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SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES 1985 - 1986

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The superposition principle in quantum mechanics and in wave optics is an important primary physical principle. However, it yields a fact which is very simple in mathematical sense, i.e. linearity of equations in quantum mechanics and wave optics. We show, that some new superposition principles used to solve some nonlinear equations including the Bellman and the Hamilton-Jacobi equations lead to "linearity" of these equations in some semi-modules, what allows to apply some formulas and results of usual linear equations to this case.
Hopf has constructed a solution of a quasi-linear equation, starting from Bürgers equation with small viscosity and then passing to the limit as the viscosity tends to zero. He used the fact, that the Bürgers equation may be reduced to the heat equation by a certain substitution.
First we consider this situation from a different viewpoint, using an integrated variant of Bürgers equation.
We show the main ideas using simple examples and omitting proofs. The latter may be found in detail in 1 .
We obtain preliminary considerations from the following elementary example. Consider the heat equation with the small parameter h:

$$\frac{\partial u}{\partial t} = \frac{h}{2}\frac{\partial^2 u}{\partial x^2}, \quad h \longrightarrow +0. \tag{1}$$

This is a linear equation. Thus, any linear combination $\lambda_1 u_1 + \lambda_2 u_2$ of two solutions u_1 and u_2 is a solution of (1) as well. By substitution

$$u = e^{-\frac{W}{h}}$$

we reduce (1) to an integrated variant of the equation considered by Hopf, namely,

$$\frac{\partial W}{\partial t} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 - \frac{h}{2}\frac{\partial^2 W}{\partial x^2} = 0. \tag{2}$$

This equation is nonlinear. However, if W_1 and W_2 are solutions of (2), then the combination

$$W = -h \ln\left(e^{-\frac{W_1+\lambda_1}{h}} + e^{-\frac{W_2+\lambda_2}{h}}\right) \tag{3}$$

of these solutions is also a solution by virtue of the above substitution.

This fact may be interpreted as follows. Consider a space of functions with values in the ring whose elements are real numbers and operations are:

$$a \oplus b = -h \ln\left(e^{-a/h} + e^{-b/h}\right) \tag{4}$$

(generalized addition) and

$$a \odot \lambda = a + \lambda \tag{5}$$

(generalized multiplication which coincides with usual addition of reals).

In this space the integrated variant (2) of Bürgers equation is linear i.e. if W_1 and W_2 are solutions, then so is

$$W = \lambda_1 \odot W_1 \oplus \lambda_2 \odot W_2 .$$

Besides we note an important fact. The above substitution induces the scalar product

$$(W_1, W_2) = -h \ln \int e^{-\frac{W_1 + W_2}{h}} dx \tag{6}$$

and the resolving operator of (2) is self-adjoint with respect to this new product, since the resolvent operator of (1) is self-adjoint in $L_2(R)$:

$$\left(e^{\frac{th}{2}\frac{\partial^2}{\partial x^2}}\right)^* = e^{\frac{th}{2}\frac{\partial^2}{\partial x^2}}.$$

The ring with the operations of addition and multiplication (4) and (5), in which we set zero to be equal to arithmetical infinity, and unity to be equal to arithmetical zero, is isomorphic to the usual arithmetic ring with usual addition and multiplication.

The situation is rather more complicated for the true Bürgers equation. Here we discuss it briefly. Consider the space of pairs $\{U(x), C\}$ $U(x)$ being a function and C being a constant. We introduce the operations:

$$\{U_1, C_1\} \oplus \{U_2, C_2\} = \Big\{ h \frac{d}{dx}\Big(\ln e^{1/h\left(\int_0^x U_1(x)dx + C_1\right)} + \tag{7}$$

$$+ e^{1/h\left(\int_0^x U_2(x)dx + C_2\right)}\Big),\ h \ln\left(e^{C_1/h} + e^{C_2/h}\right)\Big\}, \tag{8}$$

$$\{U_1, C_1\} \odot \lambda = \{U_1, C_1 + \lambda\}$$

$$\{U_1, C_1\} \odot \{U_2, C_2\} = \{U_1 + U_2, C_1 + C_2\},$$

an involution

$$\{u, c\}^* = \{-u, -c\} \tag{9}$$

and a "scalar product" - the bilinear form

$$\left(\{u_1, c_1\}^*, \{u_2, c_2\}\right) = \ln \int e^{-1/h\left(\int_0^x u_1(x)dx + c_1\right)} \cdot e^{1/h\left(\int_0^x u_2(x)dx + c_2\right)} dx. \tag{10}$$

The (matrix) resolving operator for the pair of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{h}{2}\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial c}{\partial t} = \frac{h}{2}u_x(0,t) - \frac{1}{2}u^2(0,t) \tag{11}$$

is linear in the described space, endowed with module structure (7)-(8). Note, that one of these equations is nothing else than Bürgers equation. As we see the situation is more cumbersome but essentially the same.

