcal{H}=L_2(\mathbb{R}^d)$ and H is semi-bounded from below. Let us set

$$E(\cdot)=E(\cdot;H), \quad U(t)=\exp(-iHt).$$

We prove asymptotic completeness under the assumption that V^α are short-range functions of x^α but many intermediary results (in particular, radiation conditions-estimates) are as well true for long-range potentials.

The spectral theory of the operator H starts with the following geometrical construction. Let us introduce the set $\mathcal{X}$ of linear sums

$$X^a=X^{\alpha_1}\vee X^{\alpha_2}\vee\ldots\vee X^{\alpha_k}$$

of subspaces X^{α_j}. The zero subspace $X^0=\{0\}$ is included in the set $\mathcal{X}$ and X itself is excluded. The index a (or b) labels all subspaces $X^a\in\mathcal{X}$ and can be interpreted as the collection of all those α_j for which $X^{\alpha_j}\subset X^a$. Let x^a and x_a be the orthogonal projections of $x\in X$ on the subspaces X^a and

$$X_a:=X\ominus X^a=X_{\alpha_1}\cap X_{\alpha_2}\cap\ldots\cap X_{\alpha_k}$$

respectively. Since $X=X_a\oplus X^a, \mathcal{H}$ splits into a tensor product

$$L_2(X)=L_2(X_a)\otimes L_2(X^a). \tag{4}$$

Let us introduce for each a an auxiliary operator

$$H_a=T+V^a, \quad V^a=\sum_{X^\alpha\subset X^a}V^\alpha, \tag{5}$$

with a potential V^a which does not depend on x_a. In the representation (4)

$$H_a=T_a\otimes I+I\otimes H^a, \tag{6}$$

where $T_a = -\Delta_{x_a}$ acts in the space $\mathcal{H}_a = L_2(X_a)$ and

$$H^a = T^a + V^a, \quad T^a = -\Delta_{x^a}$$

are the operators in the space $\mathcal{H}^a = L_2(X^a)$. Set $\mathcal{H}^0 = \mathbb{C}, V^0 = 0, H^0 = 0$. In the multiparticle terminology, index a parametrizes decompositions of an N-particle system into noninteracting clusters. The operator H^a corresponds to the Hamiltonian of clusters with their centers-of-mass fixed at the origin; T^a is the kinetic energy of the center-of-mass motion of these clusters. Thus the operator H_a describes an N-particle system with interactions between different clusters neglected.

Scattering theory for the Hamiltonian H is formulated in terms of eigenvalues λ_n^a and eigenfunctions $\psi_n^a(x^a)$ of the operators H^a. Denote by P^a the orthogonal projection in $\mathcal{H}^a$ on the subspace $\mathcal{H}^{(p)}(H^a)$ and let $P_a = I \otimes P^a$. Clearly, the orthogonal projection P_a commutes with H_a and its functions. Set also $H_0 = T, P_0 = I$. The union over all a

$$\Upsilon_0 = \bigcup \sigma^{(p)}(H^a)$$

is called the set of thresholds for the operator H.

We need the following basic result (see [16, 17, 18]) of spectral theory of multiparticle Hamiltonians. It is formulated in terms of the auxiliary operator

$$A = \sum_j (x_j D_j + D_j x_j), \quad D_j = -i\partial_j, \quad \partial_j = \partial/\partial x_j, \quad j = 1, \dots, d.$$

Proposition 4 *Let Assumption 2 hold. Then the set Υ_0 is closed and countable and eigenvalues of H may accumulate only at Υ_0 so that the "exceptional" set $\Upsilon = \Upsilon_0 \cup \sigma^{(p)}(H)$ is also closed and countable. Furthermore, for every $\lambda \in \mathbf{R} \setminus \Upsilon$ there exists a small interval $\Lambda_\lambda \ni \lambda$ such that the estimate (the Mourre estimate) for the commutator holds:*

$$i([H, A]u, u) \geq c\|u\|^2, \quad c = c_\lambda > 0, \quad u \in E(\Lambda_\lambda)\mathcal{H}. \tag{7}$$

