SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

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Hidden symmetries of integrable systems in Yang-Mills theory and Kähler geometry

Séminaire Équations aux dérivées partielles (Polytechnique) (1990-1991), exp. nº 8, p. 1-15

<http://www.numdam.org/item?id=SEDP_1990-1991____A8_0>

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Séminaire 1990-1991

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HIDDEN SYMMETRIES OF INTEGRABLE SYSTEMS IN YANG-MILLS THEORY AND KÄHLER GEOMETRY

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Exposé n°VIII

22 Janvier 1991

HIDDEN SYMMETRIES OF INTEGRABLE SYSTEMS IN YANG-MILLS THEORY AND KÄHLER GEOMETRY

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1. Introduction

We now have a huge list of "nonlinear integrable systems." Most of them are more or less related to so called "soliton" phenomena, thereby referred to as "soliton equations." In the last decade, various soliton equations came to be reorganized in a unified framework, the Kadomtsev-Petviashvili (KP) hierarchy and its family [Sa-Sa] [Se-Wi].

Beside these soliton equations, there are a few exceptional cases that are considered as "higher (or multi) dimensional" nonlinear integrable systems, whereas soliton equations are to describe nonlinear waves in one dimensinal space like a canal. Typical and the most important examples are the self-duality equations in the Yang-Mills theory of gauge fields and in the Einstein theory of gravity (see reviews [Ch] [Bo] and references cited). As first pointed out by Ward [Wa1] for the Yang-Mills theory and by Penrose [Pe] for the Einstein theory, the self-duality equations and their family are in close relationship with "twistor theory" [Tw1-6], and this fact lies in the heart of their "integrability" (see [Be-Za] [Ch-Pr-Si] [Fo-Ho-Pa] [Po] for earlier work in that direction.)

"Integrability" is a magic word that summarizes a number of aspects observed in these nonlinear systems; we now focus on their "symmetries" on the space of solutions. The existence of a large set of such symmetries is strong evidence of integrability. Suppose that the set of symmetries is so large that its acts, in some sense, transitively on the space of solutions – then all solutions, in principle, can be obtained from some special solution by the action of symmetries. It would be reasonable to call such a system of equations an integrable system. This is indeed the case, literally or approximately, for most nonlinear integrable systems known so far. Further, those symmetries frequently exhibit, at infinitesimal level, some remarkable Lie algebraic structures. For soliton equations, infinite dimensional Lie algebras called "Kac-Moody" algebras and their representation theory [Ka] [Pr-Se] give such structures [Da-Ji-Ka-Mi](see also review [Ta1]). Similar structures are discovered in the self-dual Yang-Mills equation [Ue-Na] [Ch-Ge-Wu] [Do] and the self-dual Einstein equation [Bo-Pl].

A key technique for the self-duality equations (as well as their "dimensional reductions" to lower dimensions; see the articles of Ward, Fletcher and Woodhouse in [Tw6]) is the so called "Riemann-Hilbert problem," which we rather call, in the following, the "Riemann-Hilbert factorization" because the former terminology is now used in a more general context in mathematics. Twistor people like to call this method "splitting method"; in fact, this is nothing but an analytic (or group-theoretic) expression of the twistor-theoretic approach.

This article is an résumé of the present author's work [Ta2-4] in recent years on symmetries of the self-duality equations and related equations. A central issue was to find an explicit form of *infinitesimal* symmetries of the self-dual Einstein equation that should correspond to the *finite* symmetries of Boyer and Plebanski [Bo-Pl]. Main results are presented in Section 5. In general, finite symmetries of nonlinear systems are given by very complicated nonlinear transformation of dependent (and, sometimes, of independent) variables. This is already so for soliton equations and the self-dual Yang-Mills equations whose Riemann-Hilbert factorization is still relatively well understood. The case of the self-dual Einstein equations is far worse; there is no effective way to solve the factorization, only the existence of a solution being ensured by a general theorem. Infinitesimal symmetries, on the other hand, should have a more explicit, and even beautiful expression as experiences in soliton equation and the self-dual Yang-Mills equation advocate. This indeed turns out to be the case for the self-dual Einstein equation.

Finding an explicit form of infinitemsimal symmetries have also several remarkable consequences and applications. For example, in the course of this study, potentials called "Plebanski key functions" [Pl] are shown to play a role similar to the "tau function" of soliton equations [Sa-Sa] [Se-Wi]. Further, a basic Lie algebra in this case is identified to be a Poisson algebra degenerate in one direction (or equivalently, a loop algebra of a nondegenerate Poisson algebra) rather than the loop algebra of Hamiltonian vector fields predicted by Boyer and Plebanski [Bo-Pl]. This kind of Lie algebras are known to arise in physics of membrane, extended conformal symmetries, topological field theory, etc. Further progress of this work can be expected in that direction. We shall come back to this issue in the last section.