We considered above two similar examples. In both cases nonlinear equations reduce to linear ones. But it is not so interesting. It is interesting to find such a module or, more generally, semi-module, in which given nonlinear equation will be linear itself, though we have not any reduction. If such a semi-module exists, the methods and results for linear equations may be applied to such nonlinear equations.

Next we pass to the limit $h \to 0$ in our examples. In order to obtain a quasi-linear equation one should pass to the limit h 0 in Bürgers equation (11). To simplify the calculations we consider here the variant (2) of Bürgers equation. The limit as $h \to 0$ for (2) is the Hamilton-Jacobi equation:

$$\frac{\partial S'}{\partial t} + \frac{1}{2}\left(\frac{\partial S'}{\partial x}\right)^2 = 0, \tag{12}$$

rather than a quasi-linear equation. However, all the results presented are word for word valid for the quasi-linear equation.

Examine now the behaviour of the addition and multiplication operations as $h \to 0$. The multiplication operation (5) is independent of h and remains, as above, the arithmetical summing.

$$a \odot b \longrightarrow a + b, \qquad \odot \longrightarrow + \tag{13}$$

As for addition operation (4), for positive a and b a limit exists:

$$\lim_{h\to 0} a \oplus b = \lim_{h\to 0} h \ln(e^{-a/h} + e^{-b/h}) = \min(a,b) \qquad (14)$$

i.e. the generalized sum of a and b is the minimum of a and b.

$$\oplus \longrightarrow \min \qquad (15)$$

The distributive law

$$(a \oplus b) \odot c = a \odot c \oplus b \odot c \qquad (16)$$

for these two operations holds. The limit scalar product of two functions $\varphi_1(x)$ and $\varphi_2(x)$ will be equal

$$(\varphi_1, \varphi_2) = \inf_x (\varphi_1(x) + \varphi_2(x)). \qquad (17)$$

The product (17) being defined, one may consider generalized functions (distributions). For example, the δ-function is given by

$$\delta(x-\xi) = \begin{cases} +\infty, & x \neq \xi \\ 0, & x = \xi \end{cases} \qquad (18)$$

$$\inf_\xi(\delta(x-\xi) + \varphi(\xi)) = \varphi(x); \quad \int^{\oplus} \delta(x-\xi) \odot \varphi(\xi)\,d\xi = \varphi(x).$$

The resolving operator of the Hamilton-Jacobi equation will be linear with respect to operations (13),(15). In particular it means that we have the so-called Green-type representation

$$S(x,t) = \int^{\oplus} G(x,\xi,t) \odot S_0(\xi)\,d\xi, \qquad G(x,\xi,0) = \delta(x-\xi) \qquad (19)$$

of the solution, i.e. the solution S(x,t) may be represented in the form of convolution of Green function $G(x,\xi,t)$ with initial condition $S_0(\xi)$. Since the integral is the minimum over x and the product is the arithmetical sum, this convolution reads

$$S(x,t) = \min_\xi [G(x,\xi,t) + S_0(\xi)], \qquad G(x,\xi,0) = \delta(x-\xi), \qquad (20)$$

where

$$G(x,\xi,t) = \frac{(x-\xi)^2}{2t}, \qquad (21)$$

and since it satisfies equation (12) and initial condition

$$G(x,\xi,0) = \begin{cases} 0, & x = \xi \\ \infty, & x \neq \xi \end{cases} \qquad (22)$$

G(x, 0) is simply a delta-function in the new sense. Thus we have obtained the familiar formula, which expresses the solution in small of the Hamilton-Jacobi equation in terms of the generating function $G(x,\xi,t)$. We have given here a natural from the physical viewpoint interpretation of the initial condition for the generating function. Now, by using the same simplest example we demostrate, how one can apply the ideas of weak solutions of linear equations to the nonlinear Hamilton-Jacobi equations.