Let Q be multiplication by $(x^2 + 1)^{1/2}$. *Below Λ is always an arbitrary bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$*, where $\bar{\Lambda}$ is the closure of Λ. One of the main consequences of (7) is the following

Proposition 5 *Let Assumptions 2 and 3 hold. Then for any $r > 1/2$ the operator $Q^{-r}(T + I)$ is H-smooth on Λ.*

This assertion is usually called the limiting absorption principle. Its proof under Assumptions 2 and 3 can be found in [19].

Corollary 6 *The operator H is absolutely continuous on $E(\Lambda)\mathcal{H}$. In particular, it does not have any singular continuous spectrum, i.e.*

$$\mathcal{H} = \mathcal{H}^{(p)}(H) \oplus \mathcal{H}^{(ac)}(H).$$

The basic result of the scattering theory for N-particle Hamiltonians is the following

Theorem 7 *Suppose that functions V^α satisfy Assumptions 2 and 3 and are short-range, i.e. $V^\alpha = V_s^\alpha$. Then the wave operators*

$$W_a^\pm = W^\pm(H, H_a; P_a) \tag{8}$$

exist and are isometric on $P_a\mathcal{H}$. The ranges $R(W_a^\pm)$ of $W_a^\pm$ are mutually orthogonal and the asymptotic completeness holds:

$$\sum_a \oplus R(W_a^\pm) = \mathcal{H}^{(ac)}(H). \tag{9}$$

Our assumptions on V^α are somewhat larger than those of I. M. Sigal and A. Soffer [10] or G .M. Graf [11] since we do not require anything about derivatives of V^α.

Theorem 7 gives the complete spectral analysis of the operator H. Actually, by the relation (9), H is the orthogonal sum of its restrictions on different subspaces $R(W_a^\pm)$. In virtue of the intertwining property $HW_a^\pm = W_a^\pm H_a$, each of these restrictions is unitarily equivalent to the operator H_a considered in the space $P_a\mathcal{H}$. Actually, if $f \in \mathcal{H}^{(ac)}(H)$ and $f_a^\pm = (W_a^\pm)^* f \in P_a\mathcal{H}$, then

$$f = \sum_a W_a^\pm f_a^\pm \quad \text{and} \quad Hf = \sum_a W_a^\pm H_a f_a^\pm. \tag{10}$$

Furthermore, according to (6)

$$H_a f_a^\pm = \sum_n (T_a + \lambda_n^a) f_{a,n}^\pm \otimes \psi_n^a, \quad \text{if} \quad f_a^\pm = \sum_n f_{a,n}^\pm \otimes \psi_n^a. \tag{11}$$

Thus the absolutely continuous part of H is unitarily equivalent to the orthogonal sum of the "free" operators T_a shifted by the eigenvalues of the operators H^a.

Theorem 7 describes also the asymptotics as $t \to \pm\infty$ of the evolution $U(t)f$ governed by the Hamiltonian H. Indeed, the first equality (10) ensures that

$$U(t)f \sim \sum_a U_a(t) f_a^\pm, \quad t \to \pm\infty, \quad U_a(t) = \exp(-iH_a t), \tag{12}$$

which is basically equivalent to the asymptotic completeness. In virtue of (11) functions $U_a(t)f_a^\pm$ admit an explicit representation:

$$U_a(t) f_a^\pm = \sum_n \exp(-i(T_a + \lambda_n^a)t) f_{a,n}^\pm \otimes \psi_n^a. \tag{13}$$

