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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

MODULI SPACE, HEIGHTS AND ISOSPECTRAL SETS OF METRICS.

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I would like to report on several papers by Brad Osgood, Peter Sarnak and myself ([1,2,3]). The problem was colorfully stated by Mark Kac as' can you hear the shape of a drum ? That is given a manifold Σ with metric g and corresponding Laplacian Δ_g with Dirichlet boundary conditions, how many metrics have the same Δ_g-spectrum. In this count we ignore repetitions given by isometric metrics. Little is known about plane domains where this number may well be one. In the case of closed 2-manifolds Vigneras and Sunada have shown that there are arbitrarily large sets of isospectral metrics. Our main result is

Theorem 2.— *An isospectral set of closed Riemannian 2-manifolds is compact in the C^∞-topology. Likewise an isospectral set M plane domains is also compact in the C^∞-topology.*

It is known that the spectrum of the Laplacian determines the topology of the manifold so that Σ is fixed. The C^∞-topology on nonisometric classes of metrics is defined as follows : Let $\mathcal{G}^\infty(\Sigma)$ denote the usual C^∞-topology on metrics and $D^\infty(\Sigma)$ denote the group of diffeomorphisms in Σ. Let

$$R(\Sigma) = \mathcal{G}^\infty(\Sigma)/D^\infty(\Sigma) \ .$$

Then $[g]_n \to [g]$ means that there exist $g_n \in [g]_n$ and $g \in [g]$ such that $g_n \to g$ in $\mathcal{G}^\infty(\Sigma)$:

An important ingredient in our analysis is the notion of height introduced by Singer and Ray. If $0 \le \lambda_1 < \lambda_2 \le \lambda_3 \cdots$ denotes the spectrum of $-\Delta_g$, then formally

$$\det{}'\Delta_g = \Pi_{\lambda_j \neq 0} \lambda_j \ .$$

To make sense of this one need some regularization procedure such as the zeta function :

$$Z(s) = \Sigma_{\lambda_j \neq 0} \lambda_j^{-s}.$$

This can be written in terms of the heat kernel $e^{\Delta_g t}$ as

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty tr'(e^{\Delta_g t}) t^s \frac{dt}{t},$$

where

$$tr'(e^{\Delta_g t}) = \sum_{\lambda_j \neq 0} e^{-\lambda_j t}.$$

Note that

$$Z'(s) = -\sum_{\lambda_j \neq 0} \lambda_j^{-s} \log \lambda_j \ .$$

It can be shown that $Z(s)$ is meromorphic and regular at $s = 0$. This allows as to define the **height** as

$$h(g) = -\log(\det{}'\Delta_g) \equiv Z'(0).$$

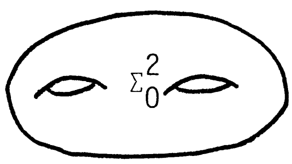
It is obvious from this that the height is an isospectral invariant. Further within a conformal class the Polyakov-Alvarez variational formula holds, that is for $g = e^{2\varphi} g_0$

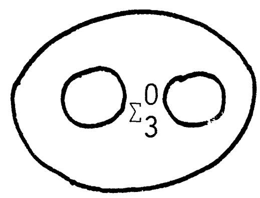
$$*) \qquad h(g) = \frac{1}{6\pi}\{\frac{1}{2}\int_{\Sigma} |\nabla_0 \varphi|^2 dA_0 + \int_{\Sigma} K_0 \varphi dA_0$$

$$+ \int_{\partial\Sigma} k_0 \varphi ds_0\} - \frac{1}{4\pi}\int_{\partial\Sigma} \partial_n \varphi ds_0 + h(g_0) \; ;$$

here ∇_0, A_0, s_0 are taken with respect to g_0 and K = Gauss curvature, k = geodesic curvature on $\partial\Sigma$.

