

# SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

E. M. STEIN

## **Some oscillatory integrals and their applications**

*Séminaire Équations aux dérivées partielles (Polytechnique)* (1984-1985), exp. n° 23,  
p. 1-5

[http://www.numdam.org/item?id=SEDP\\_1984-1985\\_\\_\\_\\_A23\\_0](http://www.numdam.org/item?id=SEDP_1984-1985____A23_0)

© Séminaire Équations aux dérivées partielles (Polytechnique)  
(École Polytechnique), 1984-1985, tous droits réservés.

L'accès aux archives du séminaire Équations aux dérivées partielles (<http://sedp.cedram.org>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

ÉCOLE POLYTECHNIQUE  
CENTRE DE MATHÉMATIQUES  
91128 PALAISEAU CEDEX - FRANCE

---

Tél. (6) 941.82.00 - Poste N°  
Telex · ECOLEX 691 596 F

S E M I N A I R E   B O N Y - S J Ö S T R A N D - M E Y E R   1 9 8 4 - 1 9 8 5

SOME OSCILLATORY INTEGRALS AND THEIR APPLICATIONS

par E.M. STEIN



SOME OSCILLATORY INTEGRALS AND THEIR APPLICATIONS

E.M. STEIN

We shall discuss several kinds of oscillatory integrals, and we describe briefly their applications to some questions related to the  $\bar{\partial}$ -Neumann problem, real-variable problems connected with certain maximal functions, and analysis on nilpotent Lie groups. The three kinds of oscillatory integrals we shall treat are in appearance quite different from each other, but in reality the ideas used to deal with each have much in common. What follows describes some joint work done with D.H. PHONG, C.D. SOGGE and F. RICCI.

### 1. OSCILLATORY INTEGRALS CONNECTED WITH RADON SINGULAR INTEGRALS.

For further details about the material of this section see the work of PHONG and the author [3], [4].

We place ourselves in  $\mathbb{R}^n$ , and we are given a real  $C^\infty$  phase function  $\Phi(x,y)$ , defined on  $\mathbb{R}^n \times \mathbb{R}^n$ .  $K(x)$  will denote a function in  $C^\infty(\mathbb{R}^n/\{0\})$ , which is homogeneous of degree  $-\mu$ ,  $0 < \mu \leq n$ . When  $\mu = n$ , we also demand that the mean-value of  $K$  on the unit sphere vanishes. We will also be given a fixed cut-off function,  $\Psi(x,y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ . With these and for each  $\lambda > 0$ , we form the family of transformations  $T_\lambda$ , defined by :

$$(1) \quad T_\lambda(f)(x) = \text{P.V.} \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} \Psi(x,y) K(x-y) f(y) dy$$

Theorem 1. Suppose that the rank of the Hessian of  $\Phi (= \{\frac{\partial^2 \Phi}{\partial x_i \partial y_j}(x,y)\})$ , is not less than  $k$ ,  $1 \leq k \leq n$ , and  $n-k < \mu \leq n$ . Then  $\|T_\lambda\|$ , the norm of  $T_\lambda$  as a bounded operator on  $L^2(\mathbb{R}^n)$  to itself, satisfies

$$\|T_\lambda\| \leq A(1+\lambda)^{-(n-\mu)/2}.$$

Note that in the limiting case (when  $k = n$ , and  $\mu \rightarrow 0$ ), and when  $K(x)$  is taken to be  $\equiv 1$ , then  $T_\lambda$  corresponds to a variant of the Fourier transform which was studied by CARLESON-SJOLIN and HÖRMANDER [2]. The meaning of this theorem is that the different oscillations due respectively to  $K(x)$  and  $e^{i\lambda\Phi(x,y)}$  do not interfere, but in fact reinforce each other as concerns their beneficial effects.

The significance of the operators  $T_\lambda$  defined by (1) is that they occur as symbols of certain pseudodifferential operators - the singular Radon integrals - which themselves play a role in to  $\bar{\partial}$ -Neumann problem.

## 2. OSCILLATORY INTEGRALS CONNECTED WITH FOURIER TRANSFORMS OF SURFACE-CARRIED MEASURES.

For further details about the work described in this section see the forthcoming paper of SOGGE and the author [6].

Let  $S$  denote a  $C^\infty$  hypersurface in  $\mathbb{R}^n$ . We shall denote by  $d\sigma(x)$  the induced (Lebesgue) measure on  $S$ , and for each  $x \in S$ ,  $K(x)$  will denote the Gaussian curvature of  $S$  at  $x$ . We fix a cut-off function  $\psi$ , with  $\psi \in C_0^\infty(S)$ .

