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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

# A DIRECTIONAL COMPACTIFICATION OF THE COMPLEX FERMI SURFACE AND ISOSPECTRALITY

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## 1. Introduction and Theorems :

The content of this report is joint work with H. Knörrer and E. Trubowitz (ETH-Zürich, Switzerland), [BKT].

We consider a lattice $\Gamma \subset \mathbf{R}^3$ of maximal rank and $L^2_{\mathbf{R}}(\mathbf{R}^3/\Gamma)$ the Hilbert-space of square-integrable real-valued functions on the torus $\mathbf{R}^3/\Gamma$. Let $q$ be in $L^2_{\mathbf{R}}(\mathbf{R}^3/\Gamma)$.

For each $k \in \mathbf{R}^3$ the self-adjoint boundary value problem

$$(-\Delta + q(x))\psi(x) = \lambda\psi(x)$$
$$\psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x) \quad \text{for all} \quad \gamma \in \Gamma$$

has discrete spectrum, customarily denoted by

$$E_1(k) \le E_2(k) \le E_3(k) \le \cdots$$

The eigenvalue $E_n(k), n \ge 1$, defines a function of $k$ called the $n$-th band function. It is continuous and periodic with respect to the lattice

$$\Gamma^\sharp := \{b \in \mathbf{R}^3 / \langle \gamma, b\rangle \in 2\pi\mathbf{Z} \quad \text{for all} \quad \gamma \in \Gamma\} \ .$$

dual to $\Gamma$.

The physical Fermi surface for energy $\lambda$ is the set

$$F_{\text{phys},\lambda}(q) := \{k \in \mathbf{R}^3 / E_n(k) = \lambda \quad \text{for some} \quad n \ge 1\} \ .$$

For example, if $q(x) =$ constant, then $F_{\text{phys},\lambda}(q)$ is the union of the spheres

$$\{k \in \mathbf{R}^3/(k_1+b_1)^2 + (k_2+b_2)^2 + (k_z+b_z)^2 = \lambda - \text{constant}\}$$

with $b = (b_1, b_2, b_3) \in \Gamma^\sharp$.

**Theorem 1.**— *If $q$ is in $L^2_{\mathbf{R}}(\mathbf{R}^3/\Gamma)$ and if for a single $\lambda$ in $\mathbf{R}$ one of the components of $F_{\text{phys},\lambda}(q)$ is a sphere (not necessarily centered at a point of the dual lattice), then $q$ is constant.*

Actually the same conclusion holds if $F_{\text{phys},\lambda}(q)$ contains an algebraic component $X$, which fulfills certain assumptions, (see section 3). These assumptions are fulfilled if $X$ is an ellipsoid.

To prove Theorem 1 we complexify the Fermi surface. The (lifted) complex Fermi surface is defined by $F_\lambda(q) := \{k \in \mathbf{C}^3/$ there exists a non trivial solution $\psi$ in $H^2_{\text{loc}}(\mathbf{R}^3)$ of $(-\Delta + q(x))\psi(x) = \lambda\psi(x)$ satisfying $\psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x)$ for all $\gamma \in \Gamma\}$.

Clearly, the dual lattice $\Gamma^\sharp$ acts on $F_\lambda(q)$ by $k \mapsto k + b$, $b \in \Gamma^\sharp$. Furthermore we have $F_\lambda(q) \cap \mathbf{R}^3 = F_{\text{phys},\lambda}(q)$.

It is easy to show, using regularized determinants (see [KT]), that $F_\lambda(q)$ is a complex analytic hypersurface in $\mathbf{C}^3$. The main purpose is to construct a directional compactification of $F_\lambda(q)$ in the sense of [KT]. The above theorem follows from the analysis of the points added at "infinity".