So, let L_t be a resolving operator for Hamilton-Jacobi equation (12), and let the initial condition

$$S\Big|_{t=0} = \varphi_0(x) \tag{23}$$

have the following properties: $\varphi_0(0)=0$, and the graph of the derivative

$$p_0(x) = \varphi_0'(x) \tag{24}$$

has the form shown on Fig.1.

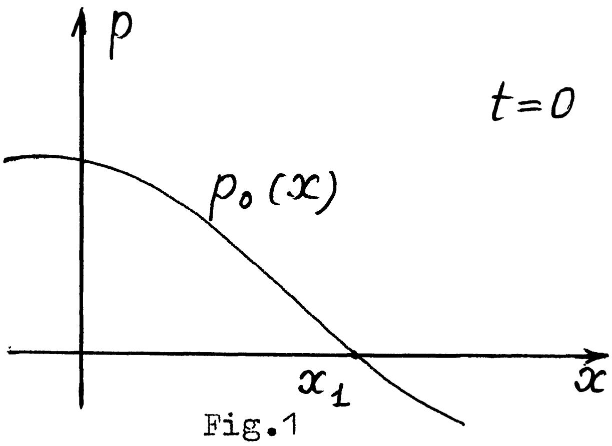

Fig.1

The corresponding Hamiltonian system has the form

$$\dot{x} = p, \qquad \dot{p} = 0. \tag{25}$$

The solution of the Hamilton-Jacobi equation may be represented in the form

$$S(x,t) = L_t\varphi_0 = \int_{x_1}^{x} p(x,t)\,dx \tag{26}$$

The function $p(x,t)$ is obtained by the shift of the curve $p_0(x)$ (see Fig.1) along the trajectories of the Hamiltonian system (25) for small t, till this curve can be projected diffeomorphically on the x-axis. One may see on Fig.2, that for $t=1$ the curve is projected non-diffeomorphically, and hence there is no classical solution for the Hamilton-Jacobi equation.

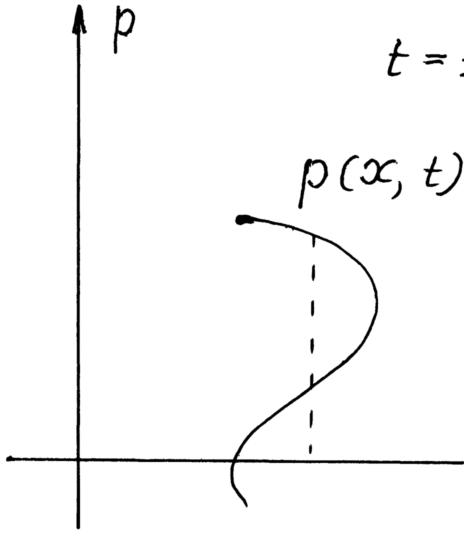

Fig.2

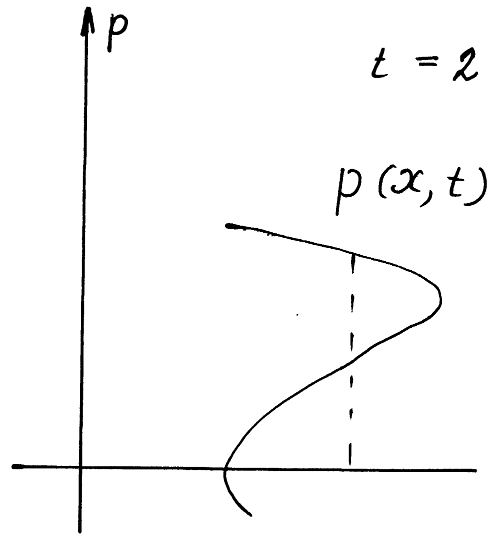

Fig.3

According to the theory of usual linear operators we shall consider the adjoint resolving operator on a dense set of such initial functions, for which classical solutions of this adjoint operator exist. In our case such a dense set is the set of functions convex downwards, since they approximate the delta-function. The derivative of such functions increase monotonically (see Fig.4).

