Séminaire Équations aux dérivées partielles – École Polytechnique

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Séminaire Équations aux dérivées partielles (Polytechnique) (1978-1979), exp. n° 3, p. 1-17

<http://www.numdam.org/item?id=SEDP_1978-1979____A3_0>

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BOUNDARY VALUE PROBLEMS AS LIMITS OF PROBLEMS IN ALL SPACE

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Exposé n° III

24 Octobre 1978

§ 1. The propagation of waves in a domain $\Omega \subset \mathbb{R}^n$ is usually described by a differential equation, $Lu = 0$ on $[0,T] \times \Omega$ and a boundary condition $Bu = 0$ on $[0,T] \times \partial\Omega$ which describes the interaction of waves with the boundary. In fact, we are so used to thinking in these terms that we do not wonder why such models are so effective. In reality no wall contains waves perfectly, there is always some leakage to the outside of Ω. A more complete description of the phenomena would involve a differential equation $\tilde{L}\tilde{u} = 0$ on $[0,T] \times \mathbb{R}^n$ where $L = \tilde{L}$ in $[0,T] \times \Omega$ and $\tilde{L}$ is very different from L outside Ω because the medium outside Ω is very different from that inside. In order for the boundary value problem to provide a good model one must have $|u - \tilde{u}|$ small in $[0,T] \times \Omega$. If the boundary value problem L,B is well posed this would follow from the equality of u and $\tilde{u}$ at $t = 0$ and an estimate Bu small on $[0,T] \times \Omega$. From this point of view boundary value problems are approximations to more precise models in all of space. They are introduced because they are easier to study than the original problem $\tilde{L}\tilde{u} = 0$. This has an important consequence :

> If a boundary value problem is too difficult to solve one should reconsider its origins. Perhaps it is not a simplification at all.

This is a path very rarely taken.

In addition it is important to have some theorems which justify the introduction of boundary conditions as models of interfaces where large changes in physical quantities occur.

In this lecture I will discuss two such theorems. More results and the details of the proof can be found in the paper [1] of Claude Bardos and myself. The underlying philosophy is discussed in an unpublished set of lecture notes [2] by K. O. Friedrichs which contain a heuristic derivation of the results described below. I would like to express my thanks to Professor Friedrichs for teaching me these, and many other things.

The point of view expressed so far is : given $\tilde{L}$ on $[0,T] \times \mathbb{R}^n$ find a boundary value problem on $[0,T] \times \Omega$ whose solutions furnish approximations to solutions of $\tilde{L}\tilde{u}$. In the last section of this paper we consider the inverse problem : given a boundary value problem on $[0,T] \times \Omega$ find a singular problem $\tilde{L}\tilde{u}$ on $[0,T] \times \mathbb{R}^n$ whose solutions are close in $[0,T] \times \Omega$ to solutions of the given problem. For symmetric hyperbolic systems and strictly dissipative boundary conditions (exactly the boundary value problems found in the original approach) we show that this is possible, and in several ways. These results are potentially useful in the numerical solution of such dissipative mixed initial boundary value problems. Typically it is not easy to include boundary conditions (or even $\partial\Omega$) in the discretizations used in numerical analysis. A way to avoid this problem is to pass from the original problem to the problem $\tilde{L}\tilde{u} = 0$ in all of space and then discretize. The price you pay is that some coefficients of $\tilde{L}$ will be large so the associated time evolution will have two natural time scales, a feature which often makes numerical analysis difficult. The results in this final section are new and complete proofs are given.

§ 2 A THEOREM

We will discuss two (singular) perturbations of the symmetric hyperbolic operator

$$L = A_o(t,x)\frac{\partial}{\partial t} - \sum_{j=1}^{n} A_j(t,x)\frac{\partial}{\partial x_j} - B.$$

The operator L is assumed to satisfy the following basic hypothesis

(i) $A_i(t.x) \in \mathrm{Hom}(\mathbb{C}^k)$ is self adjoint for $i = 0,1,2,...,n$ and all $(t,x) \in [0,T] \times \mathbb{R}^n$.

(ii) $A_o(t,x) \geq \delta I > 0$ in the sense of quadratic forms for all $t,x \in [0,T] \times \mathbb{R}^n$.

(iii) A_j and B are smooth with each derivative uniformly bounded on $[0,T] \times \mathbb{R}^n$.