The original self-duality equations, both in the Yang-Mills theory and in the Einstein theory, arise in four dimensional space-times. One can, in fact, consider their higher dimensional analogues in $4N$ ($N = 1, 2, \ldots$) dimensions . We present our results in such a generalized form, because that seems to make more clear the role of an underlying quaternionic or symplectic structure. This is also for emphasizing a remarkable similarity between the Yang-Mills case and the Einstein case.

2. Generalized self-dual Yang-Mills equation

The generalized self-dual Yang-Mills equation we now consider is a $4N$ dimensional generalization first introduced by Ward [Wa2] along with several other higher dimensional extensions; this corresponds to the "A-series" in his classification table. This series shares a number of beautiful properties with the four dimensional case such as the existence of "instanton solutions" [Co-Go-Ke] like those in four dimensions [At-Hi-Dr-Ma].

2.1. *Yang equation in generalized form*

As pointed out by Yang [Ya], the self-dual Yang-Mills equation in four dimensional Euclidean space-time has a compact expression called the "Yang equation." A key idea therein is to introduce complex linear combinations of the real space-time coordinates as new coordinates. It is then more natural to consider the equation in "complex space-time" with complexified coordinates. This is also the case for the A-series generalized self-dual Yang-Mills equation. To stress an analogy with the generalized self-dual Einstein equation, let us write a set of complexified space-time coordinates as:

$$(x,p) = (x^1,\ldots,x^{2N},p^1,\ldots,p^{2N})$$

Then the Yang equation for the generalized eqution is given by

$$\frac{\partial}{\partial x^i}\left(\frac{\partial J}{\partial p^j}J^{-1}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial J}{\partial p^i}J^{-1}\right) = 0, \tag{2.1}$$

where the unknown function $J = J(x,p)$ now takes values in the structure group G of gauge fields. (For G, we consider complex Lie groups such as $GL(n,\mathbf{C})$, $SL(n,\mathbf{C})$, etc., because we are now in complexified space-time.) In the four dimensional (i.e., $N = 1$) case, only one equation with $i = 1$ and $j = 2$ remains and gives the Yang equation.

There is another expression of the (generalized) self-dual Yang-Mills equation which is linked with the Yang equation by a simple Bäcklund-type transformation and which is equally fundamental. As we shall show later, the self-dual Einstein equation has also two such equivalent expressions. In the gerenalized form, the second expression is given by the equation

$$\frac{\partial^2 K}{\partial x^i \partial p^j} - \frac{\partial^2 K}{\partial x^j \partial p^i} + \left[\frac{\partial K}{\partial x^i}, \frac{\partial K}{\partial x^j}\right] = 0, \tag{2.2}$$

where the unknown function $K = K(x,p)$ now takes values in the Lie algebra Lie$G$ of G.

2.2. *Relation to covariant derivation*

The above two expressions are derived by a special "gauge-fixing." A general form of the generalized self-dual Yang-Mills equation in terms of the covariant derivation ∇ is, by definition, given by a set of linear relations among their commutators:

$$[\nabla_{p^i}, \nabla_{p^j}] = 0, \tag{2.3a}$$

$$[\nabla_{p^i}, \nabla_{x^j}] = [\nabla_{p^j}, \nabla_{x^i}], \tag{2.3b}$$

$$[\nabla_{x^i}, \nabla_{x^j}] = 0. \tag{2.3c}$$

This system of equations have gauge symmetries $\nabla \mapsto g^{-1} \circ \nabla \circ g$, $g = g(x,p)$ taking values in G, and (2.3c) can be "gauged away" by a suitable gauge transformation so that, without loss of generality, the coordinate components of the covariant derivation can be assumed to be

$$\nabla_{p^i} = \frac{\partial}{\partial p^i} + A_i, \quad \nabla_{x^i} = \frac{\partial}{\partial x^i}. \tag{2.4}$$

The other two, (2.3b) and (2.3c), now become equations on A_i's. One can solve one of these two by introducing a new potential. From (2.3a) one can introduce a G-valued potential, J, as

$$A_i = -\frac{\partial J}{\partial p^i} J^{-1}, \tag{2.5}$$

and (2.3b) exactly gives (2.1). If one uses (2.3b) to introduce a Lie$G$-valued potential, K, as

$$A_i = -\frac{\partial K}{\partial x^i}, \tag{2.6}$$

then (2.3a) gives (2.2). One can thus deduce the aforementioned two expressions, (2.1) and (2.2), of the generalized self-dual Yang-Mills equation. In particular, (2.5) and (2.6) give a relation (Bäcklund-type transformation) that connects J and K.