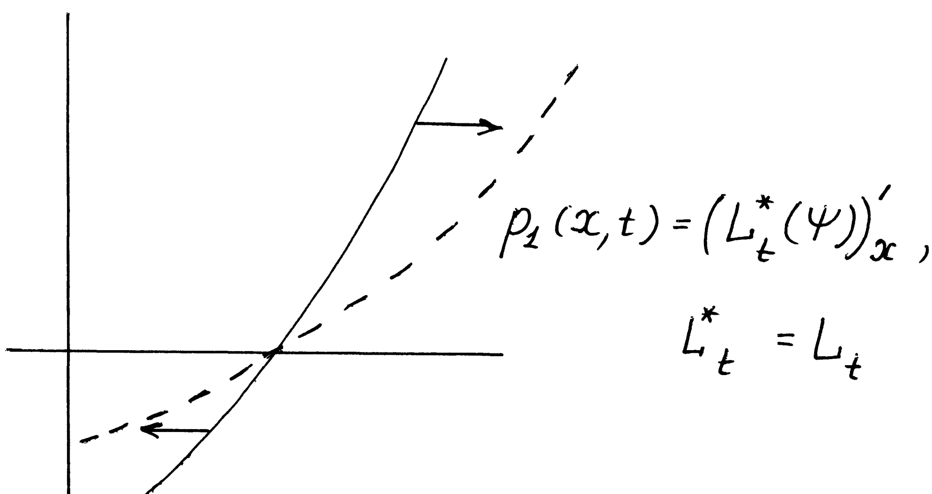

Fig.4

The adjoint operator to L_t equals L_t. Therefore, after a shift of the curve $p_1(x)$ along the Hamiltonian system, we always obtain the curve $p_1(x,t)$ which is projected diffeomorphically on the x-axis. So, $L_t^* \psi$ exists in large in the classical sense, thus it can be represented in the form

$$L_t^* \psi(x) = \inf_{\xi} \left(\frac{(x-\xi)^2}{2t} + \psi(\xi) \right) \tag{27}$$

We show, that this formula applies also in case of the initial condition $\varphi_0(x)$, if we understand the solution in the generalized sense, as it is done in the usual linear theory.
Really, we have

$$\left(L_t^* \psi(x), \varphi_0(x)\right) = \inf_x \left\{ \inf_\xi \left(\frac{(x-\xi)^2}{2t} + \psi(\xi) + \varphi_0(x)\right)\right\} =$$

$$= \inf_x \left\{ \psi(x) + \inf_\xi \left(\frac{(x-\xi)^2}{2t} + \varphi_0(\xi)\right)\right\} = \left(\psi(x), \inf_\xi \left\{ \frac{(x-\xi)^2}{2t} + \varphi_0(\xi)\right\}\right). \tag{28}$$

Thus, $\operatorname{Inf}_\xi \left\{ \frac{(x-\xi)^2}{2t} + \varphi_0(\xi) \right\}$ like in the linear theory, can be regarded as a weak solution of the Hamilton-Jacobi equation. We have interpreted the answer, obtained by Hopf.
Everything told here for this one-dimensional simple example holds for the more general Hamilton-Jacobi equation.

$$\frac{\partial S'}{\partial t} = H\left(\frac{\partial S}{\partial x}, x, t\right) = 0. \tag{29}$$

It turns out, that the resolving operator of the Hamilton-Jacobi equation is linear in the space of functions with values in the semi-ring given above. Namely, let $S_1(x,t)$ and $S_2(x,t)$ be solutions of equation (29) with the Hamiltonian H. Then

$$S(x,t) = \min\left\{ \left(S_1(x,t) + \lambda_1\right), \left(S_2(x,t) + \lambda_2\right)\right\} \tag{30}$$

is also a generalized solution of this equation.
One of the most important methods in the linear theory is that of Fourier transform. We consider its analogue in the space of functions with values in the semi-ring mentioned above. For this purpose we consider the eigenfunctions $\psi_\lambda(x)$ of the shift operator:

$$L\psi_\lambda(x) = \psi_\lambda(x+1) = \lambda \odot \psi_2(x) = \lambda + \psi_\lambda(x). \tag{31}$$

Therefore, $\psi_\lambda(x) = \lambda x$. As it is well known, the expansion in terms of eigenfunctions of the shift operator is just the Fourier transform. Thus we have:

$$\tilde{\varphi}(\lambda) = \int^{\oplus} \varphi(x) \odot \psi_\lambda(x) dx = \inf_x [\lambda x + \varphi(x)] . \tag{32}$$

