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PROPAGATION OF SINGULARITIES FOR INTERIOR MIXED HYPERBOLIC PROBLEM

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§ 1. INTRODUCTION

Let Ω_o be a domain in $\mathbb{R}^n$ with a smooth boundary γ and let $G = (0,X_o)\times\Omega_o$, $\Gamma = (0,X_o)\times\gamma$, $X_o > 0$. Consider the following mixed problem in G

$$A(x,D)u(x) = 0 \quad , \quad x\in G \tag{1.1}$$

$$u|_\Gamma = g(x') \quad , \quad x' \in \Gamma \tag{1.2}$$

$$u|_{x_o=o} = 0 \quad ; \quad \frac{\partial u}{\partial x_o}|_{x_o=o} = 0 \tag{1.3}$$

where $A(x,D)$ is a strictly hyperbolic operator of the second order , $x = (x_o,x_1,\dots,x_n)$ and x_o is the time variable.

We shall make the following assumption :

Let $\Gamma(x) = 0$ be the equation of Γ. If for $x\in\Gamma$ and $\xi \neq 0$ we have

$$A^{(o)}(x,\xi) = 0 \quad \text{and} \quad \{A^{(o)}(x,\xi),\Gamma(x)\} = 0$$

then

$$\{\{A^{(o)}(x,\xi),\Gamma(x)\},\ A^{(o)}(x,\xi)\} > 0 \ , \tag{1.4}$$

where $A^{(o)}(x,\xi)$ is the principal part of $A(x,\xi)$ and

$\{f_1,f_2\} = \sum_{k=o}^{n} \left(\frac{\partial f_1}{\partial\xi_k}\frac{\partial f_2}{\partial x_k} - \frac{\partial f_1}{\partial x_k}\frac{\partial f_2}{\partial\xi_k}\right)$ is the Poisson bracket. The assumption (1.4) is equivalent to the condition that the boundary Γ is strictly convex with respect to all null-bicharacteristics of $A(x,D)$ which are tangential to Γ. We shall describe the wave front set of $u(x)$ assuming that the wave front set of $g(x')$ is given. The propagation of singularities for hyperbolic mixed problems was investigated by Povzner and Sukharevskii [14], Lax and Nirenberg [13], Chazarain [3], Majda and Osher [11], Taylor [15] in the case where there are no singularities on the tangential bicharacteristics. Recently

Friedlander [6], Taylor [16] and Melrose [12] considered the propagation of singularities for the exterior mixed problem for hyperbolic equation of the second order in the complete form, in particular, they admitted the singularities on the tangential bicharacteristics. Their results were extended on the hyperbolic equations of the higher order by the author [4]. The works [16], [12], [4] are a development of the earlier works of Ludwig [9] and Morawetz and Ludwig [10]. We note that for the interior mixed problem with the singularities on the tangential bicharacteristics only some partial results were known (see [2]). Quite recently I have received an exposition of the lecture given by Andersson and Melrose at this seminar [1] where results closed to ours were obtained but their method is quite different.

Generalization : Everywhere below we shall consider a mixed problem for the hyperbolic equation of the second order with the Dirichlet condition on the boundary. Analogous results are valid also for the following hyperbolic mixed problem of an arbitrary order :

$$A(x,D)u(x) = 0 \; ; \; x \in G \tag{1.5}$$

$$B_j(x,D)u(x) \,\big|_{\Gamma} = g_j(x') \;, \; 1 \le j \le m \;, \; x' \in \Gamma \tag{1.6}$$

$$u\big|_{x_o=o} = \frac{\partial u}{\partial x_o}\Big|_{x_o=0} = \ldots = \frac{\partial^{2m-1} u}{\partial x_o^{2m-1}}\Big|_{x_o=o} = 0 \tag{1.7}$$

where $A(x,D)$ is a strictly hyperbolic operator of the order 2m and $B_j(x,D)$ are differential operators of the order m_j. It is supposed in addition to the condition (1.4) that

1) each component of the surface $A^{(o)}(x,\xi) = 0$ is strictly convex for x and ξ_o fixed where $A^{(o)}$ is the principal part of $A(x,\xi)$.

2) For every point $(x',\xi') \in T^*(\Gamma)$, $\xi' \neq 0$, the operators $B_j(x,D)$, $1 \le j \le m$, fulfill the Agmon condition (see [4]) (which is called also uniform Shapiro Lopatinskii condition or Kreiss condition) in the corresponding local system of coordinates. The changes in the proof needed for the case of the problem (1.5), (1.6), (1.7) will be the same as in [4].

§ 2. STATEMENT OF RESULTS

Let $T^*_o(\Gamma)$ be the cotangent space on Γ without the null-section. We denote by $N_o \subset T^*_o(\Gamma)$ the image of the surface $A^{(o)}(x,\xi) = 0$, $\{A^{(o)}(x,\xi),\Gamma(x)\} = 0$ under the natural projection $i^* : T^*_o(R^{n+1}) \to T^*_o(\Gamma)$ and by $N_+ \subset T^*_o(\Gamma)$ the image of the set $A^{(o)}(x,\xi) = 0$, $\{A^{(o)}(x,\xi),\Gamma(x)\} \neq 0$ under the projection i^* .