2.3. Linear system and auxiliary variables

A key observation that leads to "integrability" is the fact that equations (2.3) can be gathered up into a compact form

$$[\nabla_{p^i} - \lambda\nabla_{x^i}, \nabla_{p^j} - \lambda\nabla_{x^j}] = 0, \tag{2.8}$$

where λ is a free parameter that runs over all complex numbers. This implies that the linear system

$$(\nabla_{p^i} - \lambda\nabla_{x^i})\Psi(x,p,\lambda) = 0 \tag{2.9}$$

is integrable in the sense of Frobenius, and at least locally, one can find a G-valued solution. In the gauge-fixing as in (2.4), we can introduce two types of solutions $W(\lambda) = W(x,p,\lambda)$ and $\hat{W}(\lambda) = \hat{W}(x,p,\lambda)$ with the following Laurent series expansion:

$$W(\lambda) = 1 + \sum_{n\leq -1} W_n\lambda^n, \quad \hat{W}(\lambda) = \sum_{n\geq 0} \hat{W}_n\lambda^n \tag{2.10a}$$

In fact, one can further require $W(\lambda)$ and $\hat{W}(\lambda)$ to be such tbat

$$W_{-1} = -K, \quad \hat{W}_0 = J. \tag{2.10b}$$

The gauge potentials A_i, consequently, can be written as derivatives of W_{-1} and $\hat{W}_0$.

If the gauge potentials A_i are replaced by such derivatives of W_{-1} and $\hat{W}_0$, linear system (2.9) then changes into a *nonlinear* equation on the Laurent coefficients, W_n and $\hat{W}_n$. It is in this nonlinear system that we construct symmetries. Most of our symmetries cannot be written explicitly without such a set of auxiliary variables. In that sense, we call them "hidden symmetries." This situation is not specific to the self-duality equations; the same takes place in the case of soliton equations.

3. Generalized self-dual Einstein equation

A $4N$-dimensional generalization of the self-dual Einstein equation is provided by hyper-Kähler geometry. This is a special case of Kähler geometry specified by its holonomy group [Hi-Ka-Li-Ro]. In four dimensions, this is the same as Ricci-flat Kähler geometry; any self-dual Einstein space is locally a two dimensional Ricci-flat Kähler manifold. Plebanski [Pl] pointed out (or re-discovered) that self-dual Einstein spaces (also called $\mathcal{H}$-spaces) have two equivalent local pictures based upon a potential ("key function") and a system of nonlinear differential equations ("heavenly equations"). These equations are to correspond to the Yang equation for J and its equivalent for K. We now show their hyper-Kähler version.

Let us introduce a few notational conventions. Let $i, j, \ldots$ be symplectic indices with values in integers $1, \ldots, 2N$. $\in^{ij}$ and $\in_{ij}$ denote the standard symplectic $\in$-symbols normalized as $\in_{2i-1,2i} = -\in_{2i,2i-1} = 1$ and $\in^{2i-1,2i} = -\in^{2i,2i-1} = 1$. The Einstein summation convention is understood only for symplectic indices. Symplectic indices are raised and lowered as $\xi_i = \in_{ij} \xi^j, \quad \eta^j = \eta_i \in^{ij}$.

3.1. Plebanski heavenly equations

The first local expression of hyper-Kähler geometry consists of $4n$ complex coordinates

$$(p, \hat{p}) = (p^1, \ldots, p^{2N}, \hat{p}^1, \ldots, \hat{p}^{2N})$$

(we are again in *complexified* space-time), a scalar unknown function $\Omega = \Omega(p, \hat{p})$, and the "first heavenly equation"

$$\left\{ \frac{\partial \Omega}{\partial p^i}, \frac{\partial \Omega}{\partial p^j} \right\}_{(\hat{p})} = \in_{ij}, \tag{3.1}$$

where $\{ \quad , \quad \}_{(\hat{p})}$ stands for a Poisson bracket in the variables $\hat{p}$:

$$\{F, G\}_{(\hat{p})} = \in^{ij} \frac{\partial F}{\partial \hat{p}^i} \frac{\partial G}{\partial \hat{p}^j}. \tag{3.2}$$

Geometrically, Ω represents a Kähler potential, and p^i and $\hat{p}_i$ correspond to complex coordinates and their complex conjugate. In our present complexified setting, however, p and $\hat{p}$ are understood as independent variables. In four dimensions, the above equation reduces a single equation of the form

$$\det \left(\frac{\partial \Omega}{\partial p^i \partial \hat{p}^j}; \; i, j = 1, 2 \right) = 1,$$

which is the original first heavenly equation that Plebanski derived. Actually, this equation had been known in mathematics as an expression of Ricci-flat Kähler geometry, probably going back to Calabi.