Thus the Fourier in this space coincides with the well known Legendre transform, and the Legendre transform is "linear" in the space of functions with values in the semi-ring.
Now the following principal question arises. Since the Hamilton-Jacobi equation is linear in this space, one could look for difference schemes, which approximate these equations and are also linear in these spaces.
In order to write such a linear difference scheme, we shall use the analogy.
We consider a general pseudo-differential operator in the Hilbert space L_2, which is a difference operator with respect to t. It has the form

$$\psi(x, t+\Delta) = \mathcal{H}(\hat{p}, x, t)\psi = \int_V \{\mathcal{F}_{p\to x} \mathcal{H}(p, x, t)\} \psi(x-v) dv. \tag{33}$$

If we change the integral and the multiplication in this operator, as well as the Fourier transform, by the integral and by the Fourier transform in the new sense, then we obtain the following expression

$$S(x, t+\Delta) = \int^{\oplus}_{v\in\Omega} (S(x-v) \odot L(v, x, t)) dv = \min_{v\in\Omega} [S(x-v) + L(v, x, t)], \tag{34}$$

where L(v,x,t) is the Fourier transform of H(p,x,t). And it is just the general difference Bellman equation. Hence, if we take a grid with the step h in x as Ω, we obtain for the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} = \min_v \left(-v\frac{\partial S}{\partial x} + L(v, x, t)\right) \tag{35}$$

the difference scheme

$$S(x, t_j + h) = \min_{v_h \in \Omega_h} \{ S(x - hv_h, t) + hL(v_h, x, t)\}, \tag{36}$$

$$t_{j+1} = t + h$$

linear in the semi-ring with operations (13), (15).
The solution of Hamilton-Jacobi equation may be written in the form

$$S(x, t_j + h) = \min_{\xi}\{S(\xi, t_j) + G(x, \xi, t_j)\} =$$
$$= \min_{V_n \in \Omega} \{S(x - hV_n, t_j) + hL(V_n, x, t_j) + O(h^2)\}. \tag{37}$$

Explain now, what the measure in such strange integrals is. In particular, what functions differ on the set of zero measure in the sense of these new operations. First, for these new integrals, as well for usual ones, the following two properties hold:

Property 1.

$$\left(\int^{\oplus}_{x\in A} f(x)dx\right) \oplus \left(\int^{\oplus}_{x\in A} g(x)dx\right) = \int^{\oplus}_{x\in A} (f(x) \oplus g(x))\, dx. \tag{38}$$

Really,

$$\min\{\min_{x\in A} f(x), \min_{x\in A} g(x)\} = \min_{x\in A} \min\{f(x), g(x)\}.$$

Property 2.

$$\int^{\oplus}_{x\in A\cup B} f(x)\,dx = \int^{\oplus}_{x\in A} f(x)dx \oplus \int^{\oplus}_{x\in B} f(x)dx. \tag{39}$$

Really,

$$\min_{x\in A\cup B} f(x) = \min\{\min_{x\in A} f(x), \min_{x\in B} f(x)\}.$$

Let φ and ρ be two real functions on R^n, and ρ being upper semi-continuous. One can consider the integral of φ by measure $\rho(x)dx$:

$$\int^{\oplus}_{x\in A} \varphi(x) \odot \rho(x)dx = \min_{x\in A} [\varphi(x) + \rho(x)]. \tag{40}$$

We give an example of theorem about measure. Let the closure Cl of the function φ be defined by

$$(Cl\,\varphi)(x) = \sup\{\psi(x) \mid \psi \le \varphi,\ \psi \in C(R^n)\}. \tag{41}$$