We shall call outgoing bicharacteristic a null-bicharacteristic $x = x(t)$, $\xi = \xi(t)$ of the operator $A^{(o)}(x,D)$ for which the time $x_o = x_o(t)$ increases when the parameter t increases. Let $\mu(x',\xi') = 0$ be the equation of the surface N_o. We shall call outgoing limiting bicharacteristic an outgoing null-bicharacteristic of the operator $\mu(x',D')$. We shall define the following transformation $\varphi : N_+ \to N_+$. Let $(y',\eta') \in N_+$. Then the image (x',ξ') of (y',η') under the transformation φ will be the endpoint of the outgoing null-bicharacteristic of $A^{(o)}(x,D)$ which begins at the point (y',η) where $i^*\eta = \eta'$ and which touchs the boundary once more at the point (x',ξ) where $i^*\xi = \xi'$. We make the nonessential assumption that the transformation φ is defined on the whole N_+ . It may be shown that φ is a canonical transformation.

Theorem 2.1 : The wave front set $WF(\frac{\partial u}{\partial n}|_\Gamma)$ of $\frac{\partial u}{\partial n}|_\Gamma$, where $\frac{\partial}{\partial n}$ is the normal derivative, is contained in the following set

$$WF(\frac{\partial u}{\partial n}|_\Gamma) \subset WF(g) \cup (\bigcup_{k=1}^{\infty} \varphi^{(k)} \circ (WF(g) \cap N_+)) \cup M_o \qquad (2.1)$$

where $\varphi^{(k)}$ is k-th power of φ and $M_o \subset N_o$ is the union of all outgoing limiting bicharacteristics which begin at $WF(g) \cap N_o$.

We note that the propagation of singularities inside the domain G can be obtained from the Theorem 2.1 by using the Green formula, which gives expression for u(x) through $u|_\Gamma = g$, $\frac{\partial u}{\partial n}|_\Gamma$, and outgoing fundamental solution of hyperbolic equation (1.1). We don't use this way because the propagation of singularities inside G will be obtained as co-product of the proof of the Theorem 2.1 :

<u>Theorem 2.2</u> : The wave front set WF(u) of u(x) inside G is contained in the union of all broken bicharacteristics which begin in $(i^*)^{-1}(WF(g) \cap N_+)$.

We call broken bicharacteristic in G the union of an outgoing bicharacteristic and of all its multiple reflections at the boundary. The following lemma explains the name of limiting bicharacteristic :

<u>Lemma 2.1</u> : Let $(x'_o,\xi'_o) \in N_o$ be a limiting point of the sequence $(x'_m,\xi'_m) \in N_+$ when $m \to \infty$. Let γ_m be the broken bicharacteristic of the operator $A^{(o)}(x,D)$ which begins at the point (x'_m,ξ_m), where $i^*\xi_m = \xi'_m$. Then the limit of the set γ_m will be the outgoing limiting bicharacteristic γ_o which begins at the point $(x'_o,\xi'_o) \in N_o$.

Because of the local nature of the problem it is sufficient to consider the case when Ω_o is the half-space $\mathbb{R}^n_+ = \{(x'',x_n),x_n > 0\}$ $x'' = (x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}$, and $x = (x_o,x'',x_n) \in U_o \subset \mathbb{R}^{n+1}$, $\xi' = (\xi_o,\xi'') \in \Sigma_o \subset T^*_o(\mathbb{R}^n)$ where U_o is a small neighbourhood of some point $(x_o^{(o)},x''_o,0)$ and Σ_o is a small conic neighbourhood of some point $(\xi_o^{(o)},\xi''_o)$ where $(x_o^{(o)},x''_o,\xi_o^{(o)},\xi''_o) \in N_o$.

The principal part $A^{(o)}(x,\xi)$ of the operator A(x,D) can be written for $x \in U_o$, $\xi' \in \Sigma_o$ in the following form

$$A^{(o)}(x,\xi) = (\xi_n - \lambda(x,\xi'))^2 - \mu(x,\xi') , \qquad (2.2)$$

where $\lambda(x,\xi')$ and $\mu(x,\xi')$ are real.

