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EQUATIONS AUX DERIVEES PARTIELLES

AN IFF SOLVABILITY CONDITION FOR THE OBLIQUE DERIVATIVE PROBLEM

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An iff solvability condition for the oblique derivative problem

by Nicolas Lerner

Foreword

The purpose of this note is to prove the following result.

Theorem

For the oblique derivative problem, condition (ψ) is equivalent to solvability.

As a matter of fact, the proof is a modification of the argument we used to handle the two dimensional case of the Nirenberg-Treves conjecture for pseudo-differential operators in [6]. The basic remark is that the oblique derivative problem is equivalent to a pseudo-differential equation of a very particular type :

$$\partial_t + \alpha(t,x)\ \Omega(t, x, D_x)$$

where Ω is a non-negative pseudo-differential operator and α a smooth function. The main feature of this symbol is that α is a function on a lagrangean manifold, so that the spectral behaviour of its quantization is scalar. In particular, a monotone function is monotone -matrix for these kind of operators. This natural factorization yields an energy estimate , following the lines of [6]. The main reference on local solvability is chapter 26 in Hörmander's four volumes book [5] where the reader will find most of the background of this paper.

1. Condition (ψ)

Let P be a pseudo-differential operator of principal type (i.e. the hamiltonian field H_p of the principal symbol p is independent of the Liouville vector field). The symbol p is said to satisfy condition (ψ) if the imaginary part Imp does not change sign from - to + along the oriented bicharacteristic of $\mathbb{R}$ep (see definition 26.4.6 in [5]) . This condition was proven invariant by multiplication by an elliptic factor in [11] (see also lemma 26.4.10 in [5]). We 'll say that p satisfies the ($\bar{\psi}$) condition if $\bar{p}$ satisfies condition (ψ).

The importance of this geometrical condition was stressed by Nirenberg and Treves [11] who conjectured condition (ψ) is equivalent to local solvability and proved it in a number of cases. The first non solvable equation was found by Hans Lewy in 1957 and the simpler models $D_t + i\, t^{2k+1} D_x$ by Mizohata [9] later on.

The necessity of condition (ψ) for local solvability is established for general pseudo-differential equations after the works of Moyer [10] in two dimensions and Hörmander in the general case(Corollary 26.4.8 in [5]).

The sufficiency is proved for differential equations (see Nirenberg-Treves [11] with an analyticity assumption, Beals-Fefferman [1] in the general case for local solutions, Hörmander ([4] and theorem 26.11.3 in [5]) for a semi-global existence result). Note that for differential operators, condition (ψ) is equivalent to condition (P) which rules out any change of sign of Imp along the bicharacteristics of $\mathbb{R}$ep (see definition 26.5.1 in[5]). Moreover, the sufficiency in two dimensions is proved in [6] . Hörmander's work on subellipticity (theorem 27.1.11 in chapter 27 of [5])showed that if a symbol p satisfies condition ($\bar{\psi}$) and a finite type assumption (27.1.8 in [5]) then the associated operator is hypoelliptic and thus the adjoint operator is solvable. In more than three dimensions, the sufficiency of condition (ψ) for solvability is an open problem.

Since local solvability of an operator P is equivalent to an a priori estimate estimate for the adjoint operator , we 'll try to stick with the following notations :

the solvability of $\partial_t + Q(t)$ is equivalent to an a priori estimate for $D_t + i\,Q(t)$,

where $Q(t) = Q(t,x,D_x)$ is a first-order pseudo-differential operator with real principal symbol q such that

$$(1.1) \qquad q(t, x,\xi) > 0 \quad \text{and} \quad s \geq t \quad \text{imply} \quad q(s,x,\xi) \geq 0.$$

Property (1.1) is the expression of condition (ψ) for the operator $\partial_t + Q(t)$. It was shown by Nirenberg and Treves through localization and homogeneous canonical transformation that any principal type operator satisfying condition (ψ) could be reduced to $\partial_t + Q(t)$, with q satisfying (1.1). We refer the reader to theorem 21.3.6 in [5] and to the Egorov theorem ([2] , theorem 25.3.5 in [5]).

Let q be a smooth function satisfying (1.1) and define $\theta(X)$ for $X = (x,\xi)$ by

$$(1.2) \qquad \theta(X) = \inf \{ t , t \in (-1, +1) , q (t, X) > 0 \} \text{with } \theta(X) = +1 \text{ if this set is empty .}$$

Note that θ is a bounded measurable function such that, for $t \in (-1, +1)$,

(1.3) $\quad q(t,X) \le 0 \quad \text{if} \quad t \le \theta(X) \qquad \text{and} \qquad q(t,X) \ge 0 \quad \text{if} \quad t \ge \theta(X)$.