The second local expression is based upon another set of $4N$ coordinates

$$(x, p) = (x^1, \ldots, x^{2N}, p^1, \ldots, p^{2N}),$$

a scalar unknown function $\Theta = \Theta(x, p)$, and the "second heavenly equation"

$$\frac{\partial^2 \Theta}{\partial x^i \partial p^j} - \frac{\partial^2 \Theta}{\partial x^j \partial p^i} + \left\{ \frac{\partial \Theta}{\partial x^i}, \frac{\partial \Theta}{\partial x^j} \right\}_{(x)} = 0, \tag{3.3}$$

where $\{\ ,\ \}_{(x)}$ is now a Poisson bracket in the variables x:

$$\{F, G\}_{(x)} = \epsilon^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}. \tag{3.4}$$

Since we consider everything in complexified space-time, the variables (x, p) are all independent and comlex valued. As oppose to $(p, \hat{p})$, these variables are geometrically less convenient in the sense that it is hard to specify in which *real slice* a Riemannian or Kählerian manifold is realized.

3.2. Exterior differential equations and auxiliary variables

What correspond to W and $\hat{W}$ are two set of Laurent series

$$u^i(\lambda) = \sum_{n \le -1} u^i_n \lambda^n + x^i + p^i \lambda, \quad \hat{u}^i(\lambda) = \hat{p}^i + \sum_{n \ge 1} \hat{u}^i_n \lambda^n \tag{3.5}$$

$(i = 1, \ldots, 2N)$ that satisfy the exterior differential equations

$$\epsilon_{ij}\, du^i(\lambda) \wedge du^j(\lambda) = \epsilon_{ij}\, d\hat{u}^i(\lambda) \wedge d\hat{u}^j(\lambda), \tag{3.6}$$

$$d\Theta = \epsilon_{ij}\, u^i_{-2} dp^j + \epsilon_{ij}\, u^i_{-1} dx^i, \tag{3.7a}$$

$$d\Omega = -\epsilon_{ij}\, u^i_0 dp^j + \epsilon_{ij}\, \hat{u}^i_1 d\hat{p}^j. \tag{3.7b}$$

Here u^i_n and $\hat{u}^i_n$ are understood as unknown functions of (x, p) (in the second heavenly picture) or of $(p, \hat{p})$ (in the first heavenly picture); λ is considered a constant under the total differential d, i.e., $d\lambda = 0$. Note that (3.6) is a system of nonlinear equations for the Laurent coefficients u^i_n and $\hat{u}^i_n$; symmetries are to be constructed for this system (and for the whole system, (3.6) and (3.7), too).

4. Symmetries of generalized self-dual Yang-Mills equation

This section is due to [Ta2]. The contents are, in a sense, an improvement of known facts scattered in the literature (see the references cited in Section 1). We still present these results in some detail because this will be helpful for understanding the corresponding results for the generalized self-dual Einstein equation.

4.1. The case of generalized Yang-Mills equation

Suppose that $W(\lambda)$ is a G-valued holomorphic funcion of λ in, say, $|\lambda| < R$ and $\hat{W}(\lambda)$, likewise, in $|\lambda| > r$ $(r < R)$. The matrix-ratio

$$g(\lambda) = W(\lambda)^{-1} \hat{W}(\lambda) \tag{4.1}$$

then obviously satisfies the linear equation

$$\left(\frac{\partial}{\partial p^i} - \lambda \frac{\partial}{\partial x^i} \right) g(\lambda) = 0,$$

so that takes such a form as

$$g(\lambda) = \mathbf{g}(\lambda, x + p\lambda), \quad x + p\lambda = (x^1 + p^1\lambda, \ldots, x^{2N} + p^{2N}\lambda). \tag{4.2}$$

One can thus obtain a G-valued holomorphic function $\mathbf{g}(\lambda, u)$ of $2N+1$ variables, λ moving in the annular domain $r < |\lambda| < R$. Conversely, if such a function is given and the correspoinding $g(\lambda)$ is factorized as in (4.1) (this is indeed possible if, for example, $\mathbf{g}(\lambda, u)$ is sufficiently close to the identity matrix), it is easy to see that $W(\lambda)$ and $\hat{W}(\lambda)$ indeed give rise to a solution of the generalized self-dual Yang-Mills equation. (For details, see the references cited in Section 1.) (4.1) is a "Riemann-Hilbert factorization." Further, $\mathbf{g}(\lambda, u)$ gives an essential part of the twistor-theoretic data in this case.