To compactify $F_\lambda(q)$ we first embed $\mathbf{C}^3$ in a quadric $Q$ lying in $\mathbf{P}^4$. For each affine line $g = \{c + tb/t \in \mathbf{R}\}$ in $\mathbf{R}^3$, where $b, c \in \Gamma^\sharp$ and $b$ is primitive, we blow-up two distinguished points of $\mathbf{P}^4$ that lie on the quadric $Q$, to get, by using inverse limits, a space $\mathcal{M}$. Denote by $E_1(g)$ and $E_2(g)$ the corresponding exceptional divisors.

**Theorem 2.**— *The directional closure of $F_\lambda(q)$ in the space $\mathcal{M}$ intersects $E_1(g)$ and $E_2(g)$ along curves both of which are isomorphic to the one-dimensional Bloch-variety*

$$\mathcal{B}(q_g) \quad \text{where} \quad q_g(x) = \sum_{n=-\infty}^{\infty} \hat{q}(nb)e^{i\langle nb,x\rangle}, x \in g \ .$$

Here $\hat{q}(b)$ is the Fourier-coefficient $\int_{\mathbf{R}^3/\Gamma} q(x)e^{-i\langle b,x\rangle}dx(b \in \Gamma^\sharp$, without loss of generality we assume that $\mathbf{R}^3/\Gamma$ has volume one). Recall that in [KT] the complex one dimensional Bloch-variety for $p(x) \in L^2(\mathbf{R}/|b|\mathbf{Z})$ is

$\mathcal{B}(p) = \{(k,\lambda) \in \mathbf{C} \times \mathbf{C}/$ there is a non-trivial function $\psi$ in $H^2_{\text{loc}}(\mathbf{R})$ satisfying $-\psi''(x) + p(x)\psi(x) = \lambda\psi(x)$ and $\psi(x+|b|n) = e^{ik|b|n}\psi(x)$ for all $n \in \mathbf{Z}\}$.

## 2. Sketch of the proof of Theorem 2

First we construct a compactification of $\mathbf{C}^3$, which serves as the ambient space for the directional compactification of $F_\lambda(q)$. This compactification will be independent of $q$. It's construction is motivaded by considering the free Fermi-surface $F_\lambda(0)$. $F_\lambda(0)$ is the union of the quadrics

$$\{k \in \mathbf{C}^3/(k_1+b_1)^2 + (k_2+b_2)^2 + (k_3+b_3)^2 = \lambda\} \quad , b = (b_1,b_2,b_3) \in \Gamma^\sharp.$$

If we compactify $\mathbf{C}^3$ in the naive way to $\mathbf{P}^3$ or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ we would have to perform many blow-up's before the components of $F_\lambda(0)$ are in general position at infinity. Instead we embed $\mathbf{C}^3$ in the complex projective 3-dimensional nonsingular quadric

$$Q := \{(k_1,k_2,k_3,y,z) \in \mathbf{P}^4/yz = k_1^2 + k_2^2 + k_3^2\}$$

by mapping $(k_1,k_2,k_3)$ to $(k_1,k_2,k_3,k_1^2+k_2^2+k_3^2,1)$.

The image of the embedding is the complement of

$$Q_\infty := \{(k_1,k_2,k_3,y,z) \in Q/z = 0\}.$$

The closures of the components of $F_\lambda(0)$ in $Q$ are the intersections of $Q$ with the hyperplanes $H_b$ in $\mathbf{P}^4$ given by

$$y + 2\langle k,b\rangle + (b^2-\lambda)z = 0 \quad , b \in \Gamma^\sharp.$$

If $b \neq b'$, then $H_b \cap H_{b'}$ is a plane in $\mathbf{P}^4$. It intersects $Q_\infty$ in the set $D_{b,b'}$, consisting of two points, given by the equations

$$z = 0, k_1^2+k_2^2+k_3^2 = 0, \langle k, b-b'\rangle = 0 \quad , y + 2\langle k,b\rangle = 0 \ .$$

One checks that $D_{b,b'}$ and $D_{b'',b'''}$ are disjoint if $b,b',b'',b'''$ do not lie on a line and that $D_{b,b'} = D_{b'',b'''}$ if these four points of $\Gamma^\sharp$ are on a line. Thus we can denote the points $D_{b,b'}$ by $D(g)$, where $g$ is the affine line through $b$ and $b'$. The group $\Gamma^\sharp$ acts by translation on $\mathbf{C}^3$. This action extends to $Q$ and it maps $D(g)$ to $D(c+g)$ for $c \in \Gamma^\sharp$.

