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EQUATIONS AUX DERIVEES PARTIELLES

# ASYMPTOTIC COMPLETENESS OF N-BODY SYSTEMS

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# Asymptotic completeness of $N$-body systems

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**Abstract**

Some of the ideas related to tbe concept of the asymptotic completeness are sketched. Both quantum and classical $N$-body systems are considered.

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# 1 Introduction

In this exposé we would like to describe a number of properties of classical and quantum scattering which are related to the concept of the asymptotic completeness. The most interesting one is probably the asymptotic completeness of quantum long range $N$-body systems. Nevertheless, we think that it is worthwile to compare this result with some other results. In particular, we would like to describe some classical analogs of various properties of quantum systems. We feel that, apart from its own interest, some results on classical systems illustrate most of the main ideas that are behind the proof of the asymptotic completeness in the quantum long range $N$-body case contained in [De2].

# 2 Quantum 2-body scattering

We would like to begin with a short description of the 2-body scattering. We start with the quantum case. The results that we describe have been known for quite a long time, references on this subject can be found eg. in [RS vol III, Hö vol II and IV, De1].

Suppose that $X$ is a finite dimensional vector space. Let $V$ be a real function on $X$ such that

$$V = V^l + V^s,$$
$$\|(1-\Delta)^{-1}V^s(x)\langle x\rangle^{1+\mu_s}\| \leq c$$

with $\mu_s > 0$ and

$$|\partial^\alpha V^l(x)| \leq c_\alpha \langle x\rangle^{-\mu-|\alpha|}$$

with $\mu > 0$. $D$ will denote the momentum operator $-i\nabla$. The full Hamiltonian is the self-adjoint operator on $L^2(X)$

$$H := -\frac{1}{2}\Delta + V(x).$$

The free Hamiltonian is defined as $H_0 := -\frac{1}{2}\Delta$.

If $B$ is a self-adjoint operator and $\Theta \subset \mathbf{R}$ then $E_\Theta(B)$ denotes the spectral measure of $B$ onto $\Theta$ and $\sigma(B)$ denotes the spectrum of $B$. $\sigma^{pp}(B)$ denotes the pure point spectrum of $B$ and $E^{pp}(B)$ denotes the corresponding spectral projection.

Let us begin with the description of the so-called asymptotic velocity.

**Theorem 2.1** *Let $J$ be a bounded continuous function on $X$. Then there exists*

$$s-\lim_{t\to\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH}. \qquad (2.1)$$

*Moreover, there exists a unique vector of commuting self-adjoint operators $P^+$ such that 2.1 equals $J(P^+)$. $P^+$ commutes with $H$,*

$$\sigma(P^+, H) = \left\{(\xi, \frac{1}{2}\xi^2) : \ \xi \in X\right\} \cup \{0\} \times \sigma^{pp}(H) \qquad (2.2)$$

*and*

$$E^{pp}(H) = E_{\{0\}}(P^+). \tag{2.3}$$

Note that the spectral measure of $(P^+, H)$ provides a kind of classification of all the states in the Hilbert space based on their asymptotic behavior for large time. Wave operators, whose construction is described in the next two theorems, provide more information on this spectral measure.

**Theorem 2.2** *(Short range case). Assume that $V^l = 0$. Then there exists*

$$s - \lim_{t\to\infty} e^{itH}e^{-itH_0}. \tag{2.4}$$

*This limit is called the wave operator and will be denoted by $\Omega^+$. Moreover, the following identity is true.*

$$P^+E_{X\setminus\{0\}}(P^+) = \Omega^+ D\Omega^{+*}. \tag{2.5}$$

**Theorem 2.3** *(Long range case). If we choose appropriately a function $S_t(\xi)$ then there exists*

$$s - \lim_{t\to\infty} e^{itH}e^{-iS_t(D)}. \tag{2.6}$$

*This limit is called a modified wave operator and, abusing somewhat the notation, it will be denoted by $\Omega^+$. The identity 2.5 is true also in this case.*

Theorem 2.3 implies that the isometry $\Omega^+*$ intertwines $P^+E_{X\setminus\{0\}}(P^+)$ and the momentum operator. By theorem 2.2 in the short range case there one can distinguish one natural isometry with this property—the wave operator 2.4.