Our results hold for 2-manifolds of two kinds : Σ_0^p, closed manifolds of genus p and Σ_n^0, plane domains of connectivity n. For example





For plane domains $K = 0$ and so for $g = e^{2\varphi} g_0$ φ will be g_0-harmonic and as a consequence in (*) we will have $\int_{\partial\Sigma} \partial_n \varphi ds_0 = 0$.

Now both area and the length of $\partial\Sigma$ are also isospectral invariants. Hence we can without loss of generality scale the metrics to normalized isospectral sets so that

$$\text{Area}(\Sigma) = 1$$

in the case of closed manifolds and

$$\text{length } (\partial\Sigma) = 1$$

in the case of plane domains.

Note that is $g = \gamma^2 g_0$ then

$$h(g) = \frac{\chi(\Sigma)}{3} \log \gamma + h(g_0),$$

where χ denotes the Euler characteristic. We denote the space of **normalized** classes of metrics by $R_0(\Sigma)$.

Next we introduce the notion of a **uniform** metric. In the case of Σ_0^p a uniform metric has constant Gauss curvature K where for plane domains Σ_n^0 the boundary $\partial\Sigma$ two constant geodesic curvature k. The following are examples of uniform metrics :

$\Sigma_0^0 = S^2$: standard round metric,

Σ_0' : flat torus ($K = 0$),

Σ_0^p : hyperbolic metric ($K < 0$),

Σ_1^0 : Euclidean metric in the unit disk,

Σ_2^0 : cylinder ($k = 0$).

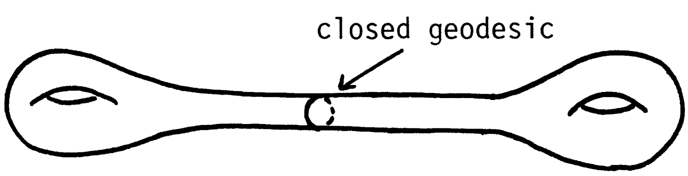
Theorem 2.— *In any conformal class of metrics in $R_0(\Sigma)$ there is a unique uniform metric and it is the unique global minimum of the height within this conformal class.*

The moduli space for Σ consists of conformal structures in $R_0(\Sigma)$.

It follows as a corollary to Theorem 2 that the set of uniform metrics in $R_0(\Sigma)$ represents the moduli space for Σ ; we denote it by $\mathcal{M}_a(\Sigma)$. It can be shown that $\mathcal{M}_a(\Sigma)$ is finite dimensional. Our principal result in $\mathcal{M}_a(\Sigma)$ is

Theorem 3.— *$h(u)$ becomes infinite as u approach the boundary of $\mathcal{M}_a(\Sigma)$.*

In the case of Σ_0^0 and Σ_1^0 where then is only one conformal class there is nothing to prove. For Σ_0^1 (torus) and Σ_2^0 (annulus) the height can be explicitely evaluated in terms of the eta function and theorem 3 proved directly. In the case of Σ_0^p (closed 2-manifolds of genus p) The degeneration of the moduli space can occur in only one way, that is when the length of a the closed geodesic approaches zero :



Theorem 3 was proved in this case by Wolpert.

For plane domains of connectivity $n \geq 3$ the boundary of $\mathcal{M}_a(\Sigma)$ is more complicated and this is reflected in the proof. The ingredients of the proof for $n \geq 3$ are as follows :

1) An explicit description of $\mathcal{M}_a(\Sigma_n^0)$ in terms of **conical** metrics in the complex plane **C** :

$$ds = \gamma \Pi |z - \tau_j|^{\alpha_j} |dx|$$

when the τ_j are n distinct points, $\alpha_j > -1$ and $\Sigma \alpha_j = -2$, and

$$\Sigma = \{z : \text{dist}_g(z, \tau_j) > 1 \quad \text{for all} \quad j\}.$$

The following types of degeneration can occur :

i) 

ii) $\gamma^{(k)} \to \infty$ or $\gamma^{(k)} \to 0$;

iii) $\alpha_i^{(k)} \to -1$;
as well as combinations of the above.