The correspondence between $(W(\lambda), \hat{W}(\lambda))$ and $\mathbf{g}(\lambda, u)$ is one-to-one. Therefore changing $\mathbf{g}(\lambda, u)$ suitably, one can obtain a deformation of a solution of the $(W(\lambda), \hat{W}(\lambda))$-system.

4.2. Explicit form of infinitesimal symmetries

To derive infinitesimal symmetries, we consider the one-paramer family of deformations

$$\mathbf{g}(\lambda, u) \mapsto e^{-\epsilon \mathbf{X}(\lambda, u)} \mathbf{g}(\lambda, u) e^{\epsilon \hat{\mathbf{X}}(\lambda, u)},$$

where $\mathbf{X}(\lambda, u)$ and $\hat{\mathbf{X}}(\lambda, u)$ are Lie$G$-valued holomorphic function of (λ, u) in the same domain of definition as $\mathbf{g}(\lambda, u)$, or, in a more algebraic formulation, Laurent series of the form

$$\mathbf{X}(\lambda, u) = \sum_{n=-\infty}^{\infty} \mathbf{X}_n(u) \lambda^n, \quad \hat{\mathbf{X}}(\lambda, u) = \sum_{n=-\infty}^{\infty} \hat{\mathbf{X}}_n(u) \lambda^n, \tag{4.3}$$

From these data, as in (4.2), we define

$$X(\lambda) = \mathbf{X}(\lambda, x + p\lambda), \quad \hat{X}(\lambda) = \hat{\mathbf{X}}(\lambda, x + p\lambda). \tag{4.4}$$

Let us write the corresponding $(W, \hat{W})$-pair as $W(\epsilon, \lambda)$ and $\hat{W}(\epsilon, \lambda)$. The Rienamm-Hilbert factorization for this pair can be rewritten

$$W(\epsilon, \lambda) e^{-\epsilon X(\lambda)} W(\lambda)^{-1} = \hat{W}(\epsilon, \lambda) e^{-\epsilon \hat{X}(\lambda)} \hat{W}(\lambda)^{-1}, \tag{4.5}$$

which is more convenient for calculating infinitesimal symmetries. The infinitesimal symmetries are defined by

$$\delta_{X,\hat{X}} W(\lambda) = \left. \frac{\partial W(\epsilon, \lambda)}{\partial \epsilon} \right|_{\epsilon=0}, \quad \delta_{X,\hat{X}} \hat{W}(\lambda) = \left. \frac{\partial \hat{W}(\epsilon, \lambda)}{\partial \epsilon} \right|_{\epsilon=0} \tag{4.6}$$

A basic result for this case is the following.

THEOREM. *The infinitesimal symmetries can be written explicitly:*

$$\delta_{X,\hat{X}} W(\lambda) \cdot W(\lambda)^{-1} = \left(W(\lambda) X(\lambda) W(\lambda)^{-1} - \hat{W}(\lambda) \hat{X}(\lambda) \hat{W}(\lambda)^{-1} \right)_{\leq -1},$$

$$\delta_{X,\hat{X}} \hat{W}(\lambda) \cdot \hat{W}(\lambda)^{-1} = \left(\hat{W}(\lambda) \hat{X}(\lambda) \hat{W}(\lambda)^{-1} - W(\lambda) X(\lambda) W(\lambda)^{-1} \right)_{\geq 0}, \tag{4.7}$$

where $(\quad)_{\geq 0}$ *and* $(\quad)_{\leq -1}$ *are linear maps on the space of Laurent series of* λ *defined by*

$$(\quad)_{\geq 0} : \lambda^n \mapsto \theta(n \geq 0) \lambda^n,$$
$$(\quad)_{\leq -1} : \lambda^n \mapsto \theta(n \leq -1) \lambda^n.$$

Further, these infinitesimal symmetries obey the commutation relations

$$\left[\delta_{X_1,\hat{X}_1}, \delta_{X_2,\hat{X}_2}\right] = \delta_{[X_1,X_2],[\hat{X}_1,\hat{X}_2]}. \tag{4.8}$$

The associated infinitesimal transformations of $J = V_0$ and $K = -W_{-1}$ can be readily derived from the above result.