4. The main analytical result of our approach is formulated as certain estimates which we call radiation conditions-estimates. Compared to the limiting absorption principle they give an additional information on the asymptotic behaviour of a quantum system for large distances and large times. To give the precise formulation let us introduce the gradient ∇_{x_a} in the variable x_a (in particular, $\nabla_{x_0} = \nabla$ is the gradient in the variable $x_0 = x$) and its angular part $\nabla^{(s)}_{x_a}$:

$$(\nabla^{(s)}_{x_a} u)(x) = (\nabla_{x_a} u)(x) - |x_a|^{-2} \langle (\nabla_{x_a} u)(x), x_a \rangle x_a. \tag{14}$$

Theorem 8 *Let Γ_a be a closed cone in $\mathbf{R}^d$ such that $\Gamma_a \cap X_b = \{0\}$ if $X_a \not\subset X_b$. Denote by χ_a the characteristic function of Γ_a. Suppose that V^α are defined by (3) where V^α_s and V^α_l satisfy Assumptions 2 and 3. Then for all a the operators*

$$G_a = \chi_a Q^{-1/2} \nabla^{(s)}_{x_a}$$

are H-smooth on Λ.

Remark. It is easy to see that

$$|\nabla^{(s)}_{x_b} u| \le |\nabla^{(s)}_{x_a} u|, \quad \text{if} \quad X_b \subset X_a.$$

Therefore Theorem 8 gives us more information about $U(t)f$ in the cone Γ_a than in Γ_b. In particular, the most complete information is obtained in the cone Γ_0 which does not intersect any $X_a \neq X$. On the contrary, the result of Theorem 8 is trivial for a such that $\dim X_a = 1$.

Remark. The notion of H-smoothness can be equivalently reformulated in terms of the resolvent of H. Thus radiation conditions-estimates given by Theorem 8 also admit a stationary formulation.

Remark. In the two-particle case the result of Theorem 8 reduces to H-smoothness of the operator $Q^{-1/2}\nabla^{(s)}$ on any bounded positive interval separated from the point 0. This corresponds to the angular part of the usual form of the radiation conditions-estimates (see e.g. [20]). Note also that the result of Proposition 5 (limiting absorption principle) definitely fails if $r = 1/2$ (even in the free case $H = H_0$). Thus the differential operator $\nabla^{(s)}$ improves the fall-off of $(U(t)f)(x)$ at infinity.

The proof of Theorem 8 relies on consideration of the commutator of H with a first-order differential operator

$$M = \sum_j (m_j D_j + D_j m_j), \quad m_j = \partial m / \partial x_j,$$

with *bounded* coefficients m_j. Actually, the function m is chosen as C^∞-homogeneous function of degree 1 (all properties of $m(x)$ are formulated for $|x| \ge 1$; in some neighbourhood of $x = 0$ we require that $m(x) = 0$). In the two-particle case we can set $m(x) = \mu_0 |x|$. In the N-particle case it should be modified in such a way that $m(x) = m(x_a)$ in some neighbourhood of each X_a. If, furthermore, x is separated from all X_b such that $X_a \not\subset X_b$, then $m(x) = \mu_a|x_a|$. Note also that, by the construction of $m(x)$, a cone $\tilde{\Gamma}_a$, where $m(x) = \mu_a|x_a|$, can be made arbitrary close to all $X_b \subset X_a$. Finally, $m(x)$ should be a convex and positive function (for $|x| \geq 1$). Such a function $m(x)$ can be constructed averaging over all ε_a of the family of functions

$$\max_a\{(1+\varepsilon_a)|x_a|\}, \tag{15}$$

where ε_a are suitably chosen small positive numbers. These functions satisfy all properties listed above except smoothness which is recovered by integration of (15) with some smooth functions of variables ε_a. We emphasize that only properties of $m(x)$ for large $|x|$ are essential. In a bounded domain $m(x)$ can be arbitrary.

The commutator $[V, M]$ is small in the following sense.