The above theorems imply also that the whole Hilbert space is the direct sum of two subspaces:

1) $Ran E^{pp}(H)$—the subspace of bound states

2) $Ran E^c(H) = Ran E_{X\setminus\{0\}}(P^+)$—the subspace of scattering states.

This statement is usually called the asymptotic completeness.

# 3 Classical 2-body scattering

Classical two-body scattering theory is also well understood [Sim,He,RS vol III].

In the classical case it is convenient to impose the following assumptions on the potentials.

$$|\partial^\alpha V(x)| \leq c\langle x\rangle^{-|\alpha|-\mu}$$

for $|\alpha| = 0, 1, 2$ and $\mu > 0$.

Our basic object of interest will be solutions (trajectories) of the equations of motion generated by the Hamiltonian

$$H(x,\xi) := \frac{1}{2}\xi^2 + V(x). \tag{3.7}$$

In the classical case one can also define the asymptotic velocity.

**Theorem 3.1** *For any trajectory $t \mapsto x(t)$ there exists*

$$\lim_{t\to\infty} \frac{x(t)}{t}. \tag{3.8}$$

Next let us state the analog of the theorems 2.2 and 2.3 about the existence and completeness of wave operators.

**Theorem 3.2** *(Short range case). Suppose that $\mu > 1$.*
*a) For any $p^+, y^+ \in X$ such that $p^+ \neq 0$ there exists a trajectory $x(t)$ such that*

$$\lim_{t\to\infty}(x(t) - tp^+ - y^+) = 0. \tag{3.9}$$

*b) Let $x(t)$ be any trajectory such that $\lim_{t\to\infty} \frac{x(t)}{t} = p^+ \neq 0$. Then there exists $y^+ \in X$ such that 3.9 holds.*

**Theorem 3.3** *(Long range case).*
*a) Let $x_1(t)$ be a trajectory such that $\lim_{t\to\infty} \frac{x_1(t)}{t} \neq 0$ and let $y^+ \in X$. Then there exists a trajectory $x_2(t)$ such that*

$$\lim_{t\to\infty}(x_2(t) - x_1(t) - y^+) = 0. \tag{3.10}$$

*b)Let $x_1(t), x_2(t)$ be two trajectories such that $\lim_{t\to\infty} \frac{x_1(t)}{t} = \lim_{t\to\infty} \frac{x_2(t)}{t} \neq 0$. Then there exists*

$$\lim_{t\to\infty}(x_1(t) - x_2(t)).$$

Note that theorem 3.1 allow us to give the following classification of trajectories based on their behavior for large time:
1) bounded trajectories (more precisely, bounded for $t > 0$);
2) "almost-bounded trajectories" (trajectories which are unbounded for $t > 0$ but for which $\lim_{t\to\infty} \frac{x(t)}{t} = 0$);
3) "scattering trajectories" (with $\lim_{t\to\infty} \frac{x(t)}{t} \neq 0$).

Clearly, scattering trajectories are classified according to their asymptotic velocity. Theorems 3.2 and 3.3 give their further classification. If we fix $p^+ \neq 0$ then the set of trajectories with the asymptotic velocity $p^+$ is naturally isomorphic to the vector space $X$ in the short range case and to the affine space $X$ in the long range case.

Note the following differences between the classical and quantum scattering.
1) The main aim of the scattering theory is to classify states in the quantum case and trajectories in the classical case according to their asymptotic behavior. But, in a certain sense there are many more scattering trajectories then scattering states. (The former are parametrized by $(X\backslash\{0\}) \times X$ whereas the latter are described by a spectral measure on

$X$).
2) In the quantum case there are no analogs of almost-bounded trajectories.

Note that in the 2-body case almost bounded trajectories always have zero energy. They have also the following property.