COROLLARY. *J and K transform under the above infinitesimal symmetry as:*

$$\begin{aligned}
\delta_{X,\hat{X}} J \cdot J^{-1} &= \operatorname*{res}_{\lambda=\infty} \lambda^{-1} W(\lambda) X(\lambda) W(\lambda)^{-1} + \operatorname*{res}_{\lambda=0} \lambda^{-1} \hat{W}(\lambda) \hat{X}(\lambda) \hat{W}(\lambda)^{-1}, \\
\delta_{X,\hat{X}} K &= \operatorname*{res}_{\lambda=\infty} W(\lambda) X(\lambda) W(\lambda)^{-1} + \operatorname*{res}_{\lambda=0} \hat{W}(\lambda) \hat{X}(\lambda) \hat{W}(\lambda)^{-1},
\end{aligned} \tag{4.9}$$

where the residues are normalized as:

$$\operatorname*{res}_{\lambda=\infty} \lambda^n = -\delta_{n,-1}, \quad \operatorname*{res}_{\lambda=0} \lambda^n = \delta_{n,-1}.$$

Thus, in particular, the contribution of $\mathbf{X}$ and $\hat{\mathbf{X}}$ are additively separated; $\mathbf{X}$ and $\hat{\mathbf{X}}$ respectiely belong to a loop algebra, i.e., a tensor product of a Lie algebra and an abelian Lie algebra of Laurent series. (4.8) shows that the linear map $(\mathbf{X}, \hat{\mathbf{X}}) \mapsto \delta_{X,\hat{X}}$ gives a Lie algebra homomorphism from a direct sum of two loop algebras. The fact that these loop algebras also contain $2N$ extra variables, $u^1, \ldots, u^{2N}$, reflects the higher dimensional nature of the generalized (as well as ordinary) self-dual Yang-Mills equations.

5. Symmetries of generalized self-dual Einstein equation

All results on the generalized self-dual Einstein equation can be stated parallel to the case of the generalized self-dual Yang-Mills equation. We present bellow only the results and omit detailed description. A main difference is that the Lie algebra struture based upon commutators $[\ ,\ ]$ are all replaced by that of Poisson brackets. The section is due to [Ta3].

5.1 Riemann-Hilbert factorization

The case of the generalized self-dual Einstein equation requires a more involved factorization, i.e., a factorization of maps (diffeomorphisms). Actually, there are several different, but equivalent ways depending on the existence of different local pictures as presened by Plebanski. More precisely, the setting depends on which Poisson bracket and corresponding canoical variables one selects. For example, for the treatment of the first heavenly equation, the Poisson bracket $\{\ ,\ \}_{(\hat{p})}$ in the canonical variables $\hat{p}$ is of course the most convenient.

Let $z = (z^1, \ldots, z^{2N})$ be such a set of canonical variables with the Poisson bracket $\{\ ,\ \}_{(z)}$. $u^i(\lambda)$ and $\hat{u}^i(\lambda)$ are then reorganized to maps

$$\begin{aligned}
u(\lambda) &: \ x \mapsto u(\lambda, x, p), \\
\hat{u}(\lambda) &: \ x \mapsto \hat{u}(\lambda, x, p)
\end{aligned} \tag{5.1}$$

from the z-space into the u-space and $\hat{u}$-space respectively. A one-parameter family of new solutions $u^i(\epsilon,\lambda)$ and $\hat{u}^i(\epsilon,\lambda)$ are then defined by the relation

$$u(\epsilon,\lambda)^{-1}\circ\exp\big(-\epsilon H_{F(\lambda)}\big)\circ u(\lambda)=\hat{u}(\epsilon,\lambda)^{-1}\circ\exp\big(-\epsilon H_{\hat{F}(\lambda)}\big)\circ\hat{u}(\lambda), \tag{5.2}$$

which corresponds to (4.5) in the generalized self-dual Yang-Mills equation. Here $H_{F(\lambda)}$ and $H_{\hat{F}(\lambda)}$ are Hamiltonian vector fields

$$H_{F(\lambda)}=\epsilon^{ij}\,\frac{\partial F(\lambda)}{\partial u^i}\frac{\partial}{\partial u^j},\quad H_{\hat{F}(\lambda)}=\epsilon^{ij}\,\frac{\partial \hat{F}(\lambda)}{\partial \hat{u}^i}\frac{\partial}{\partial \hat{u}^j}, \tag{5.3}$$

and the generating functions $F(\lambda)=F(\lambda,u)$ and $\hat{F}(\lambda)=\hat{F}(\lambda,\hat{u})$ are arbitrary functions of $2N+1$ variables with Laurent expansion

$$F(\lambda,u)=\sum_{n=-\infty}^{\infty}F_n(u)\lambda^n,\quad \hat{F}(\lambda,u)=\sum_{n=-\infty}^{\infty}\hat{F}_n(\hat{u})\lambda^n. \tag{5.4}$$

The exponentials in (5.2) stand for the (inverse) Hamiltonian flows generated by these Hamiltonian vector fields in u- and $\hat{u}$-spaces respectively.

