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A UNIQUE CONTINUATION THEOREM FOR THE WAVE EQUATION IN THE EXTERIOR OF A CHARACTERISTIC CONE

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1. The Cauchy problem for the wave equation

$$(1) \qquad Pu \equiv (\partial_n^2 - \partial_1^2 - \ldots - \partial_{n-1}^2)u = 0$$

is not well posed if $n \geqslant 3$ and Cauchy data are given on a time-like hypersurface, for instance on $\{x_1 = 0\}$. But if the solution of such a Cauchy problem exists, then it is unique by Holmgren's theorem. Furthermore, it was shown by John [5] that one then has unique continuation of the data : if $u = \partial_1 u = 0$ on a relatively open subset $\Omega \subset \{x_1 = 0\}$, then in general u and $\partial_1 u$ also vanish on a strictly larger relatively open subset $\hat{\Omega} \subset \{x_1 = 0\}$, and one has $u = 0$ on a neighbourhood of $\hat{\Omega}$ in $\mathbb{R}^n$.

Similarly, uniqueness and unique continuation theorems can be proved for improperly posed characteristic Goursat problems in $\mathbb{R}^n$, with $n \geqslant 3$, such as

$$(2) \qquad \begin{cases} Pu = 0 \quad \text{on} \quad \{x : 0 < |x_n| < x_1\}, \\ u \text{ given on } \{x, x_n = x_1 \geqslant 0\} \quad \text{and on } \{x : x_n = -x_1 \leqslant 0\} \ ; \end{cases}$$

see [2] .

In this paper, the corresponding boundary value problem for (1) will be considered in the exterior of a characteristic double cone, say in $X = \{x : x_n^2 < x_1^2 + \ldots + x_{n-1}^2\}$. Let Ω be a relatively open connected subset of $\Gamma = \partial X$ which contains the vertex $\{x = 0\}$, and suppose that $u|\Omega = 0$. It turns out that, in order to ensure that $u = 0$ on a neighbourhood of Ω in X, it is both necessary and sufficient to make the additional hypothesis that u vanishes to all orders as $x \to 0$ in X. (Compare [1] for another unique continuation theorem involving flatness of the solution at the manifold carrying the data). One then also has unique continuation of the data, which can be summarized by saying that uniqueness propagates along the bicharacteristics (the generators of Γ), towards the vertex. Similar results can be established locally for hyperbolic equations of the second order with real analytic coefficients.

It would be feasible to use the results of [5] to analyze the problem. But it is simpler to appeal to the partial analytic hypoellipticity of P, which results from Theorems 4.1 and 5.1 of [4] . This was pointed out to me by L. Hörmander, in connection with problem (2) and I want to thank him for several illuminating conversations.

Throughout we shall work in $\mathbb{R}^n$, where $n \geqslant 3$. Points $x, y\ldots$ in $\mathbb{R}^n$ will be written as (x', x_n), $(y', y_n), \ldots$, where $x', y', \ldots \in \mathbb{R}^{n-1}$. The usual inner product and norm in $\mathbb{R}^{n-1}$ are denoted by $x'.y' = x_1 y_1 + \ldots + x_{n-1} y_{n-1}$ and by $|x'| = (x'.x')^{1/2}$, respectively. In $\mathbb{R}^n$, we also introduce a Lorentzian inner product,

(3) $$\langle x,y \rangle = x_n y_n - x'y' = x_n y_n - \sum_{j=1}^{n-1} x_j y_j ,$$

and write

(4) $$\langle x,x \rangle = \|x\|^2 = x_n^2 - |x'|^2 .$$

The bicharacteristics of P (the d'Alembertian) are

(5) $$x = y + \xi t , \qquad 0 \neq \xi = \text{constant} , \qquad \langle \xi,\xi \rangle = 0 ,$$

where $y \in \mathbb{R}^n$ is fixed, and $t \in \mathbb{R}$. Their projections on $\mathbb{R}^n$ will be called bicharacteristic curves, although they are actually straight lines. The union of all bicharacteristic curves through the origin are the generators of the characteristic double cone

(6) $$\Gamma = \{x : \langle x,x \rangle = 0\}$$

This, in turn, is the union of the two characteristic cones of P at O,

(7) $$\Gamma^+ = \{x : x_n = |x'| \geqslant 0\} , \quad \Gamma^- = \{x : x_n = -|x'| \leqslant 0\} .$$

Note that Γ is a topological space with the topology induced by that of $\mathbb{R}^n$. We set

(8) $$X = \{x : \langle x,x \rangle < 0\};$$

this is usually called the exterior of Γ.