Let P(t,x) be a matrix valued function such that

(iv) $P(t,x) \geq \delta_2 I > 0 \quad \forall (t,x) \in [0,T] \times (\mathbb{R}^n \setminus \Omega)$

(v) $P = 0$ in $[0,T] \times \Omega$

(vi) P is smooth in $[0,T] \times (\mathbb{R}^n \setminus \Omega)$ with each derivative uniformly bounded.

The region Ω is assumed to be open with $\partial\Omega$ smooth compact and Ω lying on one side of $\partial\Omega$.The operators $\tilde{L}$ on $[0,T] \times \mathbb{R}^n$ which we will study are

$$\tilde{L} = L + \lambda P \quad \text{and} \quad \tilde{L} = L + \lambda P\frac{\partial}{\partial t}$$

with $\lambda \gg 1$. In particular we study the behaviour as $\lambda \to \infty$, of the solutions u_λ of

$$\tilde{L}u_\lambda = 0 \text{ on } [0,T] \times \mathbb{R}^n$$

$$u_\lambda(0,x) = g(x) \ , \ g = 0 \quad \text{outside } \Omega \ .$$

This equation is a wave equation on $\mathbb{R}^n$ with a singular term which impedes penetration of waves into $\mathbb{R}^n \setminus \Omega$. For $\lambda \gg 1$ we will show that u_λ is small outside Ω and will find a boundary operator B so that L,B defines a well posed mixed initial boundary value problem in $[0,T] \times \Omega$ and Bu_λ is small on $[0,T] \times \partial\Omega$.

It follows that inside Ω $\tilde{u} \approx u$ where u is the solution of

$$Lu = 0 \quad \text{on } [0,T] \times \Omega$$

$$Bu = 0 \quad \text{on } [0,T] \times \partial\Omega$$

$$u(0,x) = g(x) \quad \text{on } \Omega \ .$$

To carry out this program we need an additional technical hypothesis which is needed to show the well-posedness of both the limiting boundary value problem and the Cauchy problem for $L + \lambda P \frac{\partial}{\partial t}$.

(vii) There is a smooth vector field $\nu(t,x)$ on $[0,T] \times \mathbb{R}^n$ such that for $x \in \partial\Omega$, $\nu(t,x)$ is the outward pointing unit normal to $\partial\Omega$ and rank $(\Sigma A_j \nu_j)$ is constant on a neighborhood of each component of $[0,T] \times \partial\Omega$.

Imprecisely, $\partial\Omega$ is characteristic of constant multiplicity. If the wave equation $Lu = 0$ has no zero sound speeds this hypothesis is always satisfied with rank $(\Sigma A_j \xi_j) = k \ (\forall \xi \neq 0)$.

We can now proceed with the analysis of u_λ. The first step is to show that the waves are effectively confined to the region $[0,T] \times \Omega$.

<u>Computation #1</u> : Showing $u_\lambda \to 0$ in $[0,T] \times \Omega$.
First consider the problem with $\tilde{L} = L + \lambda P \frac{\partial}{\partial t}$. The standard energy identity is

$$0 = (u_\lambda, \tilde{L} u_\lambda) + (\tilde{L} u_\lambda, u_\lambda) = E(t) - E(0) + \int_0^T \int \langle Zu, u \rangle dx\, dt$$

where

$(\ , \ )$ is $L_2([0,T] \times \Omega)$ scalar product

$\langle \ , \ \rangle$ is $\mathbb{C}^k$ scalar product

$$E(s) = \int \langle A_0 u_\lambda(s), \ u_\lambda(s) \rangle + \lambda \langle P u_\lambda(s), u_\lambda(s) \rangle \, dx$$

$$Z = \frac{\partial A_0}{\partial t} + \lambda \frac{\partial P}{\partial t} - \Sigma \frac{\partial A_j}{\partial x_j} - B - B^* \ .$$

Gronwall's method yields

$$E(t) \le e^{\alpha t} E(0)$$

where α depends on the coefficients of the equation but not on λ. It follows that for $0 \le t \le T$

$$\int \langle Pu_\lambda(t,x),\ u_\lambda(t,x)\rangle\, dx = O(\lambda^{-1})$$

(this uses the fact that $u(0,x) = 0$ if $x \notin \Omega$), so $u_\lambda \to 0$ outside Ω.

