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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ON STRONG GLOBBAL SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS.

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On stronggloba1 solutions of nonlinear
hyperbolic equations.

Philip Brenner

0. Introduction

This is a version adapted for the seminar from a paper to appear in Math. Z.

The aim is to prove the existence of global solutions belonging to $L_\infty^{loc}(\mathbb{R}; L_2^2(\mathbb{R}^n))$ for nonlinear equations of the form

$$(0.1)\qquad \begin{aligned} &\partial_t^2 u - \Delta u + b(x,D)u + f(u) = 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ &u(0,x) = \varphi_0(x),\ \partial_t u(0,x) = \varphi_1(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

Here Δ devotes the Laplacian on $\mathbb{R}^n$, $b(x,D)$ is any first order differential operator on $\mathbb{R}^n$ with bounded smooth coefficients, and the nonlinearity f is assumed to satisfy

$$(0.2)\qquad f(0) = 0, f \in C^2, F(v) = \int_0^v f(u)du \geq 0, \ v \in \mathbb{R}^n,$$

and

$$(0.3)\qquad |f'(u)| \leq C(1+|u|)^{\rho-1}, \ \rho < \frac{n+2}{n-2}.$$

Since all results will be local in time, we may as well confine ourselves to discuss the result for the slightly simpler Klein-Gordon equation

$$\text{NLKG}\quad \begin{cases} \partial_t^2 u - \Delta u + m^2 u + f(u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = \varphi_0(x),\ \partial_t u(0,x) = \varphi_1(x), & x \in \mathbb{R}^n \end{cases}$$

where $m > 0$ and f satisfies (0.2) and (0.3) above and in addition $f'(0) = 0$. This will simplify the exposition (and some references to related work), with no loss of generality. The case when also the principal part of $A(x,D) = -\Delta + b(x,D)$ is allowed to have variable coefficients, is discussed in Section 3 below.

Our main result is the following.

THEOREM 1. *Let f satisfy (0.2), (0.3), and let $n \geq 3$. Let u be the (unique) solution of the NLKG with data in $L_2^2(\mathbb{R}^n) \times L_2^1(\mathbb{R}^n)$. Then $u \in L_\infty^{loc}(\mathbb{R}; L_2^2(\mathbb{R}^n))$.*

The proof will be given in section 3 below. Let us here remark that using the results of [1], also the global version of Theorem 1 holds for the NLKG, namely that $u \in L_\infty(\mathbb{R}_+; L_2^2(\mathbb{R}^n))$. Obviously this global result does not in general hold for (0.1).

Previous results on strong solutions are due to Heinz and v. Wahl [6], who proved Theorem 1 assuming the more restrictive growth-condition $\rho < \frac{n}{n-2}$, (but working with more general elliptic operators than $-\Delta + b(x,D)$). In Brenner and v.Wahl [4], the growth condition was relaxed to (0.3) for $n \leq 12$, but asymptotically the result in [4] only improved the previous established bound to $\rho < 1 + \frac{2}{n-4}$.

The bound (0.3) is a "natural" growth condition on f, since it implies (with (0.2)) that the energy (which is a conserved quantity for both the NLKG and the corresponding linear Klein-Gordon equation) is equibounded for the linear and the nonlinear equations.

This paper is structured as follows: Section 1 contains some basic estimates, with proper references. Section 2 states the results on relations between space-time means for solutions of the linear and the non-linear equations. In Section 4 (local versions of) all these results are proved, while, as mentioned above, Theorem 1 is proved in Section 3.

1. Some basic estimates

Below, $\|\cdot\|_{q,s}$ will denote the norm in the Besov space $B_q^{s,2}$. For a definition of these spaces and the related Sobolev spaces L_q^s based on L_q, see e.g. [4]. The Besov spaces may in many places be replaced by the corresponding Sobolev norms, using that for $q \le 2 \le q'$, $L_q^s \subseteq B_q^{s,2}$ and $B_{q'}^{s,2} \subseteq L_{q'}^s$. We assume, when nothing else is stated, that $1 < \rho < \frac{n+2}{n-2}$, $N \ge 3$.

We use the notation $\delta_r = \frac{1}{2} - \frac{1}{r}$, $r \ge 2$ and $\delta_r = \frac{1}{r} - \frac{1}{2}$ for $r \le 2$. Also primed indices (such as r and r') denote duality, i.e. $1/r + 1/r' = 1$. For proofs of the results below cf. [1], [4] and also [5].