If $b$ and $b' \in \Gamma^\sharp$ are different points on the line $g = c_1 + \mathbf{R}c_2 \quad (c_i \in \Gamma^\sharp)$ then $Q \cap H_b$ and $Q \cap H_{b'}$ have different tangent planes in the points of $D(g)$. Therefore we can separate $Q \cap H_b$ and $Q \cap H_{b'}$ by blowing-up the points of $D(g)$. Precisely, for each line $g = c_1 + \mathbf{R}c_2 \quad (c_i \in \Gamma^\sharp)$, let $\mathcal{M}(g)$ be the space obtained from $\mathbf{P}^4$ by blowing-up the points of $D(g)$, $Q(g)$ the strict transform of $Q$ in $\mathcal{M}(g)$ and $E_1(g), E_2(g)$ the two exceptional divisors over the two points of $D(g)$. As compactification $\mathcal{M}$ of $\mathbf{C}^3$ we take the inverse limit of all the spaces $\mathcal{M}(G)$, where $G$ is a finite set of affine lines and $\mathcal{M}(G)$ is obtained from $\mathbf{P}^4$ by blowing-up the points of $\cup_{g \in G} D(g)$, defined by the natural maps $\mathcal{M}(G_1) \to \mathcal{M}(G_2)$ for $G_2 \subset G_1$.

Using the action of $\Gamma^\sharp$ we consider $\mathcal{M}(g)$ where $g$ passes through the origin, and after rotating and scaling we further assume that $g = t(1,0,0)$.

Then

$$D(g) = \{(0, \pm i, 1, 0, 0) \in \mathbf{P}^4\}$$

Consider now the exceptional divisor $E_1 := E_1(g)$ lying above the point $(0, i, 1, 0, 0)$, the other division is treated similarly. Near this point we take coordinates $(\frac{k_1}{k_3}, \frac{k_2}{k_3} - i, \frac{y}{k_3}, \frac{z}{k_3})$. In $\mathcal{M}(g)$ we have coordinates $(\ell_1, \ell_2, y', z)$ such that

$$\frac{k_1}{k_3} = z\ell_1, \frac{k_2}{k_3} - i = z\ell_2, \frac{y}{k_3} = zy' \ , \ k_3 = \frac{1}{z}$$

For convenience we perform the change of variables

$$y' = -\mu + \ell_1^2 + \lambda$$

In these coordinates the blow-up map $\pi : \mathcal{M}(g) \to \mathbf{P}^4$ is

$$k_1 = \ell_1, k_2 = \ell_2 + \frac{i}{z}, y = -\mu + \ell_1^2 + \lambda \ , \ k_3 = \frac{1}{z} \ .$$

$Q(g)$ intersects $E_1$ in the plane $z = \ell_2 = 0$. The strict transform of the hyperplane $H_b, b \in \Gamma^\sharp$, does not meet $E_1$ if $b_2 \neq 0$ or $b_3 \neq 0$. Further, the strict transform of $H_{(b_1,0,0)}$ intersects $E_1$ in

$$(\ell_1 + b_1)^2 - \mu = 0$$

Remember that the strict transform of $Q \cap H_b$ is the closure of a component of the free Fermi-surface $F_\lambda(0)$, and that the one-dimensional Bloch-variety for potential zero is

$$\cup_{n \in \mathbf{Z}} \{(\ell, \mu) \in \mathbf{C} \times \mathbf{C} / (\ell + n)^2 - \mu = 0\} \ .$$