The surface N_o is the surface $\mu(x',0,\xi') = 0$ and N_+ is the set (x',ξ') where $\mu(x',0,\xi') > 0$. We note that N_o is a smooth surface since $A^{(o)}(x,\xi)$ is hyperbolic with respect to ξ_o and that the assumption (1.4) has the form

$$\{\xi_n - \lambda,\mu\} < 0 \quad \text{when} \quad \mu(x',0,\xi') = 0 \qquad (2.3)$$

To prove the theorems 2.1 and 2.2 we shall construct a parametrix of the problem (1.1), (1.2), (1.3), i.e. such function $u_o = R(g)$ which solves (1.1), (1.2), (1.3) modulo C^∞ functions and which will be given more ou less explicitly. It may be shown that for this purpose it is sufficient to find in the half-space $x_n \geq 0$ a function $u = R(g)$ with the following properties :

$$A(x,D)R(g) \in C^\infty \quad \text{for } x_n \geq 0 \ , \ x \in U_o \ , \tag{2.4}$$

$$R(g)|_\Gamma - g(x) \in C^\infty \text{ for } x_n = 0, \quad x \in U_o \tag{2.5}$$

$$R(g) \in C^\infty \quad \text{for } x_o < 0 \ , \ x \in U_o \tag{2.6}$$

We suppose that $g(x') = 0$ for $x_o < 0$ and that the wave-front set of $g(x')$ is contained in a small neighbourhood of $(x_o^{(o)}, x_o'', \xi_o^{(o)}, \xi_o'') \in N_o$ Now the problem of propagation of singularities reduces to the description of the singularities of $R(g)$.

§ 3. AN EXAMPLE

To clarify the situation we shall consider at first the following boundary problem in the half-space $\mathbb{R}_+^{n+1} = \{(x', x_n), \ x_n > 0\}$:

$$a(x_n, D)u(x', x_n) = 0 \quad , \ x_n > 0 \tag{3.1}$$

$$u(x', 0) \quad = g(x') \tag{3.2}$$

$$u(x', x_n) \quad = 0 \quad \text{for } x_o < 0 \ , \ x_n \geq 0 \tag{3.3}$$

where $g(x') = 0$ for $x_o < 0$, $WF(g)$ is contained in some neighbourhood $|\xi_o| \leq C|\xi''|$ and $a(x_n, \xi) = \xi_n^2 - (\xi_o|\xi''| - x_n|\xi''|^2)$, i.e. $a(x_n, \xi)$ is a particular case of $A^{(o)}(x, \xi)$ when $\lambda = 0$, $\mu = (\frac{\xi_o}{|\xi''|} - x_n)|\xi''|^2$.

We note that the problem (3.1), (3.2), (3.3) is similar to the problem considered by Friedlander [6] but it differs by the sign of x_n (in [6] the case corresponding to the exterior mixed problem was treated). In this case the surface N_o will be the surface $\xi_o = 0$, the set N_+ will be given by the inequality $\xi_o > 0$ and M_o will be the union of all rays $(x_o + t, x'', 0, \xi'')$, $\forall \ t \geq 0$ where $(x_o, x'', 0, \xi'') \in WF(g)$. We shall denote the sets N_o, N_+, M_o for the case of the operator $a(x_n, D)$ by $\hat{N}_o$, $\hat{N}_+$, $\hat{M}_o$.

By performing the Fourier transform $\tilde{u}(\xi', x_n) = \int u(x', x_n) e^{i(x', \xi')} dx'$ with respect to $x' = (x_o, x'')$ we shall obtain an ordinary differential equation

$(i\frac{\partial}{\partial x_n})^2\tilde{u}(\xi',x_n) - (\xi_o|\xi''| - x_n|\xi''|^2)\tilde{u}(\xi',x_n) = 0$ which can be reduced to the Airy equation. The only solution of the problem (3.1), (3.2), (3.3) is given by the formula

$$u(x',x_n) = \frac{1}{(2\pi)^n}\int \frac{A_o\left(\left(\frac{\xi_o+iT}{|\xi''|} - x_n\right)|\xi''|^{\frac{2}{3}}\right)}{A_o\left(\frac{\xi_o+iT}{|\xi''|}|\xi''|^{\frac{2}{3}}\right)} e^{-i(x'',\xi'')-i(\xi_o+iT)x_o} . \tag{3.4}$$

$$\cdot\ \tilde{g}(\xi_o+iT,\xi'')d\xi_o d\xi''$$

where by $A_o(z)$ we denote the Airy function which has the following asymptotics for t real :

$$A_o(t) \approx \frac{C}{|t|^{\frac{1}{4}}} e^{-\frac{2}{3}|t|^{\frac{3}{2}}} \quad \text{for } t \to -\infty$$

$$A_o(t) \approx \frac{C}{t^{\frac{1}{4}}}\sin\left(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}\right) \quad \text{for } t \to +\infty \tag{3.5}$$

We have taken $T > 0$ in (3.4) to avoid the zeros of $A_o\left(\frac{\xi_o}{|\xi''|^{\frac{1}{3}}}\right)$ on the real axis. The integral (3.4) does not depend on T because of the Paley-Wiener theorem.