Lemma 1.1 *Let $f \in C^1$, $f(0) = 0$ and $f(u) \le Cu^{\rho-1}$. Then for $0 \le s \le 1$,*

$$\|f(u)\|_{p,s} \le C\|u\|_{2,1}^{\rho-1+\eta}\|u\|_{p',s'}^{1-\eta}$$

where

$$\begin{cases}(s - s')/(1 - s') < \rho - 1, \\ 1 - \rho \le \eta \le 1 \\ 1 + 4\delta - 2\delta\eta \le \rho \le \rho_n - 2\eta(n\delta - s + s'), \\ \rho_n = (n + 4n\delta - 2(1+s) + 2s')/(n-2).\end{cases}$$

Lemma 1.2 *With f as above,*

$$\|f(u)\|_{q,1} \le C\|u\|_r^{\rho-1}\|u\|_{q',1}$$

where $\delta_{q'}(n+1) = 1, \; 2\delta_{q'}r \ge \rho - 1.$

Let $E(t)g$ denote the solution of the KG equation

(KG) $\qquad \partial_t^2 u_0 - \Delta u_0 + m^2 u_0 = 0, \text{ on } \mathbb{R}_+ \times \mathbb{R}^n,$

with $u(x,0) = 0$, $\partial_t u(x,0) = g(x)$. The operator $E(t)$ will be used frequently in the following, mainly because the solution u of the NLKG can be written in terms of the solution u_0 of the KG with the same initial data, as

$$u(t) = u_0(t) + \int_0^t E(t-\tau)f(u(\tau))d\tau. \tag{1.1}$$

This formula is the natural starting point when treating the NLKG as a perturbation of the KG.

Proposition 1.1 *Let* $n \geq 3$. *Let for some* θ, $0 \leq \theta \leq 1$, $(n+1+\theta)\delta \leq 1+s-s'$, $s,s' \geq 0$ *where* $\delta = \delta_{p'}$, $p' \geq 2$. *Then*

$$\|E(t)g\|_p \leq K(t)\|g\|_{p'}$$

where

$$K(t) \leq C \begin{cases} |t|^{-(n-1+\theta)\delta}, & |t| > 1 \\ |t|^{-(n-1-\theta)\delta}, & |t| \leq 1. \end{cases}$$

Usually we will assume that for some $\theta \geq 0$, $(n-1-\theta)\delta < 1$, so that $K \in L^{loc}_{1+\epsilon}(\mathbb{R})$, some $\epsilon > 0$.

In the next section, and in particular in section 4, we will use properties of space-time means of solutions to the linear Klein-Gordon equation. There are two results:

Proposition 1.2 *Let* E *denote the fundamental solution of the KG equation, and let* $f \in L_r(\mathbb{R}; L^s_q(\mathbb{R}^n)) = L_r(L^s_q)$, $f = f(x,t)$. *Then*

$$\|E*f\|_{L_{r'}(L^{s'}_{q'})} \leq K_0\|f\|_{L_r(L^s_q)},$$

provided $2 \leq r' \leq q'$, *and*

(a) $1 - s - s' > 0$

(b) $2\delta_{r'} + (n-1-\theta)\delta_{q'} \geq 1 \geq \delta_{q'}(n-1-\theta) + 2\delta_{r'}$

(c) $(n+1+\theta)\delta_{q'} \leq 1 - s - s'$

for some θ, $0 \leq \theta \leq 1$. *If* $\delta_{r'} = 0$, *strict inequalities have to be used in (b).*

For a proof, see e.g. Proposition 1 in [2]. The above result, and also the next, is essentially due to Segal [8] (for n = 3), Strichartz [9] and Marshall [7].

Proposition 1.3 *Let the conditions of Prop. 1.2 hold with* $s-s'$ *replaced by* $2s$ *in (a) through (c). Then any solution* u_0 *of the KG with data in* $L^{1+s}_2 \times L^s_2$, *belongs to* $L_{r'}(L^{\frac{1}{2}+s}_{q'})$.

A special case, often used is $r' = q'$, $\delta_{q'}(n+1) = 1$ (and $\theta = 0$).