In the remaining part of this section, we wish to give a few very simple examples which may help the reader to understand the geometric complexity allowed by condition (ψ). First of all, the function θ need not to be continuous as shown by

$$q = \xi_1\ \omega(t,x,\xi) \qquad \text{with } \omega \ge 0 \qquad \text{which gives} \qquad \theta = -\operatorname{sign}(\xi_1).$$

Moreover, the boundary of the set where q is positive is not a manifold in general, and it is even not possible to separate the open sets $\{ q > 0 \}$ and $\{ q < 0 \}$ by a manifold :

an example is provided by $\quad q = (t^3 + x^2) | \xi |$.

However we 'll see that the latter example, though unpleasant geometrically, is symplectically simple since the important parameters are not conjugate, i.e. q is the product of a non-negative symbol with a symbol on a lagrangean manifold. We can also check on the former example that, even when a manifold does separate the sets $\{ \pm q > 0\}$, the hamiltonian field of the real part could be tangent to this manifold.

On the other hand, the operator $D_t + i\,Q(t)$, with q given by

$$q = \xi_1 + t\ x_1^2\ |\xi_2| \qquad (\text{near } t = 0 ,\ x_1 = 0 ,\ \tau = 0 ,\ \xi_1 = 0 ,\ \xi_2 = 1)$$

is subelliptic but somehow generates the worst simple example with

$q = \alpha(t) (\xi_1 + t\ x_1^2\ |\xi_2|)$, where α is a C^∞ non-negative function.

We should also keep in mind that all the previous functions q could be multiplied by a non-negative symbol and still satisfy condition (ψ).

Let's now remark that bounds on the first derivative of the function θ imply local solvability. Note first that , since the symbol q is assumed to be positively homogeneous, the function θ is homogeneous of degree zero with respect to the ξ-variables.

Remark 1.1 : if the function θ is locally lipschitzian , then the operator $\partial_t + Q(t)$ is locally solvable(it is enough to assume $|\theta_x \cdot q_x||\xi|^{-1} + |\theta_\xi \cdot q_\xi||\xi|$ locally bounded). This follows easily from the computation of

$$\mathrm{Re} < D_t u + i\, Q(t)\, u \,,\, i\, T^{-1} (t - \theta(x,\xi))^{Wick}\, u >_{L^2(\mathbb{R}^n)} \quad ,$$

for u in $C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}\, u \subset \{ |t| < \delta T \}$. Here, a^{Wick} stands for the ordinary quantization of a regularization of a (see the proof of theorem 18.1.14 in [5] with ϕ gaussian ; the Wick quantization of a is given by the formula (18.1.18)).

2 . The oblique derivative problem

Let Ω be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary Γ. Assume Γ has finitely many connected components. Let $X \in C^\infty(\Gamma, T_\Gamma \mathbb{R}^n)$ be a smooth non-vanishing real vector field.We write $X = L + \alpha \frac{\partial}{\partial v}$, where L is a smooth real vector field tangent to Γ, $\frac{\partial}{\partial v}$ is the interior unit normal and α is a smooth real function on Γ. Let P be a second order elliptic operator with C^∞ coefficients and real principal symbol uniformly elliptic in Ω. We are interested in the following boundary value problem :

$$(2.1) \qquad \begin{cases} Pu = 0 & \text{in } \Omega \\ Xu = f & \text{on } \Gamma \end{cases} \quad .$$

Our main assumptions will be

(2.2) $\alpha(m) < 0$ imply $\alpha(\exp tL m) \le 0$ for $t > 0$ (condition (ψ)),

(2.3) The set $\{m \in \Gamma, \alpha(m) = 0\}$ does not contain a maximal integral curve of L.

We have then the following results :

Claim 2.1

Under the previous assumptions , the problem (2.1) is solvable. There exists a finite dimensional space $\mathcal{N}$ of smooth densities on Γ such that, for every $s \le +\infty$, if f is orthogonal to $\mathcal{N}$ and belongs to $H^s(\Gamma)$, then one can find $w \in H^{s-0}(\Gamma)$ so that $u = Gw$ satisfies (2.1) ,where G is the Dirichlet kernel for P in Ω .

Theorem 2.2

Under the previous assumptions (without (2.3)), condition (ψ) (2.2) is equivalent to local solvability for the pseudo-differential operator XG on the boundary Γ.