For the problem with $\tilde{L} = L + \lambda P$ a similar analysis yields

$$\int_0^T \int \langle P\, u_\lambda , u_\lambda\rangle dx\, dt = O(\lambda^{-2})$$

so again $u_\lambda \to 0$ outside Ω. ∎

For both problems we have shown that for λ large, waves cannot penetrate much into the exterior of Ω. It is reasonable to expect that the u_λ can be approximated by solutions of a boundary value problem in Ω. I will next outline an argument which identifies the boundary condition. There are many methods of doing this, matched asymptotic expansions and boundary large expansions are two from the usual toolbox of an analyst. The argument below is chosen because it leads to a proof of convergence to the appropriate limit. For maximal clarity we carry out the computation assuming that the A_j, B, P are constant and that $\Omega = \{x : x_1 < 0\}$. This contains the heart ofthe proof without the technical complications caused by the fact that the usual localizations must be handled with care since some of the "error terms" contain large parameters so must be treated with respect.

<u>Computation #2</u> : Guessing the boundary conditions.
In the simplified situations under consideration we have

$$L = A_o \frac{\partial}{\partial t} - \Sigma A_j \frac{\partial}{\partial x_j} - B$$

$$\tilde{L} = L + \lambda P \frac{\partial}{\partial t} \qquad \text{or} \qquad \tilde{L} = L + \lambda P$$

$$\Omega = \{x : x_1 < 0\}$$

and we have, from computation #1, a uniform bound on u_λ in the space $L^2([0,T] \times \mathbb{R}^n)$. We treat the case where $\tilde{L} = L + \lambda P$. From the bound on u_λ it follows that $\frac{\partial u_\lambda}{\partial t}$ and $\frac{\partial u_\lambda}{\partial x_i}$ for $i > 2$ considered as functions of x_1 are bounded in $L^2(\mathbb{R} ; H_{-1}([0,T] \times \mathbb{R}^{n-1}))$. It turns out that for $g \in H_1(\Omega)$ one gets boundedness of these "tangential derivatives" in $L^2([0,T] \times \mathbb{R}^n)$

(see § 3). We write the equation for u_λ in the form

$$-A_1 \frac{\partial u_\lambda}{\partial x_1} + \lambda P u_\lambda = \text{bounded}$$

Let $v_\lambda = P^{1/2} u_\lambda$. Then we have

$$(1) \qquad -(P^{-1/2} A_1 P^{-1/2}) \frac{\partial v_\lambda}{\partial x_1} + \lambda v_\lambda = \text{bounded} \quad .$$

By an appropriate choice of orthonormal basis in $\mathbb{C}^k$ we may assume that

$$P^{-1/2} A_1 P^{-1/2} = \begin{bmatrix} D_+ & & 0 \\ & D_- & \\ 0 & & 0 \end{bmatrix}$$

where $\pm D_\pm$ are positive diagonal matrices . Write $v_\lambda = (v_\lambda^+, v_\lambda^-, v_\lambda^o)$ corresponding to this spectral decomposition of $\mathbb{C}^k$.

Consider v_λ^+ . Taking the scalar product of (1) with v_λ^+ and integrating over $[0,T] \times [0,\infty] \times \mathbb{R}^{n-1}$ yields

$$\frac{1}{2} \int_0^T \int < D^+ v_\lambda^+, v_\lambda^+ > dx_2 \ldots dx_n dt + \lambda \int_{[0,T] \times \mathbb{R}^n_+} \int < v_\lambda^+, v_\lambda^+ > dx\, dt = (\text{bd}, v_\lambda^+).$$

Notice that the boundary term is positive. Applying Schwartz inequality on the right we first obtain that

$$\| v_\lambda^+ \|_{[0,T] \times \mathbb{R}^n} = O(\lambda^{-1})$$

which we already knew. With this bound for v_λ^+ we have from the boundary term

$$\| v_\lambda^+ \|_{L^2([0,T] \times \partial\Omega)} = O(\lambda^{-1/2})$$

For $(t,x) \in [0,T] \times \partial\Omega$ let π_+ be orthogonal projection of $\mathbb{C}^k$ onto the positive eigenspace of $P^{-1/2} A_1 P^{-1/2}$. We have shown that

$$\| \pi^+ P^{1/2} u_\lambda \|_{L^2([0,T] \times \partial\Omega)} = O(\lambda^{-1/2}).$$

Thus u_λ "almost" satisfies the boundary conditions $\pi^+ P^{1/2} u_\lambda = 0$.