For the next result we need some notation. Let $(T^\sigma h)^\wedge = (i\eta)^\sigma \hat{h}(\eta)$, $\hat{h}(\eta) = \int_{-\infty}^{\infty} e^{-i\eta t} h(t)dt$, and let $B^\sigma = (m^2-\Delta)^{\sigma/2}$. By $\|f\|_{L_h^{\sigma_1}(L_p^{\sigma_2})}$ we denote (slightlly illogical, maybe) $\|T^{\sigma_1}B^{\sigma_2}f\|_{L_r(L_p)}$.

<u>Proposition 1.4</u> *Let $E(\cdot)$ be the solution operator of Prop. 1.1, and let u be a solution of the NLKG, and let f satisfy (0.2), (0.3). Then*

$$\|E(\cdot)f(u(\cdot))\|_{L_r^{-\sigma}(L_2^{s+\sigma})} \leq K_0 \|f(u(\cdot))\|_{L_r(L_2^{s-1})},$$

where $1 > \sigma \geq 0$, $s \geq 1$ and $1 < r < \infty$.

Proof: Let

$$\mathcal{E}(t) = \exp(itB)$$

Then

$$E(t) = \mathrm{Im}(B^{-1}\mathcal{E}(t)),$$
$$u(t) = \mathrm{Re}\,[\mathcal{E}(t)\Phi], \quad \Phi = \varphi + iB^{-1}\psi.$$

It is enough to show that

$$T^{-\sigma}(iB)^\sigma(\mathcal{E}(t)f(u(t)) = \mathcal{E}(t)(1 + (iB)^{-\sigma}T^\sigma)^{-1}f(u(t))$$

and since the symbol

$$\sigma_H(\xi,\eta) = (1 + (m^2+|\xi|^2)^{-\frac{\sigma}{2}}\eta^\sigma)$$

of

$$H = (1 + (iB)^{-\sigma}T^{\sigma})$$

is $\neq 0$ for $\xi,\eta \in \mathbb{R}\times\mathbb{R}$, $0 \leq \sigma < 1$, $m > 0$, H^{-1} is bounded on $L_r(H^s)$, $s \geq 0$, $1 < r < \infty$. Since $\mathcal{E}(t)$ is an isometry on H^s, the estimate in Prop. 1.4 follows.

The following is an important application of Proposition 1.4: Let u be a solution of the NLKG, with u_0 the corresponding solution of the KG. We note that the characteristic function χ_t of $(0,t)$ belongs to $L_r^{1/r}$ for $1 < r < \infty$. Let $I = (0,t)$ and let $1/r + 1/r' = 1$, $\sigma \leq 1/r$. Then by Prop. 1.4,

$$\|u(t)\|_{2,2} \leq \|u_0(t)\|_{2,2} + \|\int \chi_t(\tau)E(t-\tau)f(u(\tau))d\tau\|_{2,2}$$

$$\leq \|u_0(t)\|_{2,2} + \|\chi_t\|_{L_r^{\sigma}}\|E(t-\cdot)f(u(\cdot))\|_{L_{r'}^{-\sigma}(I;L_2^2)}$$

$$\leq \|u_0(t)\|_{2,2} + C(\sigma,r;t)\|f(u)\|_{L_{r'}(I;L_2^{1-\sigma})}$$

Since we may take $\sigma = 1/r = 1 - 1/r'$, we get

$$\|u(t)\|_{2,2} \leq \|u_0(t)\|_{2,2} + C(r;t)\|f(u)\|_{L_{r'}(I;L_2^{1/r'})}. \tag{1.2}$$

Thus, to estimate U locally in L_2^2, we have for some $r' < \infty$ (but otherwise as larg as we wish) to prove that $f(u) \in L_{r'}^{loc}(L_2^{1/r'})$. In a sense we have gained "almost one derivative", using the oscillatory properties of the nonlinear term in(1.1).

2. Space–time means for nonlinear Klein–Gordon equations.

In this section we will state some results, most of relate boundedness of space–time means for the linear equation with those of the nonlinear equation.

As before, we assume that (0.2), (0.3) hold, and in particular that $\rho < \frac{n+2}{n-2}$, $n \geq 3$. We denote by u the solution of the NLKG and by u_0 the corresponding solution (i.e. with the same initial data) of the linear Klein–Gordon equation.

We have to relate the L_p-spaces with the behavior of the kernel K(t) in Proposition 1.1, and introduce for that purpose the condition

$(*)'_s$ $\quad (n-1)\delta_{p'} < 1$, $(n+1)\delta_{p'} \leq 1+s-s'$, some s, $0 \leq s \leq 1$.