We shall also need the forward and backward dependence domains of a point $y \in \mathbb{R}^n$,

(9) $D^+(y) = \{x : x_n > y_n + |x' - y'|\}$, $D^-(y) = \{x : x_n < y_n - |x' - y'|\}$,

and put

(10) $$D(y,z) = D^+(y) \cap D^-(z)$$

if $z \in D^+(y)$; these notations extend in the obvious way to subsets of $\mathbb{R}^n$, for example $D^+(\Sigma) = \cup \{D^+(y) : y \in \Sigma\}$. We can now state the main result of this paper.

Theorem 1 : Let Ω be a relatively open connected subset of Γ which contains the vertex $0 = \{0\}$, and put $\Omega^+ = \Omega \cap \Gamma^+$, $\Omega^- = \Omega \cap \Gamma^-$. Let $u \in C^2(X) \cap C^0(\bar{X})$, and suppose that

(i) $Pu = 0$ on X ;

(ii) $u = 0$ on Ω ;

(iii) for each integer $N \geq 0$ there are constants $C_N > 0$, $\delta_N > 0$ such that $|u(x)| \leq C_N|x|^N$ when $x \in X$ and $|x| \leq \delta_N$. Then

(a) $u = 0$ on $\hat{\Omega}$, the union of all bicharacteristic curves from the vertex 0 to points of Ω , and

(b) $u = 0$ on $D(\Omega^-, \Omega^+)$.

Remark : By definition, $D(\Omega^-,\Omega^+)$ is the union of all $D^+(y) \cap D^-(z)$ where $y \in \Omega^-$ and $z \in \Omega^+$; it is easy to show that this is not empty, and that it is the same as $D(\hat{\Omega}^-,\hat{\Omega}^+)$.

The theorem states that the conditions (i), (ii) and (iii) are sufficient for the validity of (a) and (b). To show that they are also necessary we give an example, with $\Gamma = \Omega$, in which (i) and (ii) hold, (iii) is violated, and $u \not\equiv 0$ on X. Let $H_m(x')$ be a homogeneous harmonic polynomial ($\Delta' H_m = 0$ in $\mathbb{R}^{n-1}$), of degree $m \geq 2$, and define

$$u = r^{3-m-n}(r^2 - x_n^2)^{m+\frac{1}{2}n-2} H_m(x'/r), \qquad x \in \bar{X} \setminus \{0\},$$

$$u = 0, \quad x = 0 ,$$

where $r = |x'|$. One can check by computation that $Pu = 0$ on X. It is clear that $u \in C^2(X) \cap C^0(X \setminus \{0\})$. Also ,

$$|u(x)| \leqslant C\, r^{3-m-n}\, r^{2m+n-4} = C|x'|^{m-1}\ ,\ C = \sup\{H_m(\theta')|: |\theta'| = 1\}.$$

Hence $u \in C^0(\bar{X})$ as we have taken $m \geqslant 2$, so we have a solution $u \equiv 0$ of $Pu = 0$ of class $C^2(X) \cap C^0(\bar{X})$ which vanishes on Γ. Of course, (iii) does not hold.

To state the corresponding theorem for equations with variable coefficients, some preliminary remarks are needed. (See for example [3] for a more detailed account). Let

$$P(x,\partial) = \sum_{j,k=1}^{n} a_{jk}(x)\partial_j\,\partial_k + \sum_{j=1}^{n} b_j(x)\partial_j + c(x)\ ,\quad a_{jk} = a_{kj}\ , \tag{11}$$

be a differential operator defined on an open set $M \subset \mathbb{R}^n$, where $n \geqslant 3$. We may as well assume from the outset that the a_{jk}, b_j and c are real valued real analytic functions, and that M is simply connected. Suppose that the quadratic form

$$\mathbb{R}^n \in \xi \longmapsto \sum_{j,k=1}^{n} a_{jk}(x)\ \xi_j\ \xi_k$$

is non degenerate and has signature $2-n$ for all $x \in M$, so that P is a hyperbolic operator. It is helpful to make M into a Lorentzian manifold by equipping it with the (pseudo-riemannian) metric

$$\langle v,v\rangle_x = \sum_{j,k=1}^{n} g_{jk}(x)v_j v_k\ ,\quad (x,v) \in TM \cong M \times \mathbb{R}^n \tag{12}$$

where the matrix (g_{jk}) is the inverse of the matrix (a_{jk}). We shall assume that M is such that any two distinct points of M are joined by a unique geodesic of this metric, in M; we denote this (oriented) geodesic by $\gamma(y,x)$ where $x, y \in M$ and $x \neq y$. The tangent vectors V at $x \in M$ are called space-like if $\langle v,v\rangle_x < 0$, time-like if $\langle v,v\rangle_x > 0$, and null if $\langle v,v\rangle_x = 0$. This classification carries over to curves and in particular to geodesics (on which $\langle v,v\rangle$ is constant with the usual affine parametrization). The time-like vectors constitute an open subset of TM which consists of two connected components. We label the members of one of these as forward (or future-directed) and those of the other one as backward (or past directed).