**Proposition 3.4** *Let $x(t)$ be an almost-bounded trajectory. Then*

$$|x(t)| \leq c\langle t\rangle^{2(2+\mu)^{-1}}. \tag{3.11}$$

Proof. Let $r(x) := max(1, |x|)$. Note that $r$ is convex. We easily compute

$$\frac{d^2}{dt^2}\left(t^\delta r\left(\frac{x(t)}{t^\delta}\right)\right)$$

$$= t^{-\delta}\left(\xi - \delta\frac{x(t)}{t}\right)^2 \nabla^2 r\left(\frac{x(t)}{t^\delta}\right)$$

$$+t^{-2+\delta}\delta(\delta-1)\left(r\left(\frac{x(t)}{t^\delta}\right) - \frac{x(t)}{t^\delta}\nabla r\left(\frac{x(t)}{t^\delta}\right)\right) - \nabla V(x(t))\nabla r\left(\frac{x(t)}{t^\delta}\right)$$

$$\geq O(t^{-2+\delta}) + O(t^{-\delta(1+\mu)}). \tag{3.12}$$

Setting $\delta := 2(2+\mu)^{-1}$ optimizes 3.12. For this value of $\delta$ 3.12 is greater or equal than

$$ct^{(2+2\mu)(2+\mu)^{-1}} = c\frac{d^2}{dt^2}t^\delta. \tag{3.13}$$

Hence

$$\frac{d^2}{dt^2}\left(t^\delta r\left(\frac{x(t)}{t^\delta}\right) - ct^\delta\right) \geq 0. \tag{3.14}$$

Let

$$a := \lim_{t\to\infty}\frac{d}{dt}\left(t^\delta r\left(\frac{x(t)}{t^\delta}\right)\right).$$

We consider separately two cases.
1) Let $a > a_1 > 0$. Then

$$t^\delta r\left(\frac{x(t)}{t^\delta}\right) - ct^\delta \geq a_0 + a_1 t.$$

Therefore $|x(t)| \geq a_0 + a_1 t$, which is impossible, because $\lim_{t\to\infty}\frac{x(t)}{t} = 0$.
2) Let $a \leq 0$. Then

$$t^\delta r\left(\frac{x(t)}{t^\delta}\right) - ct^\delta$$

decreases. Thus either $|x(t)| \leq t^\delta$ or

$$|x(t)| - ct^\delta = t^\delta r\left(\frac{x(t)}{t^\delta}\right) - ct^\delta \leq a_0.$$

QED

Note that the proof which we presented above gives some flavor of the proof of the asymptotic completeness of $N$-body long range systems. For instance, the number $\delta := 2(2+\mu)^{-1}$ plays an important role in both proofs.

Let us remark that one cannot improve the bound 3.11. To see this consider the Hamiltonian

$$H(x,\xi) := \frac{1}{2}\xi^2 - |x|^{-\mu}.$$

One easily checks that for an appropriate $c_0$ the function

$$x(t) := c_0 t^{2(2+\mu)^{-1}}$$

is an almost-bounded trajectory for this Hamiltonian.

# 4 Quantum $N$-body scattering

Next we would like to discuss analogous problems in the $N$-body case. We will use the formalism of the so-called generalized $N$-body Hamiltonians, introduced by S.Agmon [A].

Let $X$ be a Euclidean space. Let $\{X_a : a \in \mathcal{A}\}$ be a family of its subspaces closed with respect to intersection and containing $X_{a_{min}} := X$. The orthogonal complement of $X_a$ will be denoted $X^a$. For any $x \in X$ we will write $x_a$ and $x^a$ to denote projections of $x$ onto $X_a$ and $X^a$ respectively. We will write $a \subset b$ iff $X_a \supset X_b$. Note that $\{X_a : a \in \mathcal{A}\}$ contains $\cap_{a\in\mathcal{A}} X_a$ which will be denoted $X_{a_{max}}$.

If $a \in \mathcal{A}$ then $\#a$ denotes the maximal number of distinct $a_i \in \mathcal{A}$ such that $a = a_n \subset \ldots \subset a_1 = a_{max}$. We set $N := max\{\#a : a \in \mathcal{A}\}$.

We also define

$$Z_a := X_a \setminus \bigcup_{b \not\subset a} X_b.$$

Note that $X$ is the disjoint sum of $\{Z_a : a \in \mathcal{A}\}$.

The above geometric definitions were common for classical and quantum systems. Now let us introduce some notions specific for quantum $N$-body systems.

Let $D$, $D_a$ and $D^a$ denote the operators $\frac{1}{i}\nabla$, $\frac{1}{i}\nabla_a$ and $\frac{1}{i}\nabla^a$ on $L^2(X)$ respectively. Likewise, $\Delta$, $\Delta_a$ and $\Delta^a$ denote the Laplacians corresponding to the variables $x$, $x_a$ and $x^a$ respectively.