2) To sort these degenerations out we introduced the notion of a **valuation** :
Let $\tau_{ij} = |\tau_i - \tau_j|$: Euclidean distance,

$$\beta(i,r) = \Sigma_{\tau_{ij} \leq r} \alpha_j \ ,$$

$$L(i,r) = \gamma r^{1+\beta(i,r)} \Pi_{\tau_{ij} > r} \tau_{ij}^{\alpha_j} \ ,$$

$$\sigma(\tau_i) = \min_j \tau_{ij}, \bar{\sigma}(\tau_i) = \max_j \tau_{ij},$$

$$v(\tau_i) = \inf\{L(i,r), \sigma(\tau_i) < r < \bar{\sigma}(\tau_i)\}$$

$$\bar{v}(\tau_i) = \sup\{L(i,r), \sigma(\tau_i) < r < \bar{\sigma}(\tau_i)\} \ .$$

The proof is then divided into three cases

i) For some i, $\bar{v}(\tau_i) \leq$ const,

ii) For some i, $v(\tau_i) \leq$ const but $\bar{v}(\tau_i) \to \infty$,

iii) For all i, $v(\tau_i) \to \infty$.
Cases (i) and (ii) are proved inductively by mean of the

Insertion Lemma : If a Jordan curve Γ decomposes Σ into two parts $\Omega_1 \cup \Omega_2$ and Γ is "well separated" from $\partial\Sigma$, then

$$h(\Sigma(g)) \geq h(\Omega_1) + h(\Omega_2) + 0(1),$$

where the error term $0(1)$ depends only in the separation.

In case (iii) we are able to get an explicit approximation for the height and verify theorem 3 directly, to complete the overall induction.

Remark : When $p.n \neq 0$, H. Khuri showed that Theorem 3 is false (Stanford Thesis).

Finally we come back to the problem of isospectral metrics. We represent cach g in the isospectral set an terms of the uniform metric in its conformal class : $g = e^{2\varphi} u$. According to Theorem 2,

$$h(u) \leq h(q) = \text{const.}$$

By Theorem 3, u must stay away from $\partial\mathcal{M}_a(\Sigma)$ and hence lies in a compact set of uniform metrics. Using the Polyakov-Alvarez formula we can easily get a bound on the W^1 Sobolev bound in the φ's.

Next we use the heat invariants on Σ to get a grip in K (for Σ_0^p) or k (for Σ_n^0).

$$tr'(e^{\Delta_g t}) \sim \frac{1}{t} \sum_0^\infty a_j(g) t^j \quad \text{for} \quad \Sigma_0^p,$$

$$\sim \frac{1}{t}\sum_0^\infty a_j(g)t^{j/2} \quad \text{for} \quad \Sigma_n^0.$$

The $a_j's$ are isospectral invariants. They are universal polynomials in (K, ∇_0) in the Σ_0^p case and in (k, ∂_{s_0}) in the Σ_n^0 case, for which the highest ordre derivation term dominates. From this one can show that K is C^∞-compact for Σ_0^p and that k is C^∞-compact for Σ_n^0. For Σ_n^0 this result is due to R. Melrose.

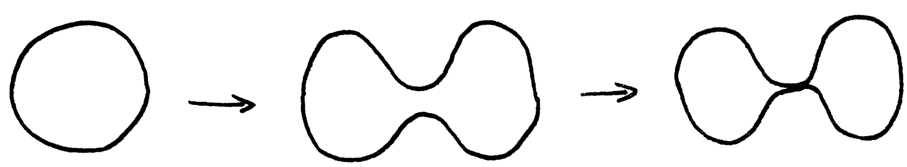
Using the relations

$$K = e^{-2\varphi}(-\Delta_0\varphi + K_0)$$

$$k = e^{-\varphi}(\partial_n\varphi + k_0)$$

together with the above two results, it is easy to show that the φ's are C^∞-compact, as need for Theorem 1.

We note that the heat invariants are not by them self enough to give the C^∞-compactness of the metrics. For example in Σ^0, we could have



with k remaining uniformly smooth in this degeneration where the metric blows up.

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