Proposition 9 *Suppose that V^α is defined by (3) where V_s^α and V_l^α satisfy Assumptions 2 and 3. Let m be C^∞-homogeneous function of degree 1 (for $|x| \geq 1$) and let $m(x) = m(x_\alpha)$ in some neighbourhood of X_α (for $|x| \geq 1$). Then*

$$|([V^\alpha, M]u, u)| \leq C\|Q^{-r}(T+I)u\|^2, \quad u \in \mathcal{D}, \quad 2r = \rho.$$

The proof of this assertion is based on the following observation. The potential V^α depends on x^α and is concentrated in a neighbourhood of X_α. In this region m depends on x_α only so that V^α and M essentially commute. This is similar to the idea of G. M. Graf applied in [11] in the time-dependent context.

The commutator of T with M equals

$$i[T, M] = 4\sum_{j,k} D_j m_{jk} D_k - (\Delta^2 m), \quad m_{jk} = \partial^2 m/\partial x_j \partial x_k. \tag{16}$$

Since the matrix $\{m_{jk}(x)\}$ is nonnegative for $|x| \geq 1$ and

$$\Delta^2 m(x) = O(|x|^{-3}), \quad |x| \to \infty, \tag{17}$$

it follows that

$$i([T, M]u, u) \geq \sum_{j,k} \int_\Gamma m_{jk} u_j \bar{u}_k dx - c\int_{|x|<1} |\nabla u|^2 dx - c\|Q^{-3/2}u\|^2, \tag{18}$$

where Γ is any region lying outside of the unit ball. Let $\tilde{\chi}_a$ be the characteristic function of the region $\tilde{\Gamma}_a$ where $m(x) = \mu_a|x_a|$. Simple calculations show that

$$\sum_{j,k} m_{jk} u_j \bar{u}_k = \mu_a |x_a|^{-1} |\nabla_{x_a}^{(s)} u|^2, \quad x \in \tilde{\Gamma}_a. \tag{19}$$

So combining Proposition 9 with (18) and (19) we arrive at the estimate (we assume here that $\rho \leq 3$)

$$i([H,M]u,u) \geq 4\mu_a \|\tilde{\chi}_a Q^{-1/2} \nabla_{x_a}^{(s)} u\|^2 - c\|Q^{-r}(T+I)f_t\|^2, \quad 2r = \rho. \quad (20)$$

The H-smoothness of the operator $\tilde{\chi}_a Q^{-1/2} \nabla_{x_a}^{(s)}$ can be standardly deduced from (20). Indeed, applying (20) to $u = f_t = U(t)f$ and integrating the identity

$$d(MU(t)f, U(t)f)/dt = i([H,M]f_t, f_t),$$

we obtain that

$$\int_{t_1}^{t_2} \|\tilde{\chi}_a Q^{-1/2} \nabla_{x_a}^{(s)} f_t\|^2 dt \leq C(|(Mf_t, f_t)\Big|_{t_1}^{t_2}| + \int_{t_1}^{t_2} \|Q^{-r}(T+I)f_t\|^2 dt).$$

If $f = E(\Lambda)f$, then the first term in the right side is bounded by $C\|f\|^2$ because the operator $ME(\Lambda)$ is bounded. The second term admits the same estimate by Proposition 5. Finally, H-smoothness of the operator $\tilde{\chi}_a Q^{-1/2} \nabla_{x_a}^{(s)}$ ensures H-smoothness of $\chi_a Q^{-1/2} \nabla_{x_a}^{(s)}$ since the region $\tilde{\Gamma}_a$ can be chosen arbitrary close to the subspaces $X_b \subset X_a$.