Perhaps surprisingly, one finds the same boundary condition when $\tilde{L} = L + \lambda P \frac{\partial}{\partial t}$. This fact is made at least plausible by taking Laplace transform of the equation $\tilde{L}u_\lambda = 0$ assuming the coefficients do not depend on t. One obtains for the transform $\hat{u}(\tau)$

$$[(\tau A_o + \lambda\tau P) - \Sigma A_j \frac{\partial}{\partial x_j} - B]\hat{u}_\lambda = (\tau A_o + \lambda\tau P)u(0).$$

Since $u(0)$ vanishes outside Ω the right hand side is bounded independent of λ so we have

$$(\lambda\tau P - A_1 \frac{\partial}{\partial x_1})\hat{u}_\lambda = \text{bounded}$$

which for $\Re e\tau > 0$ can be analysed in the same way as equation (1). The proof in [1] proceeds along different lines, as it must to treat equations with time dependent coefficients. ∎

When Ω is not a half space the role of A_1 in the description of the boundary condition is played by

$$A_\nu(t,x) \equiv \Sigma A_j \nu_j$$

where ν is the unit outward pointing normal to $\partial\Omega$. In addition, for $x \in \partial\Omega$ we let $P(t,x)$ be the limit of P from <u>outside</u> Ω (the limit from the inside vanishes).

For $(t,x) \in [0,T] \times \partial\Omega$, let $\pi_+(t,x)$ be orthogonal projection of $\mathbb{C}^k$ onto the positive eigenspace of $P^{-1/2} A_\nu P^{-1/2}$ then the solution u_λ satisfies

$$\pi_+ P^{1/2} u_\lambda = O(\lambda^{-1/2}) \quad .$$

To complete the story we must show that in $[0,T] \times \Omega$ the wave equation L together with the boundary operator $\pi_+ P^{+1/2}$ defines a well posed problem. Once that is done it will follow that

$$u_\lambda - u = O(\lambda^{-1/2})$$

in Ω where u is defined by

$$Lu = 0 \text{ in } [0,T] \times \Omega$$

$$u(0,x) = g(x) \qquad (= u_\lambda(0,x))$$

$$\pi_+ P^{1/2} u = 0 \text{ on } [0,T] \times \partial\Omega \ .$$

Now the general theory of mixed initial boundary value problems for hyperbolic systems is complicated. However, for symmetric hyperbolic systems there is one class of well set boundary conditions which is easy to describe , the maximal dissipative boundary conditions. Notice that the condition $\pi_+ P^{1/2} u = 0$ is equivalent to $u \in M(t,x) \equiv \text{nullspace } (\pi_+ P^{1/2})$. The boundary condition is called maximal dissipative if

(i) $(\forall (t,x) \in [0,T] \times \partial\Omega)(\forall m \in M(t,x))(\langle A_\nu(t,x)m, m\rangle \le 0)$

(ii) dim M = dimension of nonpositive spectral subspace of A.

The second condition implies that M cannot be enlarged without violating (1).

<u>Computation #3</u> : Showing that the boundary conditions are maximal dissipative.

If $m \in \mathbb{C}^k$ and $\pi_+ P^{1/2} m = 0$ let $v = P^{1/2} m$ then v is in the span of the eigenvectors of $P^{-1/2} A_\nu P^{-1/2}$ with nonpositive eigenvalues so

$$\langle P^{-1/2} A_\nu P^{-1/2} v, v\rangle \le 0 \ .$$

On the other hand

$$\langle A_\nu m, m\rangle = \langle A_\nu P^{-1/2} v, P^{-1/2}\rangle$$

$$= \langle P^{-1/2} A_\nu P^{-1/2} v, v\rangle \ .$$

Thus M satisfies (i). Since $P^{1/2}$ is invertible,

$$\dim M = \dim \text{null}(\pi_+ P^{1/2}) = \dim \text{null}(\pi_+)$$

and it is not difficult to show that $\dim \text{null}(\pi_+)$ is independent of the positive matrix P. Taking $P = I$ yields (ii). ∎

We have now sketched the core of a proof of the following result.