Then

$$\inf(\varphi + \rho) = \inf(Cl\,\varphi + \rho), \tag{42}$$

and we obtain the following theorem: two functions are equivalent with respect to the measure ρ , if they have the same closure.
Explain now, what the weak convergence means with respect to the scalar product (17) given above.
Let X be a compact normal topological space, B be some subset of the set of lower semi-continuous real functions. We denote by P_B the projector on the linear envelope of the set B ; the kernel of P_B is given by

$$P_B\,\delta(x-\xi) = \sup_{\psi\in B}[\psi(x) - \psi(\xi)]. \tag{43}$$

Let $\{f_n(x)\}_{n=1}^{\infty}$ be an arbitrary sequence of real-valued functions, Λ be the set of all the continuous functions which are greater than all the functions of the sequence $\{P_B\, f_n\}_{n=1}^{\infty}$, starting from a certain number.
<u>Definition</u>. The lower bound of the set Λ will be called the "upper B-envelope" $\Phi_B\{f_n\}$ of the family $\{f_n\}_{n=1}^{\infty}$.
<u>Theorem</u>. For functions $\varphi \in B$

$$\lim_{n\to\infty}(f_n, \varphi) = (\Phi_B\{f_n\}, \varphi). \tag{44}$$

There is a notion in radiophysics. It is called a modulating signal baseband. It means, that the lower frequency is imposed on the high frequency. Namely, we have a rapidly oscillating function, which has the envelopes: the upper and the lower.
And just the lower envelope is in our sense a weak limit of the rapidly oscillating function.
For example, the sequence of functions

$$\varphi(x)\cdot\sin nx + \psi(x)\cdot\cos nx, \tag{45}$$

$\varphi(x)$, $\psi(x) \in C(R^1)$, weakly converges to the envelope $\sqrt{\varphi^2+\psi^2}$ as $n\to\infty$ in the semi-ring with operations $\oplus$ = inf, $\odot$ = + .
Thus we have the mathematical interpretation of the notion: modulating signal.

Another very simple example can demostrate the strength of the method described above. Just the problems of such type showed the author that it was necessary to construct this theory.
One may approach the linear theory of the wave equation starting from the problem of perturbation propagation in a crystal lattice. In the same way the problems of behaviour of computational medium with a great number of homogeneous elements leads to the conception presented here.
Again we consider an elementary example, namely, the medium activity equation for the so called Kung processor. We assume here, that the processor can be switched on only after the upper and the left processors were switched on. And it is switched on only under the condition, that the signal has come to it.
We shall set, that if the processor is ON, the state of it is equal to unity, and if the processor is OFF, the state of it is equal to zero. Thus, there are two states. Moreover, if the signal F is applied, it is unity, if the signal is not applied, it is zero.
So, the equation, which describes the state S of the processor (m,n) at time $k+1$, has the form:

$$S^{k+1}(m,n) = \min\{S^k(m-1,n), S^k(m,n-1), \mathcal{F}^k(m,n)\}, \quad (46)$$

$$S^0(m,n) = \begin{cases} 0, & m>0,\ n>0 \\ 1, & \text{otherwise} \end{cases}$$

It may be seen, that such an elementary equation has the solution: only unity or zero. Therefore, it seems at first sight, that there is no connection with differential equations, even if there are many processors. However, it is not so. We shall see, that the weak limit of a solution of such an equation (weak in the sense mentioned above) will converge to a solution of some Hamilton-Jacobi differential equation.
Note the following: the operator T on an integer-valued lattice, given by

$$TS(n,m) = \min\{S(n,m-1), S(n-1,m)\} \quad (47)$$

is linear in the space of functions on an integer-valued lattice with values in the semi-ring:

$$\oplus = \min, \quad \odot = +, \text{ or } \odot = \max .$$

Note, that this is just the operator, which is in the medium activity equation without the signal F . The adjoint operator T has the form:

$$T^{*}\bar{S}(n,m) = \min\left\{\bar{S}(n, m+1), \bar{S}(n+1, m)\right\} \tag{48}$$

As a matter of fact we have equation (46) which still has the signal F , by analogy to the right-hand side of the differential equation. We can get rid of the right-hand side just in the same way as it is done in the usual linear theory. We can send the right-hand side into the initial conditions according to the Duhamell principle, and obtain instead of (46) with initial conditions

$$S^{0}(m, n) = 0 \tag{49}$$

another equation:

$$\begin{cases} W_{t+1,\tau}(m, n) = \min\left\{W_{t,\tau}(m-1, n), W_{t,\tau}(m, n-1)\right\} \\ W_{t,\tau}\Big|_{t=\tau} = \mathcal{F}^{(\tau)}(m, n); \qquad t \geqslant \tau \end{cases} \tag{50}$$

