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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ON FULLY NONLINEAR ELLIPTIC EQUATIONS OF SECOND ORDER.

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On fully nonlinear elliptic equations of second order

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This is a brief report on some of the developments in recent years in the use of a priori estimates in solving nonlinear elliptic equations of the form

$$F(x, u, \nabla u, \nabla^2 u) = 0 \tag{1}$$

for a real function u defined in a bounded domain Ω in R^n or on a compact manifold Ω (perhaps without boundary).

The operator F in (1) is called elliptic at a function u if the matrix is

$$\{a^{ij}(x)\} = \left\{ \frac{\partial F}{\partial u_{ij}}(x, u, \nabla u(x), \nabla^2 u(x)) \right\}$$

is definite (say positive definite) $\forall x \in \overline{\Omega}$. F may be elliptic at some u but not at others. So in seeking solutions for which F is elliptic one may have to restrict the class of functions considered.

Primarily the report is on the series of papers by Caffarelli, Spruck (and Kohn) and the author. For the convenience of the reader we have listed all the papers in the references at the end even if we mention the results of only a few. Some have been summarized in [N]. Further references may be found in these papers and in [GT]. See also the book [K] for much of the work in the Soviet Union on this subject.

Much of the recent progress is tied to developments in "elliptic machinery", in connection with the following question for solutions of (1): If one has estimates for a certain number of derivatives of a solution can one then deduce estimates for all higher order derivatives? The answer has long been known to be yes in case one has estimates for the C^2 norm $|u|_{C^2}$ of a solution as well as for some modulus of continuity of the second derivative $\nabla^2 u$. For many years the following question, for $n > 2$, has been open for solutions u:

QUESTION: Does an estimate $|u|_{C^2} \le C$ imply some modulus of continuity for $\nabla^2 u$?

The recent progress is that the answer is *yes* in case F is also concave (or convex) in its dependence on $\nabla^2 u$. This was established in compact subsets of Ω by Evans [E] and independently by Krylov; see [K] and [GT]; and up to the boundary, for the Dirichlet problem, by Krylov (see [K] and [GT]) and independently in [CNS1], [CKNS]. The proof in the interior is based on the beautiful form of Harnack's inequality for general linear elliptic equations by Krylov and Safonov (see [KS], [GT] and [K]).

KRYLOV, SAFONOV HARNARCK INEQUALITY: In the unit ball $|x| < 1$, let u be a positive solution of a linear elliptic equations

$$Lu = a^{ij}(x)u_{ij}(x) = 0.$$

Assume L is uniformly elliptic, i.e. for some constant M,

$$M^{-1}|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq M|\xi|^2 \qquad \forall x \text{ in the ball.}$$

Then $\exists C = C(n, M)$ such that

$$\max_{|x|<\frac{1}{2}} u \leq C \min_{|x|\leq\frac{1}{2}} u.$$

Since the answer to the question is yes for a wide class of equations (1) we have been able to derive new existence results for the Dirichlet problem for some of them.

The papers [CNS] and [CKNS] mainly treat two kinds of problems, for equations (1) of the form (here $\psi > 0$ in $\overline{\Omega}$ is given):

$$f(\lambda_1, \ldots, \lambda_n) - \psi(x) = 0 \qquad \text{in } \Omega \tag{2}$$

or

$$f(\kappa_1, \ldots, \kappa_n) - \psi(x) = 0 \qquad \text{in } \Omega \tag{3}$$

$$u = \phi, \qquad \text{given, on } \partial\Omega. \tag{4}$$

Here $\lambda(x) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the Hessian matrix $\nabla^2 u = \{u_{ij}(x)\}$, and $\kappa(x) = (\kappa_1, \ldots, \kappa_n)$ are the principal curvatures of the graph of $u, (x, u(x))$ in $\mathbb{R}^{n+1}$. The function $f(\lambda)$ is a smooth *concave* function, satisfying $f_{\lambda_i} > 0 \quad \forall i$, defined in an open convex cone Γ in $\mathbb{R}^n$ with vertex at the origin ($\Gamma \neq \mathbb{R}^n$) and containing the positive cone Γ^+. Γ (and f) are assumed to be symmetric under permutation of the λ^i. For convenience we suppose $f > 0$ in Γ, $f = 0$ in $\partial\Gamma$. f is also required to satisfy: $f(R\lambda) \to \infty$ as $R \to \infty$.

A function $u \in C^2(\overline{\Omega})$ is called admissible for (2) or (3) if $\lambda(u_{jk}(x))$ or κ (graph) belongs to $\Gamma \quad \forall x \in \overline{\Omega}$. We seek admissible solutions. For example, if $\Gamma = \Gamma^+$ then u is admissible if u is strictly convex; then it is natural to require that Ω is strictly convex. So in studying these problems one has to also restrict the class of domains Ω in which one expects to solve the problem.

Our results are different depending on whether

(a) Γ is a *wide cone*, i.e. the positive axes lie inside Γ,

or

(b) Γ is a *narrow cone*: the positive axes lie on $\partial\Gamma$.

For f as above we established in [CNS2] the following existence of smooth solutions of (2), (4), in case ψ and ϕ (and $\partial\Omega$) are smooth.