The proof of the theorem 2.2 is given in section 3. We won't give here a complete proof for claim 2.1 ; we refer the reader to chapter 26 in [5] and in particular to proposition 26.6.1 and theorem 26.6.2 which should be modified accordingly to our situation. In any case, theorem 2.2 is an important step in the proof of claim 2.1 and gives a modified version of the Nirenberg-Treves estimate (see section 26.8 in[5]).

Let's begin with a few remarks. The boundary value problem (2.1) reduces to a pseudo-differential equation on the boundary , $XGw = f$. The pseudo-differential operator $P = XG$ is of principal type since X is non-vanishing ; P is elliptic when α is different from zero, so this problem is generically non-elliptic except in two dimensions . In this framework, condition (ψ) for P means that if X is outgoing at one point of Γ, it should be outgoing later on the integral curve of L. That condition is necessary for local solvability of P ([10] , [5]). The global assumption (2.3) means that the integral curves of L start and end in an elliptic region .

Let's now recall some classical works on the oblique derivative problem : after the works of Egorov and Kondrat'ev [3] , Melin and Sjöstrand [8] gave a construction of a right parametrix for the oblique derivative problem, with transversality assumptions ; with the present notations, they assumed that the regions $\{ \pm\alpha > 0 \}$ are separated by a smooth manifold to which L is tranverse (and of course points into $\{ \alpha \leq 0 \}$). This is a very strong hypothesis, but they were able to give an explicit integral form for the solutions, using the powerful tool of Fourier integral operators with complex phase, introduced in [7] . They performed the construction of exp itP and considered the integral

$$i\int_0^{+\infty} \exp\text{-}itP \, dt \qquad \text{on} \quad \operatorname{Im} p \leq 0 \ .$$

Let's note here that an explicit integral form could certainly be obtained for the solutions of pseudo-differential equations satisfying the (P) condition, using the non-homogeneous reductions of Beals and Fefferman [1] and the fact that the following problem is well-posed:

$$\partial_t v + a(t)Q_0 v = f$$
$$E_0^+ v(-\infty) = \omega_+ \quad , \quad E_0^- v(+\infty) = \omega_-$$

where Q_0 is a self-adjoint operator on a Hilbert space H, $f \in L^1(\mathbb{R}, H)$, $E_0(^{+,-})$ the projections on the half-axes for Q_0 , a a non-negative function ; the unique solution v belongs to $L^\infty(\mathbb{R}, H)$ and its norm is controlled by the L^1 norm of f and the norms of $\omega_{+,-}$.

3. Proof of theorem 2.2

Our main reference in this section will be our paper [6] . Solvability for the pseudo-differential operator XG can be reduced locally near a non elliptic point to proving an energy estimate for

(3.1) $D_t + i\,\beta(t,x)\,\Omega(t, x, D_x)$

where $\Omega\,(t,x,\xi)$ is a non-negative classical symbol of order 1 and β a smooth function such that ($\beta = -\alpha$, where α is given in section 2)

(3.2) $\beta(t,x) > 0$ and $s > t$ imply $\beta(s,x) \geq 0$.

We compute then, for $u \in C_0^\infty$, θ standing for the function defined by (1.2) with $q = \beta$, H for the Heaviside function, $<\, ,\, >$ for the L^2 product ,

(3.3) $2\mathrm{Re} < D_t u + i\,\beta(t,x)\,\Omega(t, x, D_x)u\,,\, i\,H(t)\,H(t - \theta(x))u \;-\; i\,H(-t)\,H(\theta(x)-t)u >$.

Using the lemmas 2.3.1 - 2.3.4 in [6] , we get that the expression (3.3) is bounded from below by

(3.4) $-\,C_1\,\| u \|^2 \;+ |\, u(0)\,|^2$,

where $\| \,.\, \|$ is the L^2 norm in all the variables, $|\,.\,|$ the L^2 norm in the x variables, and C_1 a constant depending on semi-norms of the symbol $\beta\,\Omega$. Since this can be done for any point t (and not only 0), we get , if the range of t is small enough ,

(3.5) $C_2 \int |\, D_t u + i\,\beta(t,x)\,\Omega(t, x, D_x)u\,|\, dt \;\geq\; \sup |\, u(t)\,|$,

which gives local solvability for the adjoint operator (Note the estimate we get here is slightly better than (2.20) in [6] , since it provides an L^∞, L^1 inequality) and proves theorem 2.2.

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