This shows that for $q \equiv 0$ the union of the closures of the components of $F_\lambda(0)$ meets $E_1 \cap Q(g)$ along a curve isomorphic to the one-dimensional Bloch-variety for potential zero. Observe however that the closure of $F_\lambda(0)$ in $Q(g)$ is bigger than the union of the closures of its components. This indicates that it is necessary for the general case to restrict the way one takes limits to $E_1$, i.e. the **directional closure** in Theorem 2 is made precise by introducing a subset $\Sigma(g)$ of $\mathbf{C}^4$ such that the closure of $F_\lambda(q) \cap \Sigma(g)$ in $Q(g)$ intersets $E_1(g)$ and $E_2(g)$ along a curve each isomorphic to the Bloch-variety $\mathcal{B}(q_g)$.

An equation for $F_\lambda(q)$ outside of the free Fermi-surface $F_\lambda(0)$ is given by (see [KT]), assuming without loss of generality $\hat{q}(0)=0$,

$$\det{}_2(-\Delta_k + q - \lambda \mathbf{1}) \circ (-\Delta_k - \lambda \mathbf{1})^{-1} = \det{}_2(\delta_{cb} + \frac{\hat{q}(c-b)}{(k+b)^2-\lambda}) = 0 \,.$$

This determinant can be computed by taking limits of finite principal minors. (It is not difficult to get an equation for $F_\lambda(q)$ on the whole $\mathbf{C}^3$, but to get the notations as small as possible we work with the above equation). In the coordinates $(\ell_1, \ell_2, \mu, z)$ of $\mathcal{M}(g)$ the entries of the matrix for $(-\Delta_k + q - \lambda) \circ (-\Delta_k - \lambda)^{-1}$ are

$$\delta_{cb} + \frac{\hat{q}(c-b)}{\frac{2}{z}(ib_2+b_3) + [(\ell_1+b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]}$$

Block the matrix in the form

$$\begin{array}{c} c \in \mathbf{Z}(1,0,0) \; \{ \\ c \notin \mathbf{Z}(1,0,0) \; \{ \end{array} \left( \begin{array}{c|c} \overbrace{A(\ell_1,\ell_2,\mu,z)}^{b \in \mathbf{Z}(1,0,0)} & \overbrace{B(\ell_1,\ell_2,\mu,z)}^{b \notin \mathbf{Z}(1,0,0)} \\ \hline C(\ell_1,\ell_2,\mu,z) & D(\ell_1,\ell_2,\mu,z) \end{array} \right) =: \mathcal{F}(\ell_1,\ell_2,\mu,z)$$

With this notation $A(\ell_1,\ell_2,\mu,z) = (\delta_{c_1 b_1} + \frac{\hat{q}(c_1-b_1,0,0)}{(\ell_1+b_1)^2-\mu})_{c_1,b_1 \in \mathbf{Z}}$. This is the matrix whose determinant describes the Bloch-variety of the averaged potential $q_g$ outside of $\mathcal{B}(0)$. Furthermore on $Q(g) \cap E_1 = \{z = \ell_2 = 0\}$ the matrix $B = 0$ and $D = \mathbf{1}$.

The square of the Hilbert-Schmidt norm of

$$\mathcal{F}(\ell_1,\ell_2,\mu,z) - \mathcal{F}(\ell_1,0,\mu,0)$$

is bounded by

$$\|q\|_2^2 \sum_{\substack{b \in \Gamma^\sharp \\ b \notin \mathbf{Z}(1,0,0)}} \frac{1}{|\frac{2}{z}(ib_2+b_3) + [(\ell_1+b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^2}$$

**Definition :**

$$\Sigma(g) :=$$

$$\{(\ell_1,\ell_2,\mu,z) \in \mathbf{C}^4 / \sum_{\substack{b \in \Gamma^\sharp \\ b \notin \mathbf{Z}(1,0,0)}} \frac{1}{|\frac{2}{z}(ib_2+b_3) + [(\ell_1+b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^2}$$

$$+ \sum_{\substack{b \in \Gamma^\sharp \\ b \notin \mathbf{Z}(1,0,0)}} \frac{|\ell_1+b_1|^2 + b_2^2}{|\frac{2}{z}(ib_2+b_3) + [(\ell_1+b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^4} < |z|^{1/5}\}$$