Theorem : Let L be as above and $\tilde{L} = L + \lambda P$ and u_λ be the solution of

$$\tilde{L}u_\lambda = 0 \quad \text{in} \quad [0,T] \times \mathbb{R}^n$$

$$u_\lambda(0,x) = g(x) \ , \quad g \in H_1(\Omega) .$$

Then as $\lambda \to \infty$,

$$\|u_\lambda\|_{C([0,T] \ ; \ L_2(\mathbb{R}^n \setminus \Omega))} = O(\lambda^{-1}) \ ,$$

and

$$\|u_\lambda - u\|_{C([0,T] \ ; \ L_2(\Omega))} = O(\lambda^{-1/2})$$

where u is the unique solution of

$$Lu = 0 \quad \text{in} \quad [0,T] \times \Omega$$

$$u(0,x) = g(x)$$

$$\pi_+ P^{1/2} u = 0 \quad \text{on} \quad [0,T] \times \partial\Omega .$$

Here the value $P^{1/2}(t,x)$ is the limit from $\mathbb{R}^n \setminus \Omega$ and π_+ is the spectral projection of $P^{-1/2} A_\nu P^{-1/2}$ corresponding to positive eigenvalues.

It $g \in L_2(\Omega)$ the above result is true with $O(\lambda^{-1})$ and $O(\lambda^{-1/2})$ replaced by $o(1)$.

If $\tilde{L} = L + \lambda P \frac{\partial}{\partial t}$ the above assertions are true with $O(\lambda^{-1})$ and $O(\lambda^{-1/2})$ replaced by $O(\lambda^{-1/2})$ and $O(\lambda^{-1/4})$ respectively.

§ 3. SOME TECHNIQUES FROM THE PROOF

In addition to the central core of ideas described in § 2 there are some interesting techniques required in the proof which, I believe, are critical for the study of problems where boundary layers develope. The most important of these involve the derivations of a priori estimates for "tangential derivatives". The idea is that for a function with a surface across which there is a rapid transition the derivatives parallel to the surface may be bounded independent of the speed of the transition.

Typical behavior is $u_\varepsilon = u(\frac{x_1}{\varepsilon}, x_2, \ldots, x_n)$ for which $\frac{\partial u_\varepsilon}{\partial x_j}$ for $j > 1$ is bounded independent of ε. A natural approach to such estimates is to change coordinates so that the surface is $\{x : x_1 = 0\}$ and to try to estimate the derivatives in the directions $x_2, \ldots, x_n$. The underlying norm would then be $\|u\|_{L^2} + \sum_{j>2} \|\frac{\partial u}{\partial x_j}\|_{L^2}$. There is a basic flaw in this strategy : different choices of coordinates $x_1 \ldots, x_n$ lead to inequivalent norms. A clearly invariant space of functions is $H_{1,tan}(\Omega)$ defined (when Ω is compact) as the set of $u \in L_2(\Omega)$ such that for any smooth vector field $V(x)$ on Ω with $\langle V, \nu \rangle = 0$ on $\partial\Omega$ we have $Vu \in L_2(\Omega)$. In local coordinates the norm looks like

$$\|u\|_{L^2} + \|\frac{x_1}{\sqrt{1+x_1^2}} \frac{\partial u}{\partial x_1}\|_{L^2} + \sum_{j>1} \|\frac{\partial u}{\partial x_j}\|.$$

The critical change is the inclusion of the $\frac{x_1}{\sqrt{1+x_1^2}} \frac{\partial u}{\partial x_1}$ term. Notice that for the "typical behavior" $u(\frac{x_1}{\varepsilon}, x_2, \ldots, x_n)$ such an expression is bounded independent of ε. These "tangential spaces" play a critical role in [1]. In the framework of our basic theorem in § 2 the main result is

Theorem : If $g \in H_{1,tan}(\Omega)$ then $\{u_\lambda\}_{\lambda \geq 0}$ is bounded in each of the following spaces

$$C([0,T] ; H_{1,tan}(\Omega))$$

$$C([0,T] ; H_{1,tan}(\mathbb{R}^n \setminus \Omega))$$

$$C^1([0,T] ; L_2(\mathbb{R}^n)) .$$

Remarks : 1) In the theorem of § 2 the condition $g \in H_1(\Omega)$ may be weakened to $g \in H_{1,tan}(\Omega)$.