If y is a point of M, we set

$$\begin{cases} D^+(y) = \{x \in M : \ \gamma(y,x) \text{ is time-like and forward}\}, \\ D^-(y) = \{x \in M : \ \gamma(y,x) \text{ is time-like and backward}\}\ , \end{cases} \tag{13}$$

and we set

(14) $$D(y,z) = D^+(y) \cap D^-(z)$$

one has $D(y,z) \neq \emptyset$ if $z \in D^+(y)$ or equivalently, $y \in D^-(z)$. We now make the further hypothesis that

(15) $$D(y,z) \subset\subset M \ ;$$

one can then call M a causal domain.(Every point of a Lorentzian manifold has a causal neighbourhood).

The null geodesics of the metric are the bicharacteristic curves of P. (The bicharacteristics are the null geodesics, lifted to T^*M). So

(16) $$\Gamma^+(y) = \partial D^+(y) \quad , \quad \Gamma^-(y) = \partial D^-(y)$$

are the two characteristic cones with vertex y. Let us assume also that M contains the origin $\{x = 0\}$; this entails no loss of generality. Write

(17) $$\Gamma^+ = \Gamma^+(0) \quad , \quad \Gamma^- = \Gamma^-(0) \ , \ \Gamma = \Gamma^+ \cup \Gamma^-$$

and set

(18) $$X = \{x \in M : \gamma(y,X) \text{ is space-like}\} = M \setminus (\overline{D^+(0)} \cup \overline{D^-(0)}).$$

With these hypotheses and conventions, one has

Theorem 2 : If the hyperbolic operator P has real analytic coefficients, and M is causal with respect to the associated Lorentzian manifold with metric (12), then Theorem is valid, provided that X, Γ and D(y,z) are understood to be defined by (18), (17) and (19) respectively.

Note that in hypothesis (iii), $|x|$ can be replaced by some equivalent distance fom O, for instance one derived from an auxiliary Riemannian metric on M.

2. The proof of Theorem 1 will now be given in outline. First, some lemmas :

Lemma 1 : Set

(19) $$\tilde{X}_\Omega = (\mathbb{R}^n \setminus \Gamma) \cup \Omega$$

and define

(20) $$\tilde{u} = u \quad \text{on } X \quad , \tilde{u} = 0 \text{ on } \tilde{X}_\Omega \setminus X \ .$$

Then the (continuous) function $\tilde{u}$ satisfies $P\tilde{u} = 0$ when regarded as a member of $\mathcal{D}'(\tilde{X}_\Omega)$.

Proof : If $\phi \in C_c^\infty(\tilde{X}_\Omega)$ then (as $Pu = 0$ in X)

$$(P\tilde{u}, \phi) = (\tilde{u}, P\phi) = \lim_{\varepsilon \downarrow 0} \int_{X_\varepsilon} (uP\phi - \phi Pu)\, dx$$

where $X_\varepsilon = \{x : |x'| > \varepsilon, |x_n| < |x'| - \varepsilon\}$. Applying the divergence theorem, and then removing the derivatives of u by partial integration (on bicharacteristic generators of ∂X_ε), one obtains $(P\tilde{u}, \phi) = 0$, $\phi \in G^\infty(\tilde{X}_\Omega)$, as claimed

Lemma 2 :

(21) $$WF_A(\tilde{u}) \in \text{char } P = \{(x,\xi) \in \tilde{X}_\Omega \times \mathbb{R}^n \setminus 0 : \langle\xi,\xi\rangle = 0\}.$$

Proof : WF_A is the analytic wave front set of u. The lemma follows from Lemma 1 and Theorem 5.1 of [4] .

Lemma 3 : There is a $\delta > 0$ such that $u = 0$ on

(22) $$X_\delta = \{x : |x_n| < |x'| < \delta\}.$$

Proof : As Ω is open in Γ and contains the vertex, one has

(23) $$\Omega_\delta = \{x : |x_n| = |x'| < \delta\} \subset \Omega$$

if $\delta > 0$ is sufficiently small. Clearly, $\{x_i : |x'| < \delta\} \subset \tilde{X}_\Omega$. One can now easily deduce from Theorem 4.1 of [4] that

$$F_p(x') = (\tilde{u}(x',x_n), p(x_n)) = \int_{-|x'|}^{|x'|} u(x',x_n)p(x_n)dk_n ,$$

where p is (say) a polynomial, is real analytic on $\{|x'| < \delta\}$.