The restriction of $\det_2 \mathcal{F}$ to $\Sigma(g)$ is continous at $z = 0$ :

$$\|\mathcal{F}(\ell_1,\ell_2,\mu,z) - \mathcal{F}(\ell_1,0,\mu,0)\|^2_{\text{Hilbert-Schmidt}} = \mathcal{O}(\|q\|_2^2 |z|^{1/5})$$

Therefore we have :

$$\overline{F_\lambda(q) \cap \Sigma(g)} \cap (Q(g) \cap E_1) \subset \mathcal{B}(q_g). \tag{1}$$

To prove the converse we need information about the structure of $\Sigma(g)$ in the neighbourhood of any point of $Q(g) \cap E_1$ :

**Lemma 1.**— *For every point $p = (\ell_1^*, \ell_2^*, \mu^*, 0)$ of $E_1(g)$ and for all $A > 0$ there is a neighbourhood $\mathcal{U}$ of $p$ in $\mathcal{M}(g)$ and an open set $Z \subset \mathbf{C}$ having 0 as a cluster point such that*

$$T := \{(\ell_1, \ell_2, \mu, z) \in \mathcal{U}/z \in Z, |\ell_2 - \ell_2^*| \leq A|z|\} \subset \Sigma(g)$$

The proof of Lemma 1 is technical, very long and done by contradiction. One has to estimate the functions in the sums defining $\Sigma(g)$ outside of little discs centered at

$$z_b(\ell_1, \mu) := 2i(1 + \frac{(\ell_1 + b_1)^2 - \mu}{b_2^2 + b_3^2})^{-1}(-b_2 + ib_3)^{-1}$$

since

$$|\frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^2 =$$

$$= (b_2^2 + b_3^2)|\frac{2i}{z} - (1 + \frac{(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2}{b_2^2 + b_3^2})(-b_2 + ib_3)|^2 .$$

We do not know if $\Sigma(g)$ is path-connected, i.e. if $Z$ is.

Let us fix now a smooth point $p = (\ell_1^*, 0, \mu^*, 0)$ of $Q(g) \cap E_1 \cap \mathcal{B}(g_g)$. For simplicity we assume that $p$ doesn't lie on the free Bloch-variety $\mathcal{B}(0)$ in $Q(g) \cap E_1$. By Lemma 1 there is a neighbourhood $\mathcal{U}$ of $p$ in $\mathcal{M}(g)$ and an open subset $Z \subset \mathbf{C}$ having 0 as a cluster point such that $T \subset \Sigma(g)$.

It is easy to see (using the definition of $\Sigma(g)$ and the fact that $\det_2$ is continuous in Hilbert-Schmidt norm) that we have

**Lemma 2.**— *The restriction of the function*

$$f(\ell_1, \ell_2, \mu, z) := \det{}_2\mathcal{F}(\ell_1, \ell_2, \mu, \mathrm{z})$$

*to $\overline{T}$ has the following properties :*

*i)* $f(p) = 0$

*ii) There is a constant $C$, such that*

$$|f(\ell_1, \ell_2, \mu, z) - f(\ell_1, \ell_2, \mu, 0)| \leq C|z|^{1/5}$$

*for all $z \in Z$, $(\ell_1, \ell_2, \mu, z) \in \mathcal{U}$*

*iii) For any $z \in \bar{Z}$ the mapping $f(., z)$ is differentiable and $(\ell_1, \ell_2, \mu, z) \mapsto (\nabla_{(\ell_1, \ell_2, \mu)} f)(\ell_1, \ell_2, \mu, z)$ is continuous on $\overline{T}$.*

*iv) $\frac{\partial f}{\partial \ell_1}(p)$ and $\frac{\partial f}{\partial \mu}(p)$ are not both equal to zero.*

We apply this lemma as follows :