2) We already used such estimates in Computation #2.

Sketch of proof : The idea is quite simple. One reduces to the case $\Omega = \{x_1 < 0\}$ then with D equal to $\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial x_j}$ with $j > 1$ or $\frac{x_1}{\sqrt{1+x_1^2}} \frac{\partial}{\partial x_1}$ one differentiates the equation $\tilde{L} u_\lambda = 0$ to obtain

$$\tilde{L}(Du_\lambda) = [\tilde{L}, D] u_\lambda .$$

If $[L,D]$ was a linear combination of terms of the form $a(t,x)D$ then one could apply the standard energy method. The difficulty arises from $[A_1 \frac{\partial}{\partial x_1}, D]$ which may have a $\frac{\partial}{\partial x_1}$ part. However if A_1 were independent of (t,x) the above sketch would work.

Lemma : By a nonsingular change of dependent variable $u = U(t,x)w$ we may reduce to the case $A_1 = \text{constant}$.

Computation #4 : Showing why the lemma is true.

For the new function w we have the equation $\tilde{L}(Uw) = 0$. Multiply this equation on the left by U^* (in most cases U is not unitary) we find an equation

$$U^* A_1 U \frac{\partial w}{\partial x_1} + (\text{terms without } \frac{\partial w}{\partial x_1}) = 0 .$$

The assumption that A_ν has constant rank near each component of $\partial\Omega$ means that if we choose coordinates so that ν is parallel to $\frac{\partial}{\partial x_1}$ then A_1 has constant rank near $x_1 = 0$. It follows that we may choose U so that

$$U^* A_1 U = \begin{bmatrix} I_{n_1 \times n_1} & & \bigcirc \\ & -I_{n_2 \times n_2} & \\ \bigcirc & & 0_{n_3 \times n_3} \end{bmatrix}$$

with n_1, n_2, n_3 independent of (t,x). Notice that it was important to multiply by U^* not U^{-1}. This choice is also indicated by a desire to preserve the symmetry of the coefficients. ∎

§ 4. AN INVERSE PROBLEM

Given operator L and a maximal dissipative boundary space M one can ask if the solution to

$$\begin{aligned} Lu = 0 \quad & \text{on } [0,T] \times \Omega \\ u \in M \quad & \text{on } [0,T] \times \partial\Omega \\ u(0,x) = g \quad & \text{on } \Omega \end{aligned}$$

arises as a limit as $\lambda \to \infty$ of solutions to the Cauchy problem

$$\tilde{L}\tilde{u}_\lambda = 0 \qquad u(0,X) = y$$

$$\tilde{L} = L + \lambda P \frac{\partial}{\partial t} \qquad \text{or} \qquad \tilde{L} = L + \lambda P$$

for some P (depending on M).

The results just discussed show that this question is equivalent to the following problem in linear algebra : given a hermitian matrix A and a maximal dissipitative subspace M for A can you find a positive hermitian matrix P such that

$$(4.1) \qquad m \in M \Leftrightarrow P^{1/2} m \in E_{\leq 0}\ (P^{-1/2} A P^{-1/2})$$

where $E_{\leq 0}\ (P^{-1/2} A P^{-1/2})$ denotes the non positive spectral subspace of $P^{-1/2} A P^{-1/2}$ (symbols $E_{>0}$, $E_{<0}$ are defined similarly).

The answer to this question is no for a simple reason. Consider first the case where A is non singular and M is defined by (4.1) for some P. If $m \in M\backslash\{0\}$, then $P^{1/2} m \in E_{<0}\ (P^{-1/2} A P^{-1/2})$ ($E_{<0} = E_{\leq 0}$ since A is non singular) so with $v = P^{1/2}$

$$<Am, m> = <P^{-1/2} A P^{-1/2} v, v> < 0$$

It follows that there is $c > 0$ so that

$$<Am, m> \leq -c \|m\|^2 \quad \forall m \in M .$$

Thus A is not only non positive in M, it is strictly negative.