(Here, the distribution pairing refers to the pullback of $\tilde{u}$ under the map $x_n \mapsto (x',x_n)$ for fixed x !) It follows from hypothesis (iii) that F_p has a zero of infinite order at $x' = 0$. Hence $F_p = 0$ on $\{|x'| < \delta\}$; as p may be any polynomial and u is continuous, the lemma follows.

<u>Proof of part (a) of Theorem 1</u> : Let $\Lambda = \{\theta' \in \mathbb{R}^{n-1} : |\theta'| < 1/\sqrt{2}\}$. With δ as in Lemma 3, take a fixed $\varepsilon \in (0,2\delta)$ and set

$$(24) \qquad f(\theta',r) = (r\theta',-\varepsilon + r(1-|\theta'|^2)^{\frac{1}{2}}), \quad \theta' \in \Lambda \quad , \quad r \in \mathbb{R}^+ .$$

Then $f : \Lambda \times \mathbb{R}^+ \to D^+(0,-\varepsilon)$ is a real analytic diffeomorphism. (The curves $\mathbb{R}^+ \ni r \mapsto f$ are just the forward time-like half-lines from $(0,-\varepsilon)$.) Let Σ be the projection of $f^{-1}(\Omega^+)$ on Λ . Clearly, Ω^+ is connected; so Σ is a connected open subset of Λ . It readily follows from Lemma 3 that $f(\Sigma \times \mathbb{R}^+) \subset \tilde{X}_\Omega$. One also easily checks that $f^*\tilde{u} \mid \Sigma \times \mathbb{R}^+$ satisfies the hypotheses of Theorem 4.1 of [4]. Hence

$$F_p(\theta') = (f^*\tilde{u},p) = \int f^*\tilde{u}(\theta',r)p(r)dr , \quad \theta' \in \Sigma$$

is real analytic when p is a polynomial. Lemma 3 also implies that $F_p = 0$ on a neighbourhood of $\theta' = 0$ (the projection of $f^{-1}(\Omega^+)$ on Λ). Hence $F_p = 0$ on Σ , and one deduces that $u = 0$ on $f(\Sigma \times \mathbb{R}^+)$.

In particular, if y is a point of Ω^+, then $\tilde{u} = 0$ on the straight segment with end points $(0,-\varepsilon)$ and y. This being so for any $\varepsilon \in (0,2\delta)$ it follows by continuity that $u = 0$ on the bicharacteristic segment from the vertex to y (make $\varepsilon \downarrow 0$).

Hence $u = 0$ on Ω^+. Repeating the argument, with Ω^+ and ε replaced by Ω^- and $-\varepsilon$, respectively, one finds that $u = 0$ on Ω^- too, and so part (a) of Theorem 1 is proved.

For the proof of part (b) one can now assume from the outset that

$$\hat{\Omega} = \Omega \tag{25}$$

Two more lemmas are needed.

<u>Lemma 4</u> : Let $\bar{y}$ and $\bar{z}$ be points in Ω^- and in Ω^+ respectively, not on the same generator of Γ , and set

$$y = (\bar{y}' , -|\bar{y}'| - \varepsilon), \ z = (\bar{z}' , |z'| + \varepsilon), \tag{26}$$

where $\varepsilon > 0$. Then there is an $\varepsilon_o > 0$ such that $D(y,z) \subset \tilde{X}_\Omega$ if $0 < \varepsilon < \varepsilon_o$.

<u>Proof</u> : Since $\Gamma^- \subset D^-(z)$, one has $D^-(z) \cap \Gamma = D^-(z) \cap \Gamma^+$. A simple geometrical arguments shows that $D^-(z) \cap \Gamma^+$ is a relatively open neighbourhood, in Γ , of the bicharacteristic segment γ with end points 0 and $\bar{y}$, and that it shrinks to γ as $\varepsilon \downarrow 0$. So it follows from (25) that $D^-(z) \cap \Gamma^+ \subset \Omega^+$ for $0 < \varepsilon < \varepsilon_1$, say. Similarly there is an $\varepsilon_2 > 0$ such that $D^+(y) \cap \Gamma = D^+(y) \cap \Gamma^- \subset \Omega^-$ if $0 < \varepsilon < \varepsilon_2$, and the lemma follows if one takes $\varepsilon_o = \min(\varepsilon_1,\varepsilon_2)$.
<u>Remark</u> : If $\bar{y}$ is a point of Γ^- and $\bar{z}$ is a point of Γ^+ then $D(\bar{y},\bar{z})$ is not empty, unless $\bar{y}$ and $\bar{z}$ are on the same (bicharacteristic) generator of Γ . So the lemma actually holds trivially in the excluded case.