This implies in particular that $K \in L_1^{loc}(\mathbb{R})$.

Proposition 2.1 *Assume that $(*)'_s$ holds, and that $(s-s')/(1-s') < \rho - 1$. Then $u_0 \in L_r^{loc}(L_p^{s'})$ implies that $u \in L_r^{loc}(L_p^{s'})$.*

Remark Using the stronger assumption

$(*)_s$ $\quad$ For some θ, $0 < \theta \leq 1$ and some s, $0 \leq s \leq 1$,

$(n-1-\theta)\delta_{p'} > 1$ $(n-1-\theta)\delta_{p'}$, $(n+1+\theta)\delta_{p'} \leq 1+s-s'$.

We find that $K \in L_1(\mathbb{R})$ globally, and in that case Proposition 2.1 holds also globally in time (L_r^{loc} is replaced by L_r). The proof, given in [3], is much more complicated than that for the local case.

Proposition 2.2 *Let $\delta = 1/2 - 1/q'$, $\delta(n+1) = 1$ and let $0 \leq \sigma \leq 1$. Then $u_0 \in L_r^{loc}(L_{q'}^{\sigma})$ implies that $u \in L_r^{loc}(L_{q'}^{\sigma})$.*

Again, there is a global version of Proposition 2.2. Both the local and the global versions may be derived from the corresponding forms of Proposition 2.1.

Proposition 2.3 *Let $\sigma < 4/(n-2)$. If $u_0 \in L_\infty^{loc}(L_2^2) \cap L_r^{loc}(L_{q'}^{1+\sigma})$ then $u \in L_r^{loc}(L_{q'}^{1+\sigma})$.*

Again there is a global version available. Notice that the assumptions in Proposition 2.3 seems to be non–optimal. It is precisely the result we need to prove our main theorem, however.

3. On the existence of global strong solutions of the nonlinear Klein-Gordon equation

Let us begin this section by stating the main result: As before $\rho < \frac{n+2}{n-2}$, $n \geq 3$.

Theorem 3.1. *Let* u *be the unique global solution of the NLKG with data in* $L_2^2(\mathbb{R}^n) \times L_2^1(\mathbb{R}^n)$. *Then* $u \in L_\infty^{loc}(\mathbb{R}; L_2^2(\mathbb{R}^n))$.

Proof. In view of (1.2), it is enough to bound the $L_{r'}^{loc}(L_2^{1/r'})$-norm of f(u), where $r' < \infty$ can be chosen as large as may become necessary. A straightforward application of Hölder's inequality and Sobolev's embedding theorem gives

$$\|f(u)\|_{2,\sigma} \leq C\|u\|_{q',1+\hat{\sigma}}\|u\|_{2,2}^{\rho-1},$$

where $1/q' - (1+\hat{\sigma})/n = 1/2 = (2+\epsilon)/n$ and provided

(3.2) $$\rho < 2\,\frac{2+\epsilon-\sigma}{n-4+2\sigma}.$$

To prove our theorem, we will first prove that $u \in L_{r'}(L_{q'}^{1+\hat{\sigma}})$, where $\hat{\sigma} \geq 1/(n+1) + 1/r'$ by (3.2), and $r' < \infty$ is large (which assumes less smoothness of u). But by Proposition 2.3, if $u_0 \in L_\infty^{loc}(L_2^2) \cap L_{r'}^{loc}(L_{q'}^{1+\hat{\sigma}})$ then $u \in L_{r'}^{loc}(L_{q'}^{1+\hat{\sigma}})$, and our result follows, assuming that $\hat{\sigma} < 4/(n-2)$, which we clearly may do. That $u_0 \in L_\infty(L_2^2)$ is clear, and so by Proposition 1.3, $u_0 \in L_{q'}(L_{q'}^{3/2}) \subseteq L_{q'}^{loc}(L_{q'}^{3/2})$. By Sobolev's embedding theorem, we find that $u_0 \in L_{r'}(L_{q'}^{1+\hat{\sigma}})$ for $r' \geq q'$, $1/r' \geq 1/q' - (1/2 - \hat{\sigma}) = \hat{\sigma} - 1/(n+1)$, i.e. $\hat{\sigma} \leq 1/(n+1) + 1/r'$, which is exactly what we needed. Since now $u \in L_{r'}(L_{q'}^{1+\hat{\sigma}})$, for $\hat{\sigma} = 1/r + 1/(n+1)$,