Denote by $A(z)$ the Airy function with the following asymptotics for t real :

$$A(t) \approx C\,t^{-\frac{1}{4}} e^{\left(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}\right)i} \quad \text{for } t \to +\infty$$

$$A(t) \approx C\,|t|^{-\frac{1}{4}} e^{\frac{2}{3}|t|^{\frac{3}{2}}} \quad \text{for } t \to -\infty \tag{3.7}$$

It can be shown $A_0(z) = A(z) - A_1(z)$ (3.8)
where $A_1(z) = \overline{A(\bar{z})}$.
We shall use the following estimate for the Airy functions .

<u>Proposition 3.1</u> : Let $K(\zeta) = \dfrac{A_1(\zeta)}{A(\zeta)}$ where $\zeta = (\alpha + \dfrac{iT}{|\xi''|})|\xi''|^{\frac{2}{3}}$, $\alpha = \dfrac{\xi_o}{|\xi''|}$. Then for $\alpha \geq 0$ and $|\xi''|$ large we have

$$1 - |K(\zeta)| \geq \frac{C}{|\xi''|^{\frac{1}{3}}} \quad (1 + \sqrt{\alpha}\, |\xi''|^{\frac{1}{3}}) \tag{3.9}$$

<u>Proof</u> : For $\alpha|\xi''|^{\frac{2}{3}}$ large the estimate (3.9) follows from the asymptotics of $A_1(\zeta)$ and $A(\zeta)$. Now let $0 \leq \alpha|\xi''|^{\frac{2}{3}} \leq C$. Then

$$\frac{d}{d\zeta}|K(\zeta)|\Big|_{\zeta=\alpha|\xi''|^{\frac{2}{3}}} = \frac{\overline{A'(\alpha|\xi''|^{\frac{2}{3}})}A(\alpha|\xi''|^{\frac{2}{3}}) - A'(\alpha|\xi''|^{\frac{2}{3}})\overline{A(\alpha|\xi''|^{\frac{2}{3}})}}{|A(\alpha|\xi''|^{\frac{2}{3}})|^2}$$

because $A_1(t) = \overline{A(t)}$ for the real t. But $w(t) = A_1'(t)A(t) - A'(t)A_1(t)$ is the wronskian of the Airy equation and so it is a constant

$$w(t) = w(0) = \frac{4\pi i}{9\,\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}$$

Thus it follows from the Taylor formula that

$$1 - |K(\alpha|\xi''|^{\frac{2}{3}} + \frac{iT}{|\xi''|^{\frac{1}{3}}})| \geq \frac{C}{|\xi''|^{\frac{1}{3}}}$$

Now we shall describe the wave front set of $\dfrac{\partial u(x',0)}{\partial x_n}$. If $v(x') \overset{def}{\equiv} e^{-x_oT} \dfrac{\partial u}{\partial x_n}(x',0)$ we obtain

$$v(x') = F^{-1} \frac{A_o'(\zeta)}{A_o(\zeta)} \tilde{g}_1(\xi') \tag{3.10}$$

where F^{-1} is the inverse Fourier transform, $\tilde{g}_1(\xi') = F(e^{-x_oT} g)$. Let $\chi_o(t) \in C_o^\infty(\mathbb{R}^1)$, $\chi_o(t) = 1$ for $|t| < \frac{1}{2}$, $\chi_o(t) = 0$

for $|t| \geq 1$ and $0 \leq \chi_o(t) \leq 1$ for all t. We shall denote by $\chi_1(t)$ the C^∞ function equal to $1-\chi_o(t)$ for $t \geq 0$ and equal to zero for $t \leq 0$. By $\chi_{-1}(t)$ we denote the C^∞ function equal to $1-\chi_o(t) - \chi_1(t)$.

Let $\varepsilon > 0$ and small. We have

$$v(x') = v_1(x') + v_o(x') + v_{-1}(x') \tag{3.11}$$

where $\tilde{v}_k(\xi') = \chi_k(\alpha|\xi''|^\varepsilon)\tilde{v}(\xi')$, $k = -1, 0, +1$.

At first we shall find the wave front set of $v_1(x')$. We have

$$\tilde{v}_1(\xi') = \chi_1(\alpha|\xi''|^\varepsilon)\frac{A'(\zeta)-A_1'(\zeta)}{A(\zeta)-A_1(\zeta)}\,\tilde{g}_1(\xi') =$$

$$= \chi_1(\alpha|\xi''|^\varepsilon)\left(\frac{A'(\zeta)}{A(\zeta)} - \frac{A_1'(\zeta)}{A(\zeta)}\right)\frac{\tilde{g}_1(\xi')}{1-\chi_2(\alpha|\xi''|^\varepsilon)K(\zeta)} \tag{3.12}$$

where $\chi_2(\alpha|\xi''|^\varepsilon) = \chi_1(2\alpha|\xi''|^\varepsilon)$ so that $\chi_2 = 1$ on the supp χ_1 .