THEOREM 1. *If Γ is a wide cone there is a unique admissible solution $u \in C^\infty(\overline{\Omega})$ for any bounded domain Ω.*

THEOREM 2. *If Γ is a narrow cone, assume in addition that f satisfies*

$$f(\lambda + Re_n) \to \infty \text{ as } R \to \infty, \qquad \forall \lambda \in \Gamma,$$

$e_n = (0, \dots, 0, 1)$. *Then for every $\phi \in C^\infty(\partial\Omega)$ there exists a unique admissible solution $u \in C^\infty(\overline{\Omega})$ of (2), (4) iff*

$$(5) \qquad \begin{cases} \partial\Omega \text{ is connected and } \forall x \in \partial\Omega, \text{ if } \mu_1(x), \dots, \mu_{n-1}(x) \\ \text{are the principal curvatures of } \partial\Omega \text{ relative to the} \\ \text{interior normal then for some large } R, (\mu(x), \dots, \mu_{n-1}(x), R) \\ \text{belongs to } \Gamma. \end{cases}$$

In [CNS5] we prove analogous results for (3), (4) but only for $\phi \equiv 0$ and $\partial\Omega$ strictly convex. Li [L] has extended our results. Ivochkina [I] has announced results which are more general than some of those in [CNS5]. See also [T1] and [T2].

Recently, a number of authors have extended the known "elliptic machinery" to, so called, viscosity solutions: A continuous function u is a viscosity solution of (1) (for convenience we suppose F independent of u) if for any classical C^2 strict subsolution (supersolution) ϕ, $u - \phi$ cannot have an interior minimum (resp. maximum) ϕ being a strict subsolution means $F(x, \nabla\phi, \nabla^2\phi) > 0$.

In particular Caffarelli, see [C1], [C2], where other references may be found (also P.L. Lions and Trudinger have studied viscosity solutions), has redone the standard elliptic machinery, including the Schauder theory, $W^{2,p}$ theory etc. for equations

$$F(x, \nabla^2 u) = f(x).$$

In deriving the classical estimates for linear elliptic equations one usually approximates the operators, locally, by ones with constant coefficients, and using results for such operators. Caffarelli relies on interior estimates for solutions of equations of the form

$$F(0, D^2 w) = 0.$$

He even obtains new estimates for linear elliptic equations. In [C1–2] he treats uniformly elliptic equations, but recently he has also studied nonuniformly elliptic equations, for example

$$\det(u_{ij}) = f > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

He obtains $W^{2,p}$ estimates for the solution in case f is close to constant. Also if $M^{-1} \leq f \leq M$ and $B_1 \subset \Omega \subset B_n$, (here $B_k = \{x < k\}$ then he shows [C3] that $u \in C^1(B_{\frac{1}{2}})$. See also Urbas [U].

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References

[C1] L.A. Caffarelli, "Elliptic second order equations. Lectures of Accademia dei Lincei," Cambridge Univ. Press, to appear.

[C2] L.A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, preprint.

[C3] L.A. Caffarelli, *A localization property of viscosity solutions to the Monge–Ampère equations and their strict convexity*, preprint.

[CKNS] L. Caffarelli, J.J. Kohn, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations II. Complex Monge–Ampère and uniformly elliptic equations*, Comm. Pure Appl. Math. **38** (1985), 209–252.

[CNS1] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations I. Monge–Ampère equations*, Comm. Pure Appl. Math. **37** (1984), 369–402. See also correction in **40** (1987) 659–662.

[CNS2] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations III. Functions of eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.

[CNS3] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for the degenerate Monge–Ampère equation*, Rev. Mat. Iberoamericana **2** (1986), 19–27.

[CNS4] L. Caffarelli, L. Nirenberg, J. Spruck, *Nonlinear second order equations IV. Star-shaped compact Weingarten surfaces*, Current Topics in Partial Differential Equations (1986), 1–26, ed. by Y. Ohya et. al., Kinokunize Co. Tokyo.

[CNS5] L. Caffarelli, L. Nirenberg, J. Spruck, *Nonlinear second order equations V. The Dirichlet problem for Weingarten hypersurfaces*, Comm. Pure Appl. Math. **41** (1988), 47–70.

[CNS6] L. Caffarelli, L. Nirenberg, J. Spruck, *On a form of Bernstein's Theorem*, Analyse Math. et. Appl. (1988), 55–66, Gauthier–Villars, Paris.

[E] L.C. Evans, *Classical solutions of fully nonlinear convex second order elliptic equations*, Comm. Pure Appl. Math. **35** (1982), 333–363.

[GT] D. Gilbarg, N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer–Verlag, Berlin, 2nd ed'n., 1983.

[I] N.M. Ivochkina, *Solution of the Dirichlet problem for the equation of curvature of order m*, Dokl Akad. Nauk. CCCP **299** (1988), 35–38; Soviet Math. Dokl. **37** (1988) 322–325.

[K] N.V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, "Nauka" Moscow (1985), Engl. transl'n., Reidel (1987).

[KS] N.V. Krylov, M.V. Safonov, *Certain properties of solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk. SSSR **40** (1980), 161–175; Engl. transl'n. Math. USSR–Izv. **16** (1981) 151–164.

[L] Li, Yanyan, *Some existence results for fully nonlinear elliptic equations of Monge–Ampère type*, Comm. Pure Appl. Math., to appear.

[N] L. Nirenberg, *Fully nonlinear elliptic equations*, Proc. Symposium in Pure Math. AMS Vol. 48 (1988), 217–225.

[T1] N.S. Trudinger, *A priori bound for graphs with prescribed curvature*, preprint.

[T1] N.S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, in preparation.

[U] J. Urbas, *Regularity of generalized solutions of Monge–Ampère equation*, preprint.