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Propagation of singularities along gliding rays

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PROPAGATION OF SINGULARITIES
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ALONG GLIDING RAYS
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par K. G. ANDERSSON et R. MELROSE

Exposé N° I

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Let $\Omega = \{x \in \mathbb{R}^n; f(x) > 0\}$ be a domain with C^∞ boundary and let $P = P(x, D)$ be a second order differential operator of real principal type with coefficients in $C^\infty(\bar{\Omega})$. The subject of this lecture is to discuss the singularities of distributions u satisfying

$$Pu \in C^\infty(\bar{\Omega}), \quad u|_{\partial\Omega} \in C^\infty(\partial\Omega).$$

If \mathcal{P} denotes the set of zeros of the principal symbol $p(x, \xi)$, $\mathcal{F} = \partial\Omega \times \dot{\mathbb{R}}^n$ and

$$(1) \quad \{p, f\} \neq 0 \text{ in } \mathcal{P} \cap \mathcal{F} \quad (\text{microlocally}),$$

then this problem has been treated by several authors (see [2], [8]). We shall consider a case where (1) is violated. If the boundary $\partial\Omega$ is non-characteristic, then

$$(2) \quad \{p, f\}(z) = 0 \Rightarrow \{f, \{f, p\}\}(z) \neq 0, \quad z \in \mathcal{P} \cap \mathcal{F}.$$

We shall also assume that Ω is pseudo-convex with respect to P , i.e.

$$(3) \quad \{p, f\}(z) = 0 \Rightarrow \{p, \{p, f\}\} < 0, \quad z \in \mathcal{P} \cap \mathcal{F}.$$

The opposite case, when Ω is pseudo-concave, has been treated in [4], [6] and [9].

When (2) and (3) are satisfied, it is possible to define so called boundary bicharacteristics for P as follows. Consider the restriction \tilde{p} of p to the symplectic manifold $\Delta_{\mathcal{F}} = \{z \in \mathcal{F}; \{p, f\}(z) = 0\}$. The projection, onto $T^*(\partial\Omega)$ of the null-bicharacteristics for \tilde{p} are called boundary bicharacteristics for P . It is easy to check that the null-bicharacteristics for the restriction of f to $\Delta_{\mathcal{P}} = \{z \in \mathcal{P}; \{p, f\}(z) = 0\}$ give the same curves as the null-bicharacteristics for \tilde{p} . In fact, the tangent of such a curve lies in the plane spanned by H_p and H_f and is orthogonal to the gradient of $\{p, f\}$.

In order to localize the concept of "regularity up to the boundary", we assume that Ω is given by $x_n > 0$ and denote points on the boundary $\partial\Omega$ by $x' = (x', 0)$. If $(\bar{x}', \bar{\xi}') \in T^*(\partial\Omega)$, we say that $u \in H_S(\bar{\Omega})$ at $(\bar{x}', \bar{\xi}')$ if there is a homogeneous symbol $\psi(x', \xi')$ such that $\psi(x', \xi') = 1$ in a neighborhood of $(\bar{x}', \bar{\xi}')$ and $\psi(x', D')u(x', x_n)$ belongs to $H_S(\bar{\Omega})$ close to \bar{x}' . Here it is implicitly assumed that $x_n \mapsto u(\cdot, x_n)$ is well-defined. In the same manner we say that $(\bar{x}', \bar{\xi}') \notin WF(u; \bar{\Omega})$ if, for some ψ as above, $\psi(x', D')u \in C^\infty(\bar{\Omega})$ close to \bar{x}' . This definition of microlocal regularity up to the boundary has also been suggested by Chazarain [3].

Theorem : Suppose that Ω is pseudo-convex with respect to P and let γ be a boundary bicharacteristic for P . If $u \in \mathcal{D}'(\bar{\Omega})$, $Pu \in C^\infty(\bar{\Omega})$ and $\gamma \cap WF(u|_{\partial\Omega}) = \emptyset$, then either $\gamma \subset WF(u; \bar{\Omega})$ or $\gamma \cap WF(u; \bar{\Omega}) = \emptyset$.

Sketch of proof : In order to avoid technical complications, we only consider the case when P is hyperbolic. We also assume that Ω is given by $x_n > 0$. Let $(\bar{x}', \bar{\xi})$ be a point in $\mathcal{P} \cap \mathcal{F}$ where $\{p, f\} = 0$ and denote by $\gamma = \gamma(\bar{x}', \bar{\xi}')$ the boundary bicharacteristic through the projection $(\bar{x}', \bar{\xi}')$ of $(\bar{x}', \bar{\xi})$ onto $T^*(\partial\Omega)$. The main step in the proof is to construct a suitable symbol $a(x, \xi)$ of order zero such that

$$(4) \quad \{p, a\}(x, \xi) = 0$$

$$(5) \quad a(x', \xi) = r(x', \xi)p(x', \xi) + q(x', \xi')$$

and $a(x, \xi) = 1$ on a conic neighborhood of $(\bar{x}', \bar{\xi})$. Here the condition (5) is required to be satisfied for some choice of r and q .