In the singular case observe that

$$E_{\leq 0}\ (P^{-1/2} A P^{-1/2}) = P^{1/2}(\text{nullspace } A) \oplus_{\perp} E_{<0}(P^{-1/2} A P^{-1/2})$$

and correspondingly

$$M = \text{nullspace } A \oplus P^{-1/2} E_{<0}(P^{-1/2} A P^{-1/2}).$$

Corresponding to this direct sum decomposition we write $m = m_1 + m_2$ and $P^{1/2} m = v_1 + v_2$ then

$$\langle Am, m\rangle = \langle P^{-1/2} A P^{-1/2} v_2, v_2\rangle$$

so there is a $c > 0$ such that

$$\langle Am, m\rangle \leq -c\,\|m_2\|^2 .$$

If $\pi_{\text{null }A}$ is orthogonal projection on null A then $I - \pi_{\text{null }A}$ is injective on $P^{-1/2} E_{<0}(P^{-1/2} A P^{-1/2})$ so there is a $c > 0$ such that

(4.2) $$\langle Am, m\rangle \leq -C\,\|(I - \pi_{\text{null }A})m\|^2 \qquad \forall m \in M .$$

This is the appropriate definition of strict dissipativity in the singular case.

<u>Definition</u> : If $A \in \mathrm{Hom}(\mathbb{C}^k)$ is selfadjoint and $M \subset \mathbb{C}^k$ is a maximal dissipative subspace then M is strictly dissipative for A iff (4.2) holds for some $c > 0$.

<u>Examples</u> : Choose orthogonal coordinates so that

$$A = \begin{bmatrix} A_+ & 0 & 0 \\ 0 & A_- & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $\pm A_\pm$ hermitian and positive. For any vector v write $v = (v_+, v_-, v_o)$ as the associated spectral decomposition, that is, v_+ consists of the first ℓ_+ components of v, if A_+ is $\ell_+ \times \ell_+$ ...etc. The space $M = E_{\leq 0}(A) = \{v : v_+ = 0\}$ is strictly dissipative. Also the set $M = \{v : v_+ = Sv_-\}$ is strictly dissipative if $S \in \mathrm{Hom}(E_{<0}(A), E_{>0}(A))$ is small enough. In fact we have the following simple proposition.

<u>Proposition</u> : Suppose that A_+ is $\ell_+ \times \ell_+$ and A_- is $\ell_- \times \ell_-$ and S is an $\ell_+ \times \ell_-$ matrix then

(4.3) $$M = \{v : v_+ = Sv_-\}$$

is strictly dissipative for A if and only if $\|A_+^{1/2} S(-A_-)^{-1/2}\|_{\mathrm{Hom}(\mathbb{C}^{\ell_-}, \mathbb{C}^{\ell_+})} < 1$

Every strictly dissipative subspace is of this form.

Proof : Consider M defined by (4.3). Then $m \in M \Leftrightarrow m = (Sv_-, v_-, v_o)$ for some v_-, v_o and then

$$\langle Am, m\rangle = \langle A_+ Sv_-, Sv_-\rangle_{\mathbb{C}^{\ell_+}} + \langle A_- v_-, v_-\rangle_{\mathbb{C}^{\ell_-}}$$

so M is dissipative if and only if

$$S^* A_+ S < -A_- \quad \text{as operators on } \mathrm{Hom}(\mathbb{C}^{\ell_-}).$$

Multiply left and right by $(-A_-)^{-1/2}$ to see that M is strictly dissipative if and only if

$$(-A_-)^{-1/2} S^* A_+^{1/2} A_+^{1/2} S(-A_-)^{-1/2} < I .$$

The left hand side is B^*B with $B = A_+^{1/2} S(-A_-)^{-1/2}$, and the first assertion of the theorem is proved.

To prove the second, suppose M is given and is dissipative. For $m \in M$ with $m_+ \neq 0$ one must have $m_- \neq 0$ otherwise $\langle Am, m\rangle = \langle Am_+, m_+\rangle > 0$. It follows that the map $M \ni m \mapsto (m_-, m_o)$ is injective. Counting dimensions we see that the map is an isomorphism so that there exist S and T so that

$$M = \{v : v_+ = Sv_- + Tv_o\} .$$

However as remarked above, $m \in M$ and $m_+ \neq 0 \Rightarrow m_- \neq 0$ so we must have $T = 0$. ∎

The main result asserts that it is exactly the strictly dissipative boundary conditions which arise in the Theorem of § 2.