5. Our proof of asymptotic completeness demands preliminary consideration of auxiliary wave operators with identifications which are first-order differential operators

$$M^{(a)} = \sum_j (m_j^{(a)} D_j + D_j m_j^{(a)}), \quad m_j^{(a)} = \partial m^{(a)}/\partial x_j.$$

We set $m^{(a)}(x) = \eta^{(a)}(x) m(x)$ where the function m was constructed in the previous part and

$$\sum_a \eta^{(a)}(x) = 1. \quad (21)$$

We require that each $\eta^{(a)} \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $\eta^{(a)}$ is homogeneous of degree 0 and $\eta^{(a)}(x) = 0$ in some neighbourhoods of all X_b such that $X_a \not\subset X_b$. If $X_a \subset X_b$, then $\eta^{(a)}$ should not depend on x^b in some neighbourhod of X_b. The partition of unity with such properties can be constructed by the same procedure as the function m. Actually, wc define first a non-smooth function

$$\theta((1+\varepsilon_a)|x_a| - \max_{b, b \neq a}\{(1+\varepsilon_b)|x_b|\}), \quad (22)$$

where $\theta(s) = 1$ for $s \geq 0$ and $\theta(s) = 0$ for $s < 0$, and then obtain $\eta^{(a)}(x)$ by means of averaging of (22) over all ε_b and ε_a.

Theorem 10 *Suppose that functions V^α satisfy the assumptions of Theorem 7 and $E_a(\Lambda) = E_a(\Lambda; H_a)$. Then for all a the wave operators*

$$W^\pm(H, H_a; M^{(a)}E_a(\Lambda)), \quad W^\pm(H_a, H; M^{(a)}E(\Lambda)) \tag{23}$$

exist.

To prove this theorem we verify that the triple $H^{(a)}, H, M^{(a)}$ satisfies on Λ the conditions of Proposition 1. Actually, according to (5)

$$HM^{(a)} - M^{(a)}H_a = [T, M^{(a)}] + \sum_{X_a \subset X_\alpha} [V^\alpha, M^{(a)}] + \sum_{X_a \not\subset X_\alpha} V^\alpha M^{(a)}.$$

The second and third terms can be estimated with the help of limiting absorption principle. Indeed, by the construction of m and $\eta^{(a)}$, the function $m^{(a)}$ depends only on x_α in some neighbourhood of X_α. Therefore by Proposition 9

$$[V^\alpha, M^{(a)}] = (T + I)Q^{-r}BQ^{-r}(T + I), \quad 2r = \rho,$$

with a bounded operator B. The operator $Q^{-r}(T + I)$ is H- and H_a-smooth according to Proposition 5. A similar representation for $V^\alpha M^{(a)}$, $X_a \not\subset X_\alpha$, can be obtained due to the fact that $\eta^{(a)}(x) = 0$ in some neighbourhood of X_α where V^α is concentrated. We emphasize that the short-range assumption is used for the estimate of this term only.

The commutator $i[T, M^{(a)}]$ is again defined by (16) with m replaced by $m^{(a)}$. The function $\Delta^2 m^{(a)}(x)$ satisfies (17) and hence can be taken into account by Proposition 5. In order to estimate the operator

$$L^{(a)} = \sum_{j,k} D_j m^{(a)}_{jk} D_k$$

we use Theorem 8. Let $\lambda^{(a)}_n(x)$ and $p^{(a)}_n(x)$ be eigenvalues and normalized eigenvectors of the symmetric matrix $\mathbf{M}^{(a)}(x) = \{m^{(a)}_{jk}(x)\}$. Clearly, $\lambda^{(a)}_n(x)$ are homogeneous (for $|x| \geq 1$) functions of order -1 and $p^{(a)}_n(x)$ - of order 0. Diagonalizing $\mathbf{M}^{(a)}(x)$ we find that $L^{(a)} = (K^{(a)}_2)^* K^{(a)}_1$, where

$$(K^{(a)}_j u)(x) = \sum_n \nu^{(a)}_{n,j}(x) \langle \nabla u(x), p^{(a)}_n(x) \rangle p^{(a)}_n(x), \quad j = 1, 2,$$

and $\nu^{(a)}_{n,1} = |\lambda^{(a)}_n|^{1/2}$, $\nu^{(a)}_{n,1}\nu^{(a)}_{n,2} = \lambda^{(a)}_n$. To prove smoothness of the operators $K^{(a)}_j$ we need the following elementary observation.