(3.3) $$\|f(u)\|_{L_{r'}^{loc}(L_2^{1/r'})} \leq C \sup_{0\leq\tau\leq t} \|u(\tau)\|_{2,2}^{\rho-1}.$$

By (1.2) and (3.3)

(3.4) $$\|u(t)\|_{2,2} \leq \|u_0(t)\|_{2,2} + C \sup_{0\leq t\leq t} \|u(t)\|_{2,2}^{\rho-1}.$$

We may assume that $n \geq 6$, and so $\rho - 1 < 1$, since the theorem is known for $n \leq 6$ (cf. [4]). But then (3.4) implies that $u \in L_\infty^{loc}(L_2^2)$, which completes the proof of our theorem.

If $-\Delta + b(x,DS) = A(x,D)$ is replaced by a positive elliptic operator $A(x,D)$

with smooth coefficients, and which has a principal part that coincides with $-\Delta$ for $|x|$ large, then it would be natural to expect Theorem 1 to hold also in this case.

However, in the proof of Theorem 1 we used that $u_0 \in L^{loc}_{q'}(L^{1+\sigma}_{q'})$, $\sigma \geq 1/r' + \delta_{q'}$, with $r < \infty$ large and $\sigma_{q'}(n+1) = 1$. An application of Sobolev's inequality to $u_0 \in L_\infty(L^2_2)$ unfortunately only gives $u_0 \in L^{loc}_{r'}(L^{1+\epsilon}_{q'})$ with $\epsilon = \delta_{q'}$. This means that we should need an additional, although marginal, smoothness assumption on the data in order to make the present proof work in the more general case. The missing part is a version of the estimate of space–time means (Proposition 1.2) for solutions of the wave equation $\partial^2_t u - A(x,D)u = 0$. Possibly the results and techniques of [1], [2] could be combined to give a (local version of) the space–time estimates in Prop 1.2.

4. Proof of the results in section 2.

Let K_M denote the M–fold convolution of $K(t)$ with itself, where K is the kernel in Proposition 1.1. We have already remarked that under assumption $(*)_s'$, $K \in L_1^{loc} \cap L_{1+\epsilon}^{loc}$, for some $\epsilon > 0$ (and $K \in L_1 \cap L_{1+\epsilon}$ if also $(*)_s$ holds). Thus $K_M \in L_1 \cap L_{r(M)}$, where by suitable choice of M, we may get any $r(M) \leq \infty$. This is an immediate consequence of Young's inequality.

As before, we let u denote the solution of the NLKG, and u_0 the corresponding solution of the linear Klein–Gordon equation (with the same initial data as in the nonlinear case). We assume (0.2), (0.3) and $f'(0) = 0$ to hold, and that the data of the NLKG are at least in $L_2^1 \times L_2$ on $\mathbb{R}^n$.

Lemma 4.1 *Assume that* $(*)_s'$ *holds with* $(s-s')/(1-s') < \rho - 1$. *Assume also that* $U_0 \in L_r^{loc}(L^{s'}_{p'})$. *Then for* $M \geq M_0$, *some* $M_0 = M_0(r)$,

$$\int_0^\infty K_M(t-\tau)\|u(\tau)\|_{p',s'}d\tau \in L_\infty^{loc}.$$

Proof. Let $g_0(t) = K_M * \|u_0(\tau)\|_{p',s'}$, with M so large that $K_M \in L_{r'}$. Then $g_0 \in L_\infty^{loc}$ and using Lemma 1.1, and Proposition 1.1,

$$g(t) = K_M * \|u\|_{p',s'} \leq g_0(t) + C\int_0^t K_{M+1}(t-\tau)\|u\|_{p',s'}^{1-\eta}d\tau,$$

some (small) $\eta > 0$. The last term may be estimated by

$$\int_0^t (K(t-t')(\int_0^{t'} K_M(t'-\tau)\|u\|_{p',s'}dt)^{1-\eta}(\int_0^t K_M(\tau)d\tau)^\eta dt'$$

and hence

$$g \leq g_0 + Cg^{1-\eta}$$

which implies that $g \in L_\infty^{loc}$.

It's clear that a corresponding global result holds if we assume $(*)_s$ rather than $(*)_s'$. The proof of the next lemma is however quite different in the local and global case (cf. [3], where the more difficult global result is proved.)