If (x', ξ') is close to $(\bar{x}', \bar{\xi}')$ then there is either no root or else two roots ξ_n^\pm of the equation $p(x', \xi', \xi_n) = 0$. These roots coincide when $\{p, f\} = 0$. Since q is independent of ξ_n it follows from (5) that one must have

$$(5') \quad a(x', \xi', \xi_n^+) = a(x', \xi', \xi_n^-).$$

Now (4) means that a is constant along the bicharacteristics for P so (5') implies that a has to be constant along the successively reflected

bicharacteristics. In particular it follows that a must be constant along the lifting $\tilde{\gamma}$ to $\mathcal{P} \cap \mathcal{F}$ of the boundary bicharacteristic $\gamma = \gamma(\bar{x}', \bar{\xi}')$. If we denote by $\Gamma(\bar{x}', \bar{\xi}')$ the flow-out along H_P from $\tilde{\gamma}$, the results of [7] imply that a can be chosen to satisfy (4) and (5) and to have support in an arbitrary small conic neighborhood of $\Gamma(\bar{x}', \bar{\xi}')$. Since $x_n = 0$ is non-characteristic, we can in particular assume that a has the transmission property (see [1]).

Let r be a homogeneous symbol of order -2 which has the transmission property and satisfies (5) and denote by R the corresponding operator. Let furthermore Q and A be operators with symbols q and $a + a_{-1}$, where

$$(6) \quad a_{-1} = p_1 r + i \left[\frac{\partial}{\partial \xi_n} \frac{\partial}{\partial x_n} (pr - a) - \sum_{k=1}^n \frac{\partial p}{\partial \xi_k} \frac{\partial r}{\partial x_k} \right]$$

Here p_1 denotes the symbol of order one of P .

Assume now that $u \in H_s(\bar{\Omega})$ along γ and put

$$v = Au^0 + R((Pu)^0 - Pu^0),$$

where u^0 denotes the extension of u which vanishes outside Ω . Since $\gamma \cap \text{WF}(u|_{\partial\Omega}) = \emptyset$, it follows from (4) - (6) that

$$(7) \quad Pv = A((Pu)^0) \quad \text{mod } H_s(\bar{\Omega}).$$

Moreover (5) gives that

$$(8) \quad v|_{\partial\Omega} = (R((Pu)^0) + Qu)|_{\partial\Omega} \quad \text{mod } H_{s+1/2}(\partial\Omega).$$

Because A and R have the transmission property, $Pu \in C^\infty(\bar{\Omega})$ and $\gamma \cap \text{WF}(u|_{\partial\Omega}) = \emptyset$, we get from (7) and (8) that

$$Pv \in H_s(\bar{\Omega}), \quad v|_{\partial\Omega} \in H_{s+1/2}(\partial\Omega).$$

If now some point on $\gamma(\bar{x}', \bar{\xi}')$ is outside $\text{WF}(u; \bar{\Omega})$ and $a(x, \xi)$ has support sufficiently close to $\Gamma(\bar{x}', \bar{\xi}')$, it follows that we can assume that v has initial data in C^∞ . Note that the pseudo-convexity

means that the bicharacteristics in $\Gamma(\bar{x}', \bar{\xi}')$ leave Ω . Well-known regularity theorems for the mixed problem now implies that $v \in H_{s+1/2}(\bar{\Omega})$.

Since $a(x, \xi) = q(x', \xi') = 1$ in a neighborhood of the zeros of $p(x, \xi)$, when (x', ξ') is close to $\gamma(\bar{x}', \bar{\xi}')$ and x_n is small, we can assume that $r(x, \xi)$ satisfies

$$(9) \quad a(x, \xi) = r(x, \xi)p(x, \xi) + q(x', \xi')$$

there. Remember that (5) is only required to be satisfied when $x_n = 0$. From (9) it follows that

$$v = R((Pu)^0) + Qu \text{ mod } H_{s+1}(\bar{\Omega}),$$

close to γ . Thus

$$Qu = H_{s+1/2}(\bar{\Omega}),$$

close to γ , and the proof is finished.

Remark : Just before this lecture we received a manuscript from G. Eskin [5] where a parametrix is constructed for certain mixed hyperbolic problems with gliding rays. For second order Dirichlet problems such a parametrix has also been constructed by one of the authors (Melrose). This construction as well as a fuller account of the argument described above will be published elsewhere.

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