Lemma 11 *Suppose that $m(x) = m(x_b)$ is a smooth homogeneous (for $|x| \geq 1$) function of degree 1 in some cone Γ. Let $\lambda_n(x)$ and $p_n(x)$ be eigenvalues and eigenvectors of the symmetric matrix $\mathbf{M}(x) = \{m_{jk}(x)\}$. Then vectors $p_n(x)$, $x \in \Gamma$, $|x| \geq 1$, corresponding to $\lambda_n(x) \neq 0$, belong to X_b and are orthogonal to x_b.*

By our construction of m and $\eta^{(a)}$ we can find cones Γ_b such that $\bigcup_b \Gamma_b = X$, Γ_b satisfy the condition of Theorem 8 and $m^{(a)}(x) = m^{(a)}(x_b)$ if $x \in \Gamma_b$. According to Lemma 11 and the definition (14)

$$\langle(\nabla u)(x), p_n^{(a)}(x)\rangle = \langle(\nabla_{x_b}^{(s)} u)(x), p_n^{(a)}(x)\rangle, \quad x \in \Gamma_b \quad |x| \geq 1, \quad \lambda_n^{(a)}(x) \neq 0.$$

It follows that

$$|(K_j^{(a)} u)(x)| \leq C|x|^{-1/2} |(\nabla_{x_b}^{(s)} u)(x)|, \quad x \in \Gamma_b, \quad |x| \geq 1.$$

Thus H- and H_a-smoothness of the operators $K_j^{(a)}$ is ensured by Theorem 8. Putting all things together we arrive at Theorem 10.

6. Our goal now is to deduce Theorem 8 from Theorem 9. To that end we introduce the observable

$$M^{\pm}(\Lambda) := W^{\pm}(H, H; ME(\Lambda)). \tag{24}$$

Its basic properties are formulated in the following

Theorem 12 *Let M be the same operator as in part 4. Suppose that functions (3) satisfy Assumptions 2 and 3. Then the wave operators (24) exist, are self-adjoint and commute with H. Furthermore, their ranges*

$$R(M^{\pm}(\Lambda)) = E(\Lambda)\mathcal{H}. \tag{25}$$

Existence of the wave operators (24) can be verified quite similarly to Theorem 10. The intertwining property of wave operators implies that $M^{\pm}(\Lambda)$ commutes with H. We shall show that $\pm M^{\pm}(\Lambda)$ is positively definite on the subspace $E(\Lambda)\mathcal{H}$. Note the identity

$$d(mf_t, h_t)/dt = i([H, m]f_t, h_t) = i([T, m]f_t, h_t) = (Mf_t, h_t), \tag{26}$$

where $f_t = U(t)f$ and $h_t = U(t)h$. Element $h \in \mathcal{H}$ is arbitrary and f belongs to some dense in $\mathcal{H}$ set so that mf_t are well-defined. Integrating (26) and taking into account existence of the wave operators (24) we find that

$$U^*(t)mU(t)f = t\, M^{\pm}(\Lambda)f + o(|t|), \quad t \to \pm\infty, \tag{27}$$

and, in particular,

$$\|mf_t\| = \|M^{\pm}(\Lambda)f\|\ |t| + o(|t|). \tag{28}$$

Since $m(x) \geq 0$, (27) ensures that $\pm M^{\pm}(\Lambda) \geq 0$. To prove that $\pm M^{\pm}(\Lambda)$ is positively definite we use Proposition 4. In virtue of the identity $i[H, Q^2] = 2A$, it follows from (7) that

$$2^{-1} d^2(Q^2 f_t, f_t)/dt^2 = d(Af_t, f_t)/dt = (i[H, A]f_t, f_t) \geq c\,\|f\|^2,$$