Let

$$\tilde{w}_1(\xi') = \frac{\tilde{g}_1(\xi')}{1-\chi_2(\alpha|\xi''|^\varepsilon)K(\zeta)} \tag{3.13}$$

It follows from (3.9) that

$$\|w_1\|_s \leq C\|g_1\|_{s+\frac{\varepsilon}{2}} \quad , \quad \forall s \tag{3.14}$$

The equality (3.13) can be written in the following form

$$w_1 - bw_1 = g_1 \tag{3.15}$$

where $bw_1 = F^{-1}\chi_2 K\tilde{w}_1$ is a Fourier integral operator with the phase function $\hat{\varphi}(x',\xi') = (x',\xi') - \frac{4}{3}\alpha^{\frac{3}{2}}|\xi''|$.

The canonical transformation $\hat{\varphi}$ generated by the phase function $\hat{\varphi}(x',\xi')$ has the following form for $\alpha \in \operatorname{supp} \chi_2(\alpha|\xi''|^{\varepsilon})$

$$\begin{aligned} y_o &= \hat{\varphi}_{\xi_o}(x',\xi') = x_o - 2\sqrt{\alpha} \\ y'' &= \hat{\varphi}_{\xi''}(x',\xi') = x'' + \frac{2}{3}\alpha^{\frac{3}{2}} \frac{\xi''}{|\xi''|} \\ \eta' &= \xi' \end{aligned} \tag{3.16}$$

It is easy to verify that the canonical transformation (c.t.)$\hat{\varphi}$ coïncide with the c.t. φ defined in the section 2.

<u>Lemma 3.1</u> : There is the following inclusion

$$WF(w_1) \cap \hat{N}_o \subset \hat{M}_o \tag{3.17}$$

We shall make the following general remark before we begin the proof of the lemma 3.1.

<u>Remark 3.1</u> : Let $S \subset T_o^*(\mathbb{R}^n)$ be a conic domain which is invariant under the c.t.$\hat{\varphi}^{-1}$ for $\alpha \in \operatorname{supp} \chi_2(\alpha|\xi''|^{\varepsilon})$. Then for each point $(x_o',\xi_o') \in S$, there exists a C^∞-function $\beta(x',\xi')$ with support in S and homogeneous in ξ' of order zero such that $\beta(x_o',\xi_o') > 0$ and $\beta(x',\xi')$ is monotonic with respect to the c.t. $\hat{\varphi}^{-1}$, for $\alpha \in \operatorname{supp} \chi_2(\alpha|\varepsilon''|^{\varepsilon}$, i.e. $\hat{\varphi}^{-1}$ $(x',\xi') = (y',\eta')$ implies $\beta(y',\eta') \leq \beta(x',\xi')$. Then $w_1 \in C^\infty$ microlocally in S if $g_1 \in C^\infty$ microlocally in S. To prove the remark 3.1 we multiply the equation (3.15) by $\beta(x',D')$. Then we obtain $\beta b = b\beta_1 + b_1$ where $\beta_1(x',\xi') = \beta(y',\eta')$, $(y',\eta') = \hat{\varphi}^{-1}$ (x',ξ') and b_1 is an operator of a lower order. Now taking the scalar product of the equation $\beta w_1 - \beta b w_1 = \beta g_1$,with $\Lambda^{2s}\beta w_1$, where Λ^{2s} is the pseudo-differential operator (Ψ d O) with the symbol $(1+|\xi'|^2)^s$, and using the sharp Garding inequality we shall obtain an estimate for $\|\beta g_1\|_{s-\frac{\varepsilon}{2}}$ through $\|\beta g_1\|_s$ modulo lower order norm of w_1. Therefore we can obtain that $w_1 \in C^\infty$ microlocally in S because we can repeat such estimates many times. As a simple consequence of the remark 3.1 we note that the half-space $x_o < 0$ is an invariant domain in $T_o^*(\mathbb{R}^n)$ .under the c.t.$\hat{\varphi}^{-1}$ so that $w_1 \in C^\infty$ for $x_o < 0$ because $g_1 \in C^\infty$ for $x_o < 0$. Now we are able to prove the lemma 3.1. Let $(x^{(o)},\xi^{(o)}) \in \hat{N}_o$ and $(x^{(o)},\xi^{(o)}) \notin \hat{M}_o$. Then the

whole ray $(x_o^{(o)} - t, x_1^{(o)}, \ldots, x_{n-1}^{(o)}, \xi^{(o)})$, $\forall\, t \geq 0$, does not belong to $WF(g_1)$. Let S_o be a domain given by the inequalities

$$\delta^2(x_o - x_o^{(o)} - \delta)^2 - \sum_{k=1}^{n-1}(x_k - x_k^{(o)})^2 - \sum_{k=1}^{n-1}\left(\frac{\xi_k}{|\xi'|} - \frac{\xi_k^{(o)}}{|\xi^{(o)}|}\right)^2 - \frac{1}{\delta^2}\ \frac{\xi_o^2}{|\xi'|^2} > 0 \tag{3.18}$$

for $0 \leq x_o < x_o^{(o)} + \delta$ and (x'', ξ') are arbitrary for $x_o < 0$. If δ is small then $\overline{S_o} \cap WF(g_1) = \emptyset$. It follows from (3.16) that S_o is invariant under the c.t. $\hat{\varphi}^{-1}$ for $\alpha \in \text{supp}\ \chi_2(\alpha|\xi''|^{\varepsilon})$. It is easy to construct a family of functions $\beta(x', \xi')$ with support in S_o and monotonic with respect to $\hat{\varphi}^{-1}$, such that $\beta(x^{(o)}, \xi^{(o)}) > 0$. Then the remark 3.1 gives that $(x^{(o)}, \xi^{(o)}) \notin WF(w_1)$, thus we have proved (3.17).