### Theorem 2.

$$\int_S e^{ix \cdot \xi} (K(x))^{2n-2} \psi(x) d\sigma(x) = O(|\xi|^{-(n-1)/2})$$

as  $|\xi| \rightarrow \infty$ .

Note that when  $K(x)$  is non-vanishing in the support of  $\psi$ , (then the factor  $(K(x))^{2n-2}$  is irrelevant), this is a classical result of VAN DER CORPUT when  $n = 2$ , and HLAWKA for  $n \geq 3$ . These earlier results were used to study the number of lattice points contained in dilates of  $S$ . The present result can be used in the study of the maximal function

$$\mathcal{M}(f)(x) = \sup_{0 < t < \infty} \left| \int_S f(x-ty) \psi(y) d\sigma(y) \right|.$$

It can be shown that whenever  $S$  is such that its curvature  $K(x)$  does not vanish of infinite order at any  $x \in S$ , then there is an  $L^p$ -inequality

$$(2) \quad \|\mathcal{M}(f)\|_{L^p(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)},$$

for a range of  $p$ ,  $p_0 < p \leq \infty$ , with  $p_0 = p_0(S)$ , if also  $n \geq 3$ . Earlier inequalities of this type for the sphere, or for  $S$  with nowhere vanishing curvature are in [8] and [1]. The maximal inequality (2) holds in particular when  $S$  is compact and real-analytic.

### 3. SOME OTHER OSCILLATORY INTEGRALS.

Finally we mention some results obtained jointly with F. RICCI [5], which have some connection with theorem 1 and also with some questions in harmonic analysis of nilpotent Lie groups.

We let  $K(x)$  denote a function in  $C^\infty(\mathbb{R}^n/\{0\})$ , which is homogeneous of degree  $-n$ , with vanishing mean-value on the unit sphere. We denote by  $P(x,y)$  a real polynomial on  $\mathbb{R}^n \times \mathbb{R}^n$  of total degree  $\leq d$ , and consider the operator  $T$  defined by

$$(3) \quad (Tf)(x) = \text{P.V.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y) dy$$

Theorem 3. *The operator  $T$  given by (3) is bounded on  $L^2(\mathbb{R}^n)$  to itself, with a bound which is independent of the polynomial  $P$ , as long as  $P$  is restricted to be of degree  $\leq d$ , (here  $d$  is arbitrary).*

There are also similar results when one replaces  $\mathbb{R}^n$  by any homogeneous nilpotent group, with  $K(x-y)$  replaced by  $K(x.y^{-1})$ , where  $K$  has the critical degree of homogeneity, together with the vanishing of its mean-value on the unit sphere. Also the  $L^p$  results hold, when  $1 < p < \infty$ .

The special case of (3) when  $P(x,y)$  is a bilinear form in  $x$  and  $y$ , (an operator having close relation to (1)) is dealt with in [3]. Another special case, when  $P(x,y) = P(x-y)$ , i.e. when the operator is translation invariant, is in [7].

REFERENCES

- [1] A. GREENLEAF, "Principal curvature in harmonic analysis".  
Ind. Univ. J. 30 (1981), 519-537.
- [2] L. HÖRMANDER, "Oscillatory integrals and multipliers on  $F L^p$ ".  
Ark. Mat. 11 (1973), 1-11.
- [3] D.H. PHONG and E.M. STEIN, "Singular integrals related to the Radon transform and boundary value problems".  
Proc. Nat. Acad. Sci. U.S.A. 80 (1983), 7697-7701.
- [4] D.H. PHONG and E.M. STEIN, "Hilbert integrals, singular integrals and Radon transforms".  
To appear in Acta Math.
- [5] F. RICCI and E.M. STEIN, Manuscript in preparation.
- [6] C.D. SOGGE and E.M. STEIN, "Averages of functions over hypersurfaces in  $\mathbb{R}^n$ ".  
To appear in Invent. Mat.
- [7] E.M. STEIN, "Oscillatory integrals in Fourier analysis".  
To appear in "Beijing lectures on harmonic analysis", Annals of Math. Studies.
- [8] E.M. STEIN and S. WAINGER, "Problems in harmonic analysis related to curvature".  
B.A.M.S. 84 (1978), 1239-1295.