if f belongs to the subspace $E(\Lambda)\mathcal{H}$. Integrating twice this inequality we find that for sufficiently large $|t|$

$$\|Qf_t\| \geq c\,|t|\,\|f\|. \tag{29}$$

Comparing (28) with (29) and considering that $m(x) \geq m_0|x|,\ m_0 > 0$, for $|x| \geq 1$, we obtain the inequality

$$\|M^{\pm}(\Lambda)f\| \geq c\,\|f\|, \quad c = c(\Lambda) > 0.$$

Thus $\pm M^{\pm}(\Lambda)$ is positively definite on $E(\Lambda)\mathcal{H}$. In particular, (24) holds.

7. The difficult part of Theorem 7 is, of course, asymptotic completeness. We start with its proof in the form (12).

Theorem 13 *Under the assumptions of Theorem 7 for every $f = E(\Lambda)f$ there exist elements $f_a^{\pm}$ such that the relation (12) is fulfilled.*

Proof can be easily deduced from the results obtained. Let M and $M^{(a)}$ be the operators introduced in parts 4 and 5 respectively. According to (21)

$$\sum_a M^{(a)} = M. \tag{30}$$

By Theorem 12 every $f \in E(\Lambda)\mathcal{H}$ admits the representation $f = M^{\pm}(\Lambda)f^{\pm}$, $f^{\pm} \in E(\Lambda)\mathcal{H}$, so that the asymptotic relation

$$U(t)f \sim MU(t)f^{\pm}, \quad t \to \pm\infty, \tag{31}$$

holds. On the other hand, Theorem 10 ensures that for every a

$$M^{(a)}U(t)f^{\pm} \sim U_a(t)f_a^{\pm}, \quad t \to \pm\infty, \tag{32}$$

where

$$f_a^{\pm} = W^{\pm}(H_a, H; M^{(a)}E(\Lambda))f^{\pm}.$$

Summing up the relations (32) and taking into account (30) we find that

$$MU(t)f^{\pm} \sim \sum_a U_a(t)f_a^{\pm}, \quad t \to \pm\infty.$$

Comparing it with (31) we arrive at (12).

It remains to derive Theorem 7 from Theorem 13. Note that in the proof of Theorem 13 we have used only existence of the second set of the wave operators (23). Using existence of the first set of these operators and the equality (13) we can prove that the wave operators $W^{\pm}(H, H_a; ME_a(\Lambda))$ also exist. This implies existence of $W^{\pm}(H, H_a; E_a(\Lambda))$ because, according to Theorem 12, for every $g \in E_a(\Lambda)\mathcal{H}$ we can find elements $g_a^{\pm}$ such that

$$U_a(t)g \sim MU_a(t)g_a^{\pm}, \quad t \to \pm\infty.$$

Elements $f = E_a(\Lambda)f$ for different Λ, $\bar{\Lambda} \cap \Upsilon = \emptyset$, are dense in $\mathcal{H} = \mathcal{H}^{(ac)}(H_a)$ so that operators $W^{\pm}(H, H_a; I)$ exist. In virtue of (13) orthogonality of ranges of the wave operators (8) is an automatic consequence (see e.g. [14]) of their existence.

Admitting that Theorem 7 is already proven for all operators H^a and taking into account (6) we can rewrite (12) as

$$U(t)f \sim \sum_a U_a(t)\tilde{f}_a^{\pm}, \quad t \to \pm\infty, \quad \text{where} \quad \tilde{f}_a^{\pm} \in P_a\mathcal{H}.$$

Multiplying both sides by $U^*(t)$ we obtain that every $f \in E(\Lambda)\mathcal{H}$ belongs to the left side of (9). Since elements $f = E(\Lambda)f$ are dense in the subspace $\mathcal{H}^{(ac)}(H)$, this concludes our proof of the asymptotic completeness.

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