We assume that for every $a \in \mathcal{A}$ we are given a real function $X^a \ni x^a \mapsto V^a(x^a)$. Let $\mu_s > 0$ and $1 \geq \mu > 0$. We will assume that $V^a(x^a) = V^{sa}(x^a) + V^{la}(x^a)$ and the following conditions are satisfied:

$$\|(1-\Delta^a)^{-1} V^{sa}(x^a)\langle x^a\rangle^{1+\mu_s}\| \leq c; \tag{4.15}$$

and

$$|\partial^\alpha V^{la}(x^a)| \leq c\langle x^a\rangle^{-|\alpha|-\mu}. \tag{4.16}$$

We set

$$V(x) := \sum_{a\in\mathcal{A}} V^a(x^a)$$

and

$$V_a(x) := \sum_{b\subset a} V^b(x^b).$$

Our basic Hamiltonian will be the self adjoint operator $H$ defined by

$$H := -\frac{1}{2}\Delta + V(x). \tag{4.17}$$

It will be also useful to introduce subsystem Hamiltonians

$$H_a := -\frac{1}{2}\Delta + V_a(x).$$

Clearly, $H = H_{a_{max}}$.

We may identify $L^2(X)$ with $L^2(X_a)\otimes L^2(X^a)$. Then we can write:

$$H_a = -\frac{1}{2}\Delta_a \otimes 1 + 1 \otimes H^a$$

where

$$H^a := -\frac{1}{2}\Delta^a + V_a(x^a)$$

is the internal Hamiltonian of the subsystem $a$.

It turns out (which is almost a miracle) that also in the $N$-body case one can introduce the observable of the asymptotic velocity. The proof of the following theorem, based on the ideas of [Graf], can be found in [Del,2].

**Theorem 4.1** *Let $J$ be a bounded continuous function on $X$. Then there exists*

$$s-\lim_{t\to\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH}. \tag{4.18}$$

*Moreover, there exists a unique vector of commuting self-adjoint operators $P^+$ such that 4.18 equals $J(P^+)$. $P^+$ commutes with $H$,*

$$\sigma(P^+, H) = \bigcup_{a\in\mathcal{A}} \left\{(\xi_a, \tau + \frac{1}{2}\xi_a^2) :\ \xi_a \in X_a,\ \tau \in \sigma^{pp}(H^a)\right\} \tag{4.19}$$

*and*

$$E^{pp}(H) = E_{\{0\}}(P^+). \tag{4.20}$$

Next let us state the theorem on the existence and completeness of wave operators in the short range case. This theorem was first shown by V.Enss [E1,2] in the $N = 3$ case and by I.Sigal and A.Soffer in the case of an arbitrary $N$ [SigSof1].

**Theorem 4.2** *(Short range case). Assume that $V^l = 0$. Then for every $a \in \mathcal{A}$ there exists*

$$s - \lim_{t\to\infty} e^{itH} e^{-itH_a} E^{pp}(H^a). \tag{4.21}$$

*This limit is called the wave operator corresponding to the subsystem a and will be denoted by $\Omega_a^+$. Moreover*

$$P^+ E_{Z_a}(P^+) = \Omega^+ D_a \Omega^{+*}. \tag{4.22}$$

*and*

$$\sum_{a\in\mathcal{A}}^{\oplus} Ran\Omega_a^+ = L^2(X). \tag{4.23}$$

The following theorem was first proven by V.Enss in the $N = 3$ case [E1,2], by I.Sigal and A.Soffer in the case $N = 4$, $\mu = 1$ [SigSof2] and by the author in the general case [De2].

**Theorem 4.3** *(Long range case). Let $\mu > \sqrt{3} - 1$. If we choose for instance*

$$S_{at}(\xi_a) := \frac{t}{2}\xi_a^2 + \int_0^t I_a^l(s\xi_a)ds.$$

*then there exists*

$$s - \lim_{t\to\infty} e^{itH} e^{-itH^a + S_{at}(D_a)} E^{pp}(H^a). \tag{4.24}$$

*This limit is called a modified wave operator and, abusing somewhat the notation, it will be denoted by $\Omega_a^+$. 4.22 and 4.23 are true also in this case.*

Note that theorems 4.2 and 4.3 on the existence and completeness of wave opereators give a kind of a classification of all states in the Hilbert space which is valid for $\mu > \sqrt{3}-1$. Theorem 4.1 contains a somewhat poorer classification—in terms of the joint spectral measure of $(P^+, H)$. This classification has the advantage of being valid for any $\mu > 0$.