Lemma 4.2 *If* $(*)_s'$ *holds, and* $(s-s')/(1-s') < \rho - 1$, *then* $u_0 \in L_r^{loc}(L_p^{s'})$ *implies that*

$u \in L_r^{loc}(L_{p'}^{s'})$.

Proof. Let K_j as above be the convolution of the kernel K with itself j times. Then $K_j \in L_1^{loc} \cap L_{r(j)}^{loc}$, $r(j) > 1$, and by Lemma 1.1 (with $\eta = 0$) and Proposition 1.1, $K_{j-1} * \|u_0\|_{p',s'} \in L_r^{loc}$ by assumption, and so if $K_j * \|u\|_{p',s'} \in L_r^{loc}$ then also $K_{j-1} * \|u\|_{p',s'} \in L_r^{loc}$. If we let $j = M = M(r)$ as in Lemma 4.1, then $K_M * \|u\|_{p',s'} \in L_\infty^{loc} \subseteq L_r^{loc}$. Thus Lemma 4.2 follows by induction over j.

Remark *One allowed combination in Lemma 4.2 is* $\delta_{p'}(n-1) < 1$ *close to 1, and* $s' = 1/2$.

Since the arguments used in Lemma 4.1 and 4.2 will be used also in some of the following lemmas, a formal statement of one part of the "machine" used above will be convenient.

Let $K \in L_{r_0}^{loc} \cap L_{r_0+\epsilon}^{loc}$ and $h \in L_{r_0'}^{loc}$ on $\mathbb{R}$, where (as usual) $1/r_0 + 1/r_0' = 1$. Let K be the integral operator defined by

$$\mathscr{K}g(t) = \int_0^t K(t-\tau)h(\tau)g(\tau)d\tau.$$

Then $\mathscr{K}$ maps L_r^{loc} into $L_r^{loc} \cap L_{r+\epsilon'}^{loc}$, some $\epsilon' > 0$, $1 \le r \le \infty$, as follows from Young's inequality. As a consequence $\mathscr{K}^M$ maps L_r^{lo} into $L_r^{loc} \cap L_{r(M)}^{loc}$, where $r(M) \le \infty$ can be taken as large as we want, by choosing M large. Replacing the convolution operator with kernel $K(\cdot)$ used in Lemma 4.2 and 4.3, by $\mathscr{K}$, we obtain by repeating the proofs of these theorems the following lemma:

Lemma 4A. *Let* $\mathscr{K}$ *be as above, and assume that* $U_0 \in L_r^{loc}$. *If* U *satisfies*

$$U \le U_0 + C\mathscr{K}(U^{1-\eta}),\ 0 \le |\eta| < \epsilon,\ \epsilon > 0 \text{ small},$$

then also $U \in L_r^{loc}$.

Lemma 4.3 *If* $u_0 \in L_\infty^{loc}(L_2^1)$ *then* $u \in L_{q'}^{loc}(L_{p'}^{\gamma})$ *where* $\gamma = 1/2 + \delta_{q'} - \delta_{p'}$, $\delta_{q'}(n+1) = 1$ *and* $\delta_{q'} \le \delta_{p'} < 1/(n-1)$, *with* $\delta_{p'}(n-1)$ *close enough to* 1.

Proof. By Proposition 1.3, $u_0 \in L^{loc}_{q'}(L^{\gamma}_{p'})$, and since $\gamma > 1/2$, Lemma 4.3 applies in case $(n-1)\delta < 1$ is sufficiently close to 1, proving that $u \in L^{loc}_{q'}(L^{\gamma}_{p'})$.

The next result is derived from Lemma 4.3, and is valid both in local and global form. The proof given here holds also in the global case (with minor obvious changes), provided we use Lemma 4.2, and so 4.3 in a global form.

Lemma 4.4 *Let $0 \le \sigma \le 1$, $\delta_{q'}(n+1) = 1$. then $u_0 \in L^{loc}_r(L^{\sigma}_{q'})$ implies that $u \in L^{loc}_r(L^{\sigma}_{q'})$.*