__Lemma 3.2__ : The set $WF(w_1) \cap \hat{N}_+$ is contained in $\bigcup_{p=o}^{\infty} \hat{\varphi}^p \circ (WF(g_1) \cap \hat{N}_+)$

__Proof__ : If we apply the operator $\sum_{p=o}^{N} b^p$ to (3.15) we shall obtain

$$w_1 = b^{N+1}w_1 + \sum_{p=o}^{N} b^p g_1 \tag{3.19}$$

Thus

$$WF(w_1) \subset WF(b^{N+1}w_1) \cup \bigcup_{p=o}^{N} WF(b^p g_1) \subset WF(b^{N+1}w_1) \cup \bigcup_{p=o}^{\infty} WF(b^p g_1)$$

Since N is arbitrary we have

$$WF(w_1) \subset \left(\bigcap_{N=1}^{\infty} WF(b^{N+1}w_1)\right) \cup \left(\bigcup_{p=o}^{\infty} WF(b^p g_1)\right) \tag{3.20}$$

We shall show now that the intersection of $\bigcap_{N=1}^{\infty} WF(b^{N+1}w_1)$ with $\hat{N}_+$ is empty. Let $(x^{(o)}, \xi^{(o)}) \in \hat{N}_+$ be arbitrary and let $(x^{(N)}, \xi^{(N)}) = \hat{\varphi}^{-N}\ (x^{(o)}, \xi^{(o)})$. It follows from (3.16) that $x_o^{(N)} = x_o^{(o)} - 2N\sqrt{\alpha} \to -\infty$ when $N \to \infty$. We have $(x^{(o)}, \xi^{(o)}) \notin WF(b^N w_1)$ for N such that $x_o^{(N)} = x_o^{(o)} - 2N\sqrt{\alpha} < 0$ since $w_1 \in C^\infty$ for $x_o < 0$ and so that $b^N w_1 \in C^\infty$ microlocally in the neighbourhood of $(x^{(o)}, \xi^{(o)}) = \hat{\varphi}^N\ (x^{(N)}, \xi^{(N)})$. Therefore $WF(w_1) \cap \hat{N}_+ \subset \left(\bigcup_{p=o}^{\infty} WF(b^p g_1) \cap \hat{N}_+\right) \subset \bigcup_{p=o}^{\infty} \hat{\varphi}^p\ (WF(g_1) \cap \hat{N}_+)$. We note that $F^{-1}(\chi_1(\alpha|\xi''|^{\varepsilon})\,\frac{A'(\zeta)}{A(\zeta)}\,\tilde{w}_1)$ is an usual Ψ d o and $F^{-1}(\chi_1(\alpha|\xi''|^{\varepsilon})\frac{A_1'(\zeta)}{A(\zeta)}\,\tilde{w}_1)$ is a Fourier integral operator with the same phase function $\hat{\varphi}(x',\xi')$ as the operator b. So that

$$WF(v_1) \subset \hat{M}_o \cup \bigcup_{p=o}^{\infty} (\hat{\varphi}^p o(WF(g_1) \cap \hat{N}_+) \qquad (3.21)$$

Now we shall find the wave front set of $v_o(x')$.

<u>Proposition 3.1</u> : If $g_1 \in C^{\infty}$ for $x_o < t_o$ then

$$F^{-1}\,\frac{\partial^p}{\partial\zeta^p}\,\frac{A_o'(\zeta)}{A_o(\zeta)}\,\tilde{g}_1(\xi') \in C^{\infty} \quad \text{for } x_o < t_o \;,\; \forall\, p \geq 0.$$

This proposition is a consequence of the analyticity of $\frac{A_o'(\zeta)}{A_o(\zeta)}$ in ζ for $\operatorname{Im}\zeta > 0$.

<u>Proposition 3.2</u> : The following estimates are valid

$$\left|\frac{\partial^p}{\partial\xi''^p}\,\frac{A_o'(\zeta)}{A_o(\zeta)}\right| \leq C_p(1+|\xi''|)^{\frac{1}{3}-p\varepsilon} \;,\quad \forall\, p \qquad (3.22)$$

for $\alpha \in \operatorname{supp}\chi_o(\alpha|\xi''|^{\varepsilon})$.