<u>Lemma 5</u> : Let y and z be points in $\mathbb{R}^n$, such that $D(y,z)$ is not empty. Then there exists a real analytic diffeomorphism

$$\mathbb{R}^{n-1} \times \mathbb{R} \ni (w',t) \longmapsto f(w',t) = x \in D\,(y,z)$$

such that (i) $\|\partial_t f\|^2 > 0$ and (ii) for any point $x_o \in D(y,z)$, the projection $\{(w',t) : \|f\,w',t) - x_o\|^2 \leqslant 0\} \to \mathbb{R}^{n-1}$ is proper.

<u>Proof</u> : A relatively simple way to construct such a diffeomorphism is to take a vector field v which is time-like on $D(y,z)$, and to integrate the differential equation

$$\frac{\partial x}{\partial t} = v(x) \quad , \qquad x \in D(y,z). \tag{27}$$

For example, one can take

$$v(x) = \frac{\|z-x\|^2(x-y) + \|x-y\|^2(z-x)}{\|z-y\|^2} \tag{28}$$

For one then has

$$v(x)^2 = \frac{\|z-x\|^2 \ \|x-y\|^2}{\|z-y\|^2} > 0 \ , \quad x \in D(y,z)$$

With this choice of v, explicit integration of (27) is possible, and gives

$$f(w',t) = y + \frac{\|z-y\|^2 (w + (z-y)e^t)}{2\langle w, z-y\rangle + \|z-y\|^2(e^t + e^{-t})} \ ,$$

where

$$w = (w', (|w'|^2 + \|z-y\|^2)^{1/2}$$

The assertions of the lemma can then be verified directly.

<u>Proof of part (b) of Theorem 1</u> : Let $\bar{y} \in \Omega^-$, $\bar{z} \in \Omega^+$ be points such that $D(\bar{y},\bar{z}) \neq \emptyset$, and let y and z be as in Lemma 4, with $0 < \varepsilon < \varepsilon_0$ so that $D(y,z) \subset \tilde{X}_\Omega$. With f as in Lemma 5, it follows from Theorem 4.1 of [4] that

$$F_p(w') = (f^*\tilde{u}, p) = \int f^*\tilde{u}(w',t)p(t)\,dt \ , \ w' \in \mathbb{R}^{n-1} \ ,$$

is a real analytic function if p is a polynomial. Now, if $f^{-1}(\{0\}) = (w_0', t_0)$, then the curve $\mathbb{R} \ni t \mapsto x = f(w_0',t)$ is time like and goes through the vertex of Γ . It is thus disjoint from $\bar{X}\setminus\{0\}$ and <u>a fortiori</u> disjoint from $\{x : |x'| = \delta$, $|x_n| \leqslant \delta\}$, where δ is as in Lemma 3. The same lemma and a simple continuity argument then imply that $F_p(w') = 0$ on a neighbourhood of w_0' : hence $F_p \equiv 0$ and one concludes that $\tilde{u} = 0$ on D(y,z) by letting p range over all polynomials. But $D(\bar{y},\bar{z}) \subset D(y,z)$, so u = 0 on $D(\bar{y},\bar{z})$, and so we are done.

For equations with variable coefficients, that is to say for Theorem 2, the proof is along the same lines. In fact, Lemmas 1, 2 and 3 are proved virtually as above, and this is also the case for part (a). (The straight lines from $(0,-\varepsilon)$ are replaced by time-like geodesics from a point in $D^-(0)$ which is ultimately made to tend to $\{0\}$). More is needed for part (b). In particular, one has to rework Lemma 5. A possible choice for the vector field v is

$$v_i(x) = S(x,y,z) \sum_{j=1}^{n} (a_{ij}(x)(T(x,z)\frac{\partial T(x,y)}{\partial x_j} - T(x,y)\frac{\partial T(x,z)}{\partial x_j}),$$

$$i = 1,\ldots,n,$$

where $T(x,y)$ is the square of the geodesic distance of any two points y and x in M, and S (> 0) can for example be chosen such that $T(x,y)/T(x,z) = e^{2t}$. (Then v reduces to (28) if P is the d'Alembertian). The equation (27) can no longer be integrated explicitely. So one has to use what are essentially routine arguments in the theory of ordinary differential equations to establish a general version of the lemma. The final step in the proof of (b) then carries over.

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