Proof. We use the inequality

$$(4.1) \qquad \|f(u)\|_{q,\sigma} \le C\|u\|^{\rho-1}_{p',\hat{\gamma}}\|u\|_{q',\sigma}$$

where $\delta_{q'} \le \delta_{p'} < 1/(n-1)$ and $\hat{\gamma} = \frac{1}{2} - \delta_{p'} + \frac{1}{2}\frac{1}{n+1} = \frac{1}{2}\frac{n}{n+1} + \delta_{q'} - \delta_{p'}$. Now $u \in L^{loc}_{q'}(L^{\gamma}_{p'})$ with $\gamma = \hat{\gamma} + \frac{1}{2}\frac{1}{n+1}$, by Lemma 4.3 (notice that $u_0 \in L^{loc}_{\infty}(L^{1}_{2})$ is a standing assumption), and so by Sobolev's inequality, $u \in L^{loc}_{r_1}(L^{\hat{\gamma}}_{p'})$, where $1/r_1 = 1/q' - 1/2(n+1)$. If we use (1.1) and (4.1) we thus have

$$(4.2) \qquad \|u\|_{q',\sigma} \le \|u_0\|_{q',\sigma} + \int_0^t K(t-\tau)\|u\|^{\rho-1}_{p',\hat{\gamma}}\|u\|_{q',\sigma}d\tau.$$

By Proposition 1.1, $K \in L^{loc}_{r_0'}$, all r_0' such that $r_0' \frac{n-1}{n+1} < 1$. It only remains to prove that (with $1/r_0' - 1/r_0 = 1$) $r_0(\rho-1) \le r_1$ (local time-estimates), i.e.

$$(4.3) \qquad \rho - 1 < r_1(1 - 1/r_0').$$

We have to prove that any $\rho < \frac{n+2}{n-2}$ can be made to satisfy (4.3) for suitable choice of r_0'. But the upper bound in (4.3) can be made arbitrarily close to $r_1 \frac{2}{n+1}$, by proper choice of r_0'. Now

$$1/r_1 = 1/2 - 3/2(n+1)$$

so that

$$r_1 \frac{2}{n+1} = \frac{2}{n+1} * \frac{1}{1/r_1} = \frac{4}{n-2},$$

which means that for any $\rho < \frac{n+2}{n-2}$ we can find n r_0' and r_0 such that $K \in L^{loc}_{r_0'}$ and $h(\tau) = \|u\|^{\rho-1}_{p',\gamma} \in L^{loc}_{r_0}$, $1/r_0 + 1/r' = 1$. A trivial variation of this argument also proves that $h(\tau) = \|u\|^{\rho-1+\eta}_{p',\gamma}$, $|\eta|$ small, and so

(4.4) $$\|u\|_{q',\sigma} \le \|u_0\|_{q',\sigma} + C_\eta \int_0^t K(t-\tau)h(\tau)\|u\|^{1-\eta}_{q',\sigma} d\tau$$

for $|\eta|$ small, with $K \in L^{loc}_{r_0'}$, $h \in L^{loc}_{r_0}$.

An application of Lemma 4A then proves Lemma 4.4.

<u>Lemma 4.5</u> *Let* $\delta_{q'}(n+1) = 1$, *and let* $\sigma < 4/(n-2)$. *Then* $u_0 \in L^{loc}_\infty(L^2_2) \cap L^{loc}_r(L^{1+\sigma}_{q'})$ *implies that* $u \in L^{loc}_r(L^{1+\sigma}_{q'})$.

<u>Proof.</u> We have (Lemma III.6 in [4]) that

(4.5) $$\|f(u)\|_{q,1+\sigma} \le C\|u\|_{q',1+\sigma}(\|u\|_{r_0,1} + \|u\|^{\rho-1}_{r_0,1}),$$

where $1/r_0 - 1/n = 1/2 - (2+\epsilon)/n$ and

$$\sigma < \frac{4}{n-2-2\epsilon} \frac{n}{n+1}.$$

By Proposition 1.3, $u_0 \in L_{q'}(L^{3/2}_{q'})$, we have immediately $u_0 \in L_\infty(L^1_{q'})$. By Lemma 4.4 also $u \in L_\infty(L^1_{q'})$, using the "easy" case in the proof. Thus we may use $\epsilon = n/(n+1)$ in (4.5) and obtain that $u \in L^{loc}_r(L^{1+\sigma}_{q'})$ if only $u_0 \in L^{loc}_r(L^{1+\sigma}_{q'})$, using Lemma 4A.

$$\sigma < \frac{4}{n-4+2/(n+1)} \cdot \frac{n}{n+1},$$

and so in particular any $\sigma < 4/(n-2)$ will do.

The proof of Proposition 2.1 through 2.3 is now finished by reference to Lemma 4.2 through 4.5.

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