Since $Q(g)$ intersects $E_1$ transversally, we can choose $(\ell_1, \mu, z)$ as local coordinates on $Q(g) \cap \mathcal{U} =: V$ near $p$ (observe that there exists a $A > 0$ such that $|\ell_2| \leq A|z|$ for all points near $p$ in $Q(g)$). Assume $\frac{\partial f}{\partial \ell_1}(p) \neq 0$ (the other case is treated similarly using $\frac{\partial \ell_2(\mu,z)}{\partial \mu}(p) = 0$) and consider the continous mapping

$$F : V \subset \mathbf{R}^4 \times \overline{Z} \to \mathbf{R}^4$$

defined by

$$F(\ell_1, \mu, z) := (f(\ell_1, \ell_2(\mu, z), \mu, z), \mu - \mu^*) \; .$$

It is not difficult to apply the implicit function theorem to $F$, by imitating it's proof, to get a sequence $((\ell_1, \mu)_k, z_k)_{k \in \mathbf{N}}$ in $V \times Z$ with $z_k \neq 0$ converging to $((0,0),0)$ such that $F((\ell_1, \mu)_k, z_k) = 0$. Therefore $p$ lies in the closure of the zero-set of $f$ in ($Q(g)$-strict transform of $Q_\infty$) $\cap T$, hence in the closure of $F_\lambda(q) \cap \Sigma(g)$. From [Bo] one knows, that the equation defining the one-dimensional Bloch-variety $\mathcal{B}(q_g)$ is reduced. So the smooth points are dense in the zero-set of $f(\ell_1, 0, \mu, 0)$ and we get

$$\overline{F_\lambda(q) \cap \Sigma(g) \cap (Q(g) \cap E_1)} \supset \mathcal{B}(q_g) \tag{2}$$

(1) and (2) imply the Theorem 2.

## 3. Sketch of the proof of Theorem 1

First we claim :

*Assume that $q$ is a real potential and that $F_\lambda(q)$ contains an algebraic component $X$. If the closure $\overline{X}$ of $X$ in $Q$ contains of the curves $\{(k, Y, 0) \in Q_\infty / \langle k, c\rangle + y = 0\}$ with $c \in \Gamma^\sharp$, then $q$ is constant.*

**Proof :**

For $b \in \Gamma^\sharp - \{0\}$ let $g_b$ be the line $\{c + tb/t \in \mathbf{R}\}$. Then $\overline{X}$ contains all the sets $D(g_b), b \in \Gamma^\sharp$. By Lemma 1 the closure of $X \cap \Sigma(g_b)$ in $Q(g_b)$ meets $E_1(g_b)$ and $E_2(g_b)$ along a (non-empty) algebraic curve, namely the intersection of the strict transform of $\overline{X}$ with $E_1(g_b)$ resp. $E_2(g_b)$. Hence by Theorem 2 the Bloch-varieties of all the averaged potentials $q_b, b \in \Gamma^\sharp$ each contain an algebraic component. As each $q_b$ is real, Borg's Theorem [Bo] implies that $q_b$ is constant. Therefore $q$ is constant. □

The assumption of the claim is fulfilled if $F_\lambda(q)$ contains a sphere around a point of $\Gamma^\sharp$. Assume that $F_\lambda(q)/\Gamma^\sharp$ is irreducible. Then, if $X$ where any algebraic component of $F_\lambda(q)$, by Theorem 2 there would be an affine line $g$, such that $\overline{X \cap \Sigma(g)}$ intersects $E_i(g) (i = 1,2)$ along a curve, and one would deduce the fact that $q$ is constant as above.

Theorem 1' shows, under further assumptions on $X$, one does not need the irreducibility of $F_\lambda(q)/\Gamma^\sharp$ to conclude Theorem 1.

**Theorem 1'.—**

*Let $q \in L^2_{\mathbf{R}}(\mathbf{R}^3/\Gamma)$. Assume that $F_\lambda(q)$ contains an algebraic component $X$ whose closure $\overline{X} \subset Q$ is transversal to $Q_\infty$ at almost every point of $\overline{X} \cap Q_\infty$. Then $q$ is constant.*

This is the case if for example $X$ is a sphere or an ellipsoid.