Theorem : It $A \in \mathrm{Hom}(\mathbb{C}^k)$ is self adjoint and $P \in \mathrm{Hom}(\mathbb{C}^k)$ is positive self-adjoint then the subspace M of $\mathbb{C}^k$ defined by (4.1) is strictly dissipative for A. Conversely if M is strictly dissipative subspace for A then there is a positive P such that (4.1) holds.

Proof : The first part has already been proved. The second part is proved in two steps. First we treat the case where

$$A = \begin{bmatrix} I_{\ell_+ \times \ell_+} & 0 & 0 \\ 0 & -I_{\ell_- \times \ell_-} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

and then we reduce the general case to this one.

<u>First step</u> : Suppose A is the form (4.4) and M is strictly dissipative for A. Then by the above proposition there is a unique $\ell_+ \times \ell_-$ complex matrix S such that

$$M = \{v : v_+ = Sv_-\} \quad \text{and} \quad \|S\| < 1 \ .$$

The positive matrix P is defined by

$$P^{-1/2} = \begin{bmatrix} I & S & 0 \\ -S^* & I & 0 \\ 0 & 0 & I \end{bmatrix} \equiv Q$$

Now Q is clearly self adjoint and the condition $\|S\| < 1$ implies that Q is positive. Simple calculation yields

$$QAQ = \begin{bmatrix} I + SS^* & 0 & 0 \\ 0 & -(I + S^*S) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Q^{-1} = \begin{bmatrix} (I + S^*S)^{-1} & 0 & 0 \\ 0 & (I + S^*S)^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -S & 0 \\ S^* & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Thus the strictly dissipative subspace defined by P is

$$m \in M \Leftrightarrow (Q^{-1}m) \in E_{\leq 0}(QAQ)$$

which is precisely the condition $v_+ - Sv_- = 0$.

Thus the result is proved for A of the form (4.4).

<u>Second step</u> : Suppose that M is a strictly dissipative subspace for A, a general hermitian symmetric matrix. Choose a nonsingular matrix V so that

$$V^*AV = \begin{bmatrix} I_{\ell_+ \times \ell_+} & 0 & 0 \\ 0 & -I_{\ell_- \times \ell_-} & 0 \\ 0 & 0 & 0_{\ell_o \times \ell_o} \end{bmatrix} \equiv \tilde{A}$$

Then $V^{-1}M$ is a strictly dissipative for $\tilde{A}$ so by the first step there is a positive B such that

$$r \in V^{-1}M \Leftrightarrow B^{+1/2} r \in E_{\leq 0}(B^{-1/2}\tilde{A}B^{-1/2}).$$

Thus $m \in M \Leftrightarrow V^{-1} m \in V^{-1}M \Leftrightarrow$

(4.5) $$B^{+1/2}V^{-1}m \in E_{\leq 0}(B^{-1/2}V^*AVB^{-1/2}).$$

Now for any nonsingular W and self adjoint C we have

$$E_{\leq 0}(W^*CW) = W^{-1}[E_{\leq 0}(C)] .$$

Apply this identity with $W = V^*$ and $C = B^{-1/2}V^*AVB^{-1/2}$ to obtain with $Q \equiv VB^{-1/2}V^*$

$$E_{\leq 0}(P^{-1/2}V^*AVP^{-1/2}) = (V^*)[E_{\leq 0}(QAQ)].$$

Multiply both sides of (4.5) by $(V^*)^{-1}$ to obtain

$$m \in M \Leftrightarrow Q^{-1}m \in E_{\leq 0}(QAQ), \quad Q = VB^{-1/2}V^* .$$

Notice that Q is positive since B is. This identity is of the desired form with $P^{-1/2} \equiv Q$. ■

Open question : Describe all positive matrices P which yield a given strictly dissipative boundary condition M.

Remark : The boundary conditions we obtain are strictly dissipative. In particular they are not conservative (recall conservative means $\langle Am,m \rangle = 0 \quad \forall m \in M$).

The standard conservative boundary conditions, for example Dirichlet or Neumann conditions for wave equations do arise as limits of singular perturbation problems. When phrased in our terms the corresponding P's are positive semi-definite but not definite (see [2]). There is no general theory to cover semi-definite P.

REFERENCES

[1] C. Bardos and J. Rauch : Maximal positive boundary value problems as limits of singular perturbation problems, (to appear).

[2] K. O. Friedrichs : Well posed problems of mathematical physics, mimeographed lecture notes, NYU.