# 5 Classical $N$-body scattering

In this section we consider $N$-body classical systems. We would like to present a number of their properties, which we think are analogs of some of the properties of quantum $N$-body systems. Proofs of these results can be found in [De3].

We will assume that

$$|\partial^\alpha V^a(x)| \leq c\langle x^a\rangle^{-|\alpha|-\mu}$$

for $|\alpha| = 0, 1, 2$ and $\mu > 0$.

$V$ and $V_a$ are defined analogously as in the quantum case. The full Hamiltonian is

$$H(x, \xi) := \frac{1}{2}\xi^2 + V(x) \tag{5.25}$$

and the cluster Hamiltonians are defined as

$$H_a(x, \xi) := \frac{1}{2}\xi^2 + V_a(x)$$

The asymptotic velocity exists, as described by the following theorem.

**Theorem 5.1** *Let $x(t)$ be a trajectory of the Hamiltonian 5.25. Then there exists*

$$\lim_{t\to\infty} \frac{x(t)}{t}. \tag{5.26}$$

*It defines a function on the phase space $P^+(x, \xi)$ where $x, \xi$ are the initial conditions of a given trajectory.*

We think, that the theorems which we state below can be viewed as analogs of theorems 4.2 and 4.3 about wave operators.

**Theorem 5.2** *(Short range case). Suppose that $\mu > 1$. Let $x(t)$ be any trajectory such that $\lim_{t\to\infty} \frac{x(t)}{t} = p_a^+ \in Z_a$. Then there exists $y_a^+ \in X$ such that*

$$\lim_{t\to\infty} (x(t)_a - tp_a^+ - y_a^+) = 0. \tag{5.27}$$

**Theorem 5.3** *(Long range case). Let $\mu > \sqrt{3} - 1$. Let $x_1(t), x_2(t)$ be two trajectories such that $\lim_{t\to\infty} \frac{x_1(t)}{t} = \lim_{t\to\infty} \frac{x_2(t)}{t} \in Z_a$. Then there exists*

$$\lim_{t\to\infty} (x_1(t)_a - x_2(t)_a).$$

Note that the asymptotic velocity $P^+$ gives a partial classification of all the trajectories. Theorems 5.2 and 5.3 give for $\mu > \sqrt{3} - 1$ a somewhat more detailed classification of trajectories with a fixed value of $P^+ \in Z_a$ which is based on the asymptotics of the external variables. If $\mu > 1$ then they are naturally parametrized with the vector space $X_a$ and for $\sqrt{3} - 1 < \mu \leq 1$ they are naturally parametrized with the affine space $X_a$. Still, this is not a one-to-one parametrization, because it ignores the asymptotics of the internal ("super-$a$") motion.

Let us mention that one of the key idea needed in the proof of theorem 5.3 is the following generalization of proposition 3.4.

**Proposition 5.4** *Let $x(t)$ be a trajectory such that $\lim_{t\to\infty}\frac{x(t)}{t}\in Z_a$. Then*

$$|x^a(t)| \le c\langle t\rangle^{2(2+\mu)^{-1}}. \tag{5.28}$$

Sketch of the proof. To simplify notation we set $a = a_{max}$ and assume that $X_{a_{max}} = \{0\}$. Then $Z_{a_{max}} = \{0\}$. We need to modify the function $r$ used in the proof of proposition. We set

$$r(x) := max\{\sqrt{\rho_{\#a} + x_a^2} : \ a \in \mathcal{A}\}.$$

We choose $\rho_n$ appropriately to guarantee that for any $a \in \mathcal{A}$ there exists $\epsilon > 0$ such that $r$ depends just on $x_a$ on $\{x \in X : \ dist(x, X_a) < \epsilon\}$. Note also that $r$ is convex. The rest of the proof is very similar to the proof of proposition 3.4. QED

Note that many of the ideas underlying the construction of the function $r$ are due to [Graf]; in fact, the so-called Graf vector field is essentially equal to $\frac{1}{2}\nabla r^2$. The idea of using a function similar to $r$ first appeared in [Ya].

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