For details (in particular, the derivation of (5.2)) and background ideas (closely related to twistor theory), see the paper [Ta3].

5.2. Explicit form of infinitesimal symmetries

The infinitesimal transformations

$$\delta_{F,\hat{F}}u^i(\lambda)=\left.\frac{\partial u^i(\epsilon,\lambda)}{\partial\epsilon}\right|_{\epsilon=0},\quad \delta_{F,\hat{F}}\hat{u}^i(\lambda)=\left.\frac{\partial \hat{u}^i(\epsilon,\lambda)}{\partial\epsilon}\right|_{\epsilon=0} \tag{5.5}$$

have several different expressions depending on the choice of the canonical variables z that one selects. All of theme are of course equivalent. We present below two expressions with respect to $\{\ ,\ \}_{(\hat{p})}$ and $\{\ ,\ \}_{(x)}$.

THEOREM. *The infinitesimal symmetries are written, with respect to the Poisson bracket in $\hat{p}$, as*

$$\begin{aligned}\delta_{F,\hat{F}}u^i(\lambda)&=\Big\{\Big[F(\lambda,u(\lambda))-\hat{F}(\lambda,\hat{u}(\lambda))\Big]_{\le 0},u^i(\lambda)\Big\}_{(\hat{p})},\\ \delta_{F,\hat{F}}\hat{u}^i(\lambda)&=\Big\{\Big[\hat{F}(\lambda,\hat{u}(\lambda))-F(\lambda,u(\lambda))\Big]_{\ge 1},\hat{u}^i(\lambda)\Big\}_{(\hat{p})},\end{aligned} \tag{5.6}$$

and with respect to the Poisson bracket in x, as

$$\begin{aligned}\delta_{F,\hat{F}}u^i(\lambda)&=\Big\{\Big[F(\lambda,u(\lambda))-\hat{F}(\lambda,\hat{u}(\lambda))\Big]_{\le -1},u^i(\lambda)\Big\}_{(x)},\\ \delta_{F,\hat{F}}\hat{u}^i(\lambda)&=\Big\{\Big[\hat{F}(\lambda,\hat{u}(\lambda))-F(\lambda,u(\lambda))\Big]_{\ge 0},\hat{u}^i(\lambda)\Big\}_{(x)},\end{aligned} \tag{5.7}$$

Further, the infinitesimal symmetries obey the commutation relation

$$\Big[\delta_{F_1,\hat{F}_1},\delta_{F_2,\hat{F}_2}\Big]=\delta_{\{F_1,F_2\}_{(u)},\{\hat{F}_1,\hat{F}_2\}_{(\hat{u})}}. \tag{5.8}$$

In the case of the generalized self-dual Yang-Mills equations, the infinitesimal symmetries of $W(\lambda)$ and $\hat{W}(\lambda)$ automatically induce symmetries of J and K. This is not the case for Ω and Θ because they are introduced by differential equations, (3.7), allowing an integration constant to be undetermined. Remarkably, as the following result shows, one can extend the above infinitesimal symmetries to Ω and Θ retaining basic commutation relation (5.8), which, implicitly, gives a way to fix a possible change of this hidden constant under symmetry transformations.

THEOREM. *The above infinitesimal symmetries can be extended to symmetries of (2.6) and (2.7) by*

$$\begin{aligned} \delta_{F,\hat{F}}\Theta &= \operatorname*{res}_{\lambda=\infty} F(\lambda, u(\lambda)) + \operatorname*{res}_{\lambda=0} \hat{F}(\lambda, \hat{u}(\lambda)), \\ \delta_{F,\hat{F}}\Omega &= -\operatorname*{res}_{\lambda=\infty} \lambda^{-2} F(\lambda, u(\lambda)) - \operatorname*{res}_{\lambda=0} \lambda^{-2} \hat{F}(\lambda, \hat{u}(\lambda)). \end{aligned} \tag{5.9}$$

The extended symmetries obey the same commutation relation as in (5.8).