Besides

$$S^{k}(m, n) = \min_{0 \leq \tau \leq k} W_{k,\tau}(m, n) = \int\limits_{0 \leq \tau \leq k}^{\oplus} W_{k,\tau}\, d\tau . \tag{51}$$

Now we consider the adjoint medium activity equation

$$\bar{S}^{k+1}(m, n) = \min\left\{\bar{S}^{k}(m+1, n), \bar{S}^{k}(m, n+1)\right\} \tag{52}$$

We note the following: the sense of it is not so closely connected with the processors. And we can set for this equation not necessary two-valued initial conditions (zero, unity):

$$\bar{S}^{k}(m, n)\Big|_{k=0} = f_{0}(m, n) . \tag{53}$$

We shall give the initial data which vary very slowly from one point to another. Namely, we shall consider on a three-dimensional grid in the space (t,x,y) with the step h such functions, which are the traces on this grid of a smooth function $u_h(t,x,y)$:

$$u_h(t,x,y)\Big|_{\substack{t=kh\\ x=mh\\ y=nh}} = \bar{S}^k(m,n) \tag{54}$$

Then the adjoint equation will have the following form:

$$u_h(t+h,x,y) = \min\{u_h(t,x+h,u),\ u_h(t,x,y+h)\}. \tag{55}$$

If we take for this equation smooth initial data, then as $h \longrightarrow 0$ it will be reduced since

$$u_h(t,x+h,y) = u_h(t,x,y) + h\frac{\partial u}{\partial x} + O(h^2), \tag{56}$$

to a nonlinear differential equation

$$\frac{\partial u}{\partial t} = \min\left\{\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\right\} \tag{57}$$

Thus we know the convergence of adjoint operators. Further, we can act like in the linear theory. Namely, we apply the resolving operator to the initial function and multiply it scalarly by a smooth function. Then we apply the adjoint operator to the smooth function and pass to the limit. In this way we obtain the weak limit of the solution of our equation. In order to make the analogy more clear, we write the integrals in the new sense instead of infimum.
According to the Duhamell principle the solution of the medium activity equation can be represented in the form:

$$\int_{[0,t]}^{\oplus} \mathcal{L}_{t-\tau}^{(h)}\ \mathcal{F}(x,y,\tau)\,d\tau \tag{58}$$

where $\mathcal{L}_t^{(h)} f$ is the solution of the homogeneous problem with initial condition $f(x,y)$.

The usual technique leads to the following:

$$\iint^{\oplus} dx\,dy \int^{\oplus}_{[0,t]} \mathcal{L}^{(h)}_{t-\tau}\, \mathcal{F}(x,y,\tau)\,d\tau \odot \varphi(x,y) =$$

$$= \int^{\oplus}_{[0,t]} d\tau \iint^{\oplus} \mathcal{F}(x,y,\tau) \odot \mathcal{L}^{*(h)}_{t-\tau}\, \varphi(x,y)\,dx\,dy \xrightarrow[h\to 0]{} \tag{59}$$

$$\longrightarrow \int^{\oplus}_{[0,t]} d\tau \iint^{\oplus} \mathcal{F}(x,y,\tau) \odot \mathcal{L}^{*(h)}_{t-\tau}\, \varphi(x,y)\,dx\,dy .$$

Here the x and y integrals are over the entire space, $\varphi(x,y)$ is a smooth function and $\mathcal{L}^{*}_{t}$ is the resolving operator for the limit problem (57).

Thus, if we take the solution $S^k(m,n)$ of the medium activity equation and consider a smooth function $\varphi(x, t, y)$, which has the values $S^k(m,n)$ at the points $t = kh$, $x = mh$, $y = nh$, its weak limit in the new sense on the smooth function $\varphi(x,y)$ is expressed in terms of the solution of problem (57) with the initial condition:

$$U\Big|_{t=\tau} = \varphi(x,y). \tag{60}$$

The main ideas of the author's theory are described by using only elementary examples. We did not give here general theorems, in particular, the existence theorems for the solutions of quasi-linear equations. They can be found in [1] - [3] .

Further, we considered only the semi-ring, in which the addition operation is minimum, and the multiplication operation is summing. In [1] another semi-rings are considered in detail. For example, the semi-rings, in which the addition is minimum, and the multiplication is maximum, namely, the space with values in the Boolean algebra is considered.

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