For the proof of Theorem 1' it suffices to show that

$$\overline{X} \cap Q_\infty \subset \cup_{b \in \Gamma^\sharp} \{(k, y, 0) \in Q_\infty / \langle k, b\rangle + y = 0\}. \qquad (*)$$

Let $\mathcal{D} := \{(\kappa_1, \kappa_2, \kappa_3, 1, 0) \in Q_\infty /$ there are $M$, $\tau \geq 0$ such that for all $b \in \Gamma^\sharp - \{0\}$ one has

$$|\langle \kappa, b\rangle| \geq M|b|^{-\tau}, |\langle \kappa, b\rangle + 1| \geq M|b|^{-\tau}\}.$$

Then one shows (by blowing up the point $p \in \mathbf{P}^4$ and using the methods to prove the Theorem 2).

**Lemma 3.**— *Let $q \in L^2_{\mathbf{R}}(\mathbf{R}^3/\Gamma)$ and $p = (\kappa, 1, 0) \in \mathcal{D}$. Then there is no algebraic component of $F_\lambda(q)$, whose closure passes through $p$ and is transversal to $Q_\infty$ in this point.*

If $C$ is a component of $\overline{X} \cap Q_\infty$ which is not contained in $\cup_{b \in \Gamma^\sharp} \{(k, y, 0) \in Q_\infty / \langle k, b\rangle + y = 0\}$, then $C$ meets $\{(k, y, 0) \in Q_\infty / y = 0\}$ in only finitely many points, i.e.

$$C' := \{(k, 1, 0) \in Q_\infty / (k, 1, 0) \in C\}$$

is an affine curve and by Lemma 3 $C' \cap \mathcal{D}$ *consists of only finitely many points.* One shows that this leads to a contradiction :

Let $\mathcal{D}_0$ be the set of points $(y_1, y_2, y_3) \in \mathbf{P}_2(\mathbf{R})$ which fulfil a diophantine estimate

$$|\langle y, b\rangle| \geq \frac{K}{|b|^\tau} \quad \text{for all} \quad b \in \Gamma^\sharp - \{0\}$$

with some $K, \tau \geq 0$. Clearly a point $(k, 1, 0) \in Q_\infty$ with $k \neq 0$ lies in $\mathcal{D}$ if its imaginary part $Imk$ represents a point of $\mathcal{D}_0$. Consider the map

$$\pi_0 : C' - \{(0; 1, 0)\} \to \mathbf{P}_2(\mathbf{R}) \ , \ (k; 1, 0) \mapsto Imk.$$

The image of $\pi_0$ intersects $\mathcal{D}_0$ in only finitely many points. On the other hand one easily verifies that $\mathbf{P}_2(\mathbf{R}) - \mathcal{D}_0$ has measure zero. Hence by Sard's theorem $\pi_0$ does not have maximal rank anywhere. From this one can conclude that $C'$ is contained in a plane. Therefore it exists a $\gamma \in \mathbf{C}^3$ such that

$$C \subset \{(k, y, 0) \in Q_\infty / \langle k, \gamma\rangle + y = 0\}.$$

Since $\pi_0$ har rank $\leq 1$ $\gamma$ is either purely real or purely imaginary. We discuss here the case $\gamma \in \mathbf{R}^3$. We may now assume that

$$C' = \{(k, 1, 0) \in Q_\infty / \langle k, \gamma\rangle + 1 = 0\} = \{(k, 1, 0) \in \mathbf{P}^4 / k_1^2 + k_1^2 + k_3^2 = 0, \langle k, \gamma\rangle + 1 = 0\}.$$

We have to show : $\gamma \in \Gamma^\sharp$, i.e. (*) is true.