These results show several important consequences: • First, we again encounter a loop algebra structure with extra variables. Of course, one can naturally expect this from the work of Boyer and Plebanski [Bo-Pl]. In fact, we find that the corresponding loop algebra is a direct sum of two parts coming from two generators, $F(\lambda, u)$ and $\hat{F}(\lambda, u)$, of infinitesimal symmetries. All of these observations are derived along with a very explicit expression of infinitesimal symmetries. • Second, the basic Lie algebraic structure is that of Poisson brackets. Except for that, everything is entirely parallel to the case of the generalized self-dual Yang-Mills equation. The similarity between the two cases is remarkable, and sometimes useful (see, for example, the "perturbative method" mentioned in Section 6). • Third, it is shown that the symmetries are consisitently extended to the key functions Ω and Θ, which fixes some integration constants hidden in their definition; this is by no means obvious from the origin of symmetries. That the extended symmetries obey the same commutation relation as those on the level of $u^i(\lambda)$ and $\hat{u}^i(\lambda)$, is also somewhat surprising, because the tau function of soliton equations, which is more or less very similar to the key functions [Ta1,3], exhibits a different behavior. For the case of soliton equations, too, there are counterparts of $u^i(\lambda)$ and $\hat{u}^i(\lambda)$ and symmetries are constructed first at the level of these auxiliary variables. These symmetries can be then extended to the tau function, but the commutation relation takes a different form, i.e., one has to take a "central extension" [Sa-Sa] [Se-Wi] [Da-Ji-Ka-Mi] (see also review [Ta1] and references cited therein). A reason for the absense of such a phenomenon for Ω and Θ is presented in the last part of [Ta3] along with a geometric interpretation of these potentials in the context of "contact geometry."

6. Concluding remarks

6.1. "Perturbative method"

This is a method first developed by Leznov and Mukhtarov [Le-Mu] (see also the last part of the review by Leznov and Saveliev [Le-Sa]) for the self-dual Yang-Mills equation and later extended by the present author [Ta4] to the (generalized) self-dual Einstein equation. A key idea is to derive differential equations *in the parameter* ε for such quantities as

$$X(\epsilon,\lambda) = W(\epsilon)X(\lambda)W(\epsilon)^{-1},\ldots, \quad F(\epsilon,\lambda) = F(\lambda,u(\lambda)),\ldots,$$

and solve those equations by expanding in powers of ϵ. From our results on infinitesimal symmetries, one can readily find such differential equations. Analogy between the Yang-Mills case and the Einstein case now becomes a very useful guiding principle.

In the same spirit, one can solve the Riemann-Hilbert factorization, (4.5) and (5.2), in the same manner, expanding both hand sides in powers of ϵ. It is not hard to see that the Taylor coefficients of $W(\epsilon,\lambda)$ etc. are then recursively determined. This, in particular, gives a new proof of the existence of solution of the Riemann-Hilbert factorization for the generalized self-duality equations, which is in a sense constructive.

6.2. Membrane physics

"Mambranes" are surface-like extended objects just as "strings" are one dimensional extended objects. Some physicists [Bi-Fl-Sa] [Fl-Le] [Gr-Tz] [Za] pointed out that the self-dual Einstein equation is related to a model of membranes at least at "classical level". These models allow area-preserving (i.e., two dimensional canonical) maps as symmetries. At "quantum level" these symmetries may have different commutations relations (such as central extensions) due to "anomalies". Central extensions of nondegenerate Poisson algebras are studied in detail by physicists [Ar-Sa] [BA-Po-Se] [Fl-Il] [Ho1].

6.3. Extended conformal symmetries and topological field theory

The Lie algebra of area-preserving maps was, as mentioned, discussed first from the point of view of membrane physics, but later it was pointed out that it can be identified with a special limit of $SU(N)$ as $N \to \infty$ [Ho2]. This limit, simply called $SU(\infty)$, is given another characterization in the context of "extended conformal symmetries," in particular, "w_∞ algebras" [Ba1,2]. The self-dual Einstein equations (and its possible quantization) are recently discussed from that point of view by physicists [Oo-Va] [Pa1,2] [Ya-Ch].

In a recent paper [Wi], Witten pointed out a relation of these subjects with topological field theory.

6.4. $SU(\infty)$ Toda fields

Bakas [Ba2] and Park [Pa2] further pointed out a relation of the w_∞ algebras with the so called $SU(\infty)$ Toda fields. This is a three dimensional nonlinear field theory described by the equation of motion

$$\frac{\partial^2\phi}{\partial z\partial\bar{z}} + \frac{\partial^2}{\partial s^2}(e^{\phi}) = 0, \quad \phi = \phi(z,\bar{z},s).$$

It will be interesting to study this equation from our point of view.

Actually, the same equation had beed derived, independently, as an equation describing self-dual Einstein metrics with a rotational symmetry [Bo-Fi] [Ge-Da]. Another interpretation of those metrics (and the above equation) is the Einstein-Weyl theory in three dimensional Riemannian geometry; this is also pursued in detail by twistor people [Hi] [Wa3] [Jo-To] [LeBr]. Leznov and Vershik [Le-Ve] proposed an entirely different approach to the $SU(\infty)$ Toda fields.

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