The proposition 3.2 follows from the estimate (3.9) and from an obvious estimate

$$\left|\frac{\partial}{\partial\xi''}(\alpha|\xi''|^{\frac{2}{3}})\right| \leq C|\alpha|\,|\xi''|^{-\frac{1}{3}} \leq C\,|\xi''|^{-\frac{1}{3}-\varepsilon}$$

when $|\alpha| \leq \dfrac{C}{|\xi''|^{\varepsilon}}$.

The proposition 3.2 permits the localization with respect to x'' because the commutators of Ψ d o with the symbol $\chi_o(\alpha|\xi''|^{\varepsilon})\,\dfrac{A_o'(\zeta)}{A_o(\zeta)}$ and Ψ d o with the symbol $\beta(x'',\xi')$ will be of a lower order. Now the combination of the propositions 3.1 and 3.2 gives possibility to establish the following lemma.

<u>Lemma 3.3</u> : The wave front set of $v_o(x')$ is contained in $\hat{M}_o$:

$$WF(v_o) \subset \hat{M}_o \qquad (3.23)$$

We note that $WF(v_{-1}) \subset WF(g_1)$, (3.24) .

Since $v_{-1} = F^{-1}(\chi_{-1}(\alpha|\xi''|^{\varepsilon})\frac{A_o'(\zeta)}{A_o(\zeta)})$ is a Ψ d o with a symbol belonging to the class $S_{\frac{1}{3},o}$ (see [8]) on $WF(g_1)$.

Therefore, for the problem (3.1), (3.2), (3.3) the theorem 2.1 follows from (3.11), (3.21), (3.23) and (3.24), i.e.

$$WF(\frac{\partial u(x',o)}{\partial x_n}) \subset WF(g) \cup \bigcup_{p=1}^{\infty} \hat{\varphi}^{p}{}_{o}(WF(g) \cap \hat{N}_+) \cup \hat{M}_o \tag{3.25}$$

We note that if $WF(g) \cap \hat{N}_o$ is contained in the closure of $WF(g) \cap \hat{N}_+$ then it follows from the lemma 2.1 that

$$\hat{M}_o \subset \overline{\bigcup_{p=o}^{\infty} \hat{\varphi}^{p}{}_{o}(WF(g) \cap \hat{N}_+} \tag{3.26}$$

<u>Remark 3.2</u> : Let $(x_o^{(o)},\dots,x_{n-1}^{(o)},0,\xi_o'')$ be a point of $WF(g) \cap N_o$ and let γ_o be the limiting outgoing bicharacteristic $(x_o^{(o)} + t,\dots,x_{n-1}^{(o)},\xi_o')$ $t \geq 0$ which begins at this point. We assume that $g(x') \in C^{\infty}$ microlocally on γ_o for $t > 0$. The interesting question is:when for $t > 0$ the bicharacteristic γ_o is contained in $WF(\frac{\partial u(x',o)}{\partial x_n})$?

We shall consider two examples :

1) Denote $\chi^{(\mu)}(\xi') = \chi_o(\frac{\alpha|\xi''|^{\frac{2}{3}} - \mu}{\delta})$ where μ is real, $\delta > 0$ is small and $\chi_o(t)$ is the same function as above. Let μ_k, $1 \leq k < \infty$, be the zeros of the Airy function $A_o(z)$. We shall take $\mu \neq \mu_k$, $1 \leq k < \infty$, and $\delta > 0$ such that $A_o(\alpha|\xi''|^{\frac{2}{3}}) \neq 0$ on $\operatorname{supp} \chi_o(\frac{\alpha|\xi''|^{\frac{2}{3}} - \mu}{\delta})$. Let $g(x') = \chi^{(\mu)}(D')\delta(x')$ where $\delta(x')$ is the δ-function. It is obvious that : $WF(g) = \{(0;0,\xi'')\}$ where ξ'' is arbitrary. Since the symbol $\chi^{(\mu)}(\xi')\,\frac{A_o'(\zeta)}{A_o(\zeta)}$, $\zeta = \alpha|\xi''|^{\frac{2}{3}} + \frac{iT}{|\xi''|^{\frac{1}{3}}}$, belongs to the class $S_{\frac{1}{3},0}$ we have $WF(\frac{\partial u(x',o)}{\partial x_n}) \subset WF(g)$ so that $WF(\frac{\partial u(x',o)}{\partial x_n})$ does not contain for $t > 0$ the limiting bicharacteristic which begins at the point $(0,0,\xi'')$.