So let $\gamma \notin \Gamma^\sharp$. Consider for $k \in \mathbf{C}^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ $v(k)$, the unit vector in $\mathbf{R}^3$ such that $Rek, Imk, v(k)$ form an oriented orthogonal basis.

Put $\mathcal{D}_1 := \{v \in \mathbf{R}^3 / |v| = 1, v \neq \frac{b}{|b|}$ for all $b \in \Gamma^\sharp - \{0\}$ and there are only finitely many $b \in \Gamma^\sharp$ such that $|v - \frac{b}{|b|}| < \frac{1}{|b|^2}\}$.

It is easy to see that the complement of $D_1$ in the unit sphere $S^2$ has Lebesgue measure zero. Further one shows

**Lemma 4.**— *For any $k \in \mathbf{C}^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ and $v(k) \in \mathcal{D}_1$ there is a $K' > 0$ such that for all $b \in \Gamma^\sharp - \{0\}$*

$$|\langle k, b\rangle| \geq K'|b|^{-2}.$$

But the map $C' \to S^2, (k,1,0) \mapsto v(k)$ has maximal rank almost every where. Therefore for all points $(k,1,0)$ outside a set of Lebesgue measure zero in $C'$ there is a $K > 0$ such that $|\langle k, b\rangle| \geq K|b|^{-2}$ for all $b \in \Gamma^\sharp - \{0\}$.

Now the map

$$\pi : C' \to P := \{x \in \mathbf{R}^3 / \langle x, \gamma\rangle + 1 = 0\}, (k,1,0) \mapsto Re\ k$$

is surjective and submersive. Thus Theorem 1' follows immediatly (since then $C' \cap \mathcal{D}$ consists of infinitely many points) from

**Lemma 5.**— *The set of points $x \in P$ for which there is $K, \tau > 0$ such that $|\langle x, b\rangle + 1| \geq K|b|^{-\tau}$ has positive Lebesgue measure.*

## 4. Appendix

It is possible to show that $F_\lambda(q)/\Gamma^\sharp$ for split potentials of the form $q(x) = p_1(x_1, x_2) + p_3(x_3)$ for a lattice $\Gamma = a_1\mathbf{Z} + a_2\mathbf{Z} + a_3\mathbf{Z}$ with $\langle a_1, a_3\rangle = \langle a_2, a_3\rangle = 0$ is always irreducible. One uses three facts :

i) The Bloch-varieties $\mathcal{B}(p_1)$ and $\mathcal{B}(p_2)$ are irreducible (see [KT])

ii) The map $\Phi : \mathcal{B}(p_1) \times \mathcal{B}(p_2) \to \mathcal{B}(p_1 + p_2)$ is surjective

iii) Introducing

$$\pi_1^{(\lambda)} : \mathcal{B}(p_1) \to \mathbf{C}, (k_1, k_2, \lambda_1) \to \lambda_1 - \frac{\lambda}{2}$$
$$\pi_2^{(\lambda)} : \mathcal{B}(p_2) \to \mathbf{C}, (k_3, \lambda_2) \to \frac{\lambda}{2} - \lambda_2$$

the Fermi-surface $F_\lambda(q)$ is the fibered product

$$\mathcal{B}(p_1) \times_\lambda \mathcal{B}(p_2) = \{((k_1, k_2, \lambda_1), (k_3, \lambda_2)) \in \mathcal{B}(p_1) \times \mathcal{B}(p_2) / \pi_1^{(\lambda)}(k_1, k_2, \lambda_1) = \pi_2^{(\lambda)}(k_3, \lambda_2)\}$$

Therefore we have

**Theorem 3.**—

*If $q \in L^2(\mathbf{R}^3/\Gamma)$ and the Fermi-surface $F_{\text{phys},\lambda(\text{q})}$ is the same as $F_{\text{phys},\lambda}(q')$, where $q'$ is a split potential of the above form, then $q$ also splits.*

Let us close this report by the remark that for the discrete periodic Schrödinger operator $F_\lambda(q)/\Gamma^\sharp$ is always irreducible (see [B]).

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