2) Now let $\mu = \mu_k$ where μ_k is one of the zeros of $A_o(z)$ and let $\delta > 0$ be such that there is no others zeros of $A_o(z)$ on the supp $\chi^{(\mu_k)}(\xi')$. Then $\chi^{(\mu_k)}(\xi')\frac{A_o'(\zeta)}{A_o(\zeta)} = \chi^{(\mu_k)}(\xi')(\frac{C_k}{\zeta - \mu_k} + K_1(\zeta))$ where C_k is a constant and $\chi^{(\mu_k)}(\xi')K_1(\zeta)$ belongs to the class $S_{\frac{1}{3},0}$. Thus $WF(F^{-1}\chi^{(\mu_k)}(\xi')K_1(\zeta)) \subset WF(g)$ where $g = \chi^{(\mu_k)}(D')\delta(x')$.

$$\text{Also}\quad \frac{1 - \chi_o(\frac{\alpha|\xi''|^{\frac{2}{3}} - \mu_k}{\delta})}{\zeta - \mu_k} \in S_{\frac{1}{3},0} \quad \text{so that}$$

$WF(F^{-1}\frac{C_k(1 - \chi^{(\mu_k)}(\xi'))}{\zeta - \mu_k}) \subset WF(g)$. It is easy to verify that the wave front set of $F^{-1}\frac{C_k}{\zeta - \mu_k}$ is equal to $\{(x_o,0,0,\xi'')\}$ where $x_o \geq 0$ and $\xi'' \neq 0$ are arbitrary, i.e. $WF(\frac{F^{-1}C_k}{\zeta-\mu_k}) = \gamma_o$. This shows that for $\mu = \mu_k$ $WF(\frac{\partial u(x',0)}{\partial x_n}) \supset \gamma_o$.

§ 4. THE GENERAL CASE

Now we shall carry out the same program as in the section 3 to construct and investigate the parametrix $u = R(g)$, which satisfies (2.4), (2.5), (2.6).

4.1 Construction of the phase function

Consider the eikonal equation

$$(\varphi_{x_n} - \lambda(x,\varphi_{x'}))^2 - \mu(x,\varphi_{x'}) = 0 \tag{4.1}$$

For the operator $a(x_n,D)$ there exist two solutions of the eikonal equation $a(x_n,\varphi_x) = 0$ in the region $\frac{\xi_o}{|\xi''|} - x_n > 0$: $\varphi_{\pm} = (x',\xi') \pm \frac{2}{3}(\frac{\xi_o}{|\xi''|} - x_n)^{\frac{3}{2}}|\xi''|$ and both have singularity on the caustic $\frac{\xi_o}{|\xi''|} - x_n = 0$.

The similar theorem holds in the general case (see e.g. [4] section 2) :

<u>Theorem 4.1</u> : Let $\mu(x_o', 0, \xi_o') = 0$ for $x_o = (x_o', 0) \in \mathbb{R}^{n+1}$ and $\xi_o' = (\xi_o^{(o)}, \xi_o'') \in \mathbb{R}^n \setminus \{0\}$. Let $U_o \subset \mathbb{R}^{n+1}$ be some small neighbourhood of $x_o = (x_o', 0)$ and $\hat{\Sigma}_o \subset \mathbb{R}^n \setminus \{0\}$ be some small neighbourhood of $\eta_o' = (0, \xi_o'')$. Then for $x \in U_o$, $\xi' \in \hat{\Sigma}_o$ there exist real C^∞-functions $\theta(x,\xi')$, $\rho(x,\xi')$ homogeneous in ξ', $\operatorname{ord}_{\xi'} \theta(x,\xi') = 1$, $\operatorname{ord}_{\xi'} \rho(x,\xi') = \frac{2}{3}$ such that $\varphi_{\pm}(x,\xi') = \theta(x,\xi') \pm \frac{2}{3}\rho(x,\xi')$ is a solution of the equation (4.1) for $\rho \geq 0$, $x \in U_o$, $\xi' \in \hat{\Sigma}_o$. Moreover

$$\det \left\| \frac{\partial^2 \theta}{\partial x_j \partial \xi_k} \right\|_{j,k=0}^{n-1} \neq 0$$

$$\rho(x,\xi') = (\alpha + O(\alpha^\infty))|\xi'|^{\frac{2}{3}} \quad \text{for } x_n = 0$$

$$\frac{\partial \rho}{\partial x_n} < 0 \quad \text{for } |\xi'| = 1$$

where $\alpha = \frac{\xi_o}{|\xi'|}$ and $O(\alpha^\infty)$ means $O(\alpha^N)$ for arbitrary N.

4.2 <u>Parametrix for the homogeneous equation in the half-space</u>

Now following Ludwig [9] (see also [4] section 3) we shall use the phase function to construct the asymptotic solution of the equation $A(x,D)u = 0$ in the region $\rho(x,\xi') \geq 0$, $x_n \geq 0$, $x \in U_o$. We shall choose an asymptotic solution in the following form

$$G(x,\xi') = (g(x,\xi')A_o(\rho) + ih(x,\xi')A_o'(\rho))e^{-i\theta}$$

where $g(x,\xi') \sim \sum_{k=o}^{\infty} g_k(x,\xi')$, $h(x,\xi') \sim \sum_{k=o}^{\infty} h_k(x,\xi')$,

$\operatorname{ord}_{\xi'} g_k = -k$ $\operatorname{ord}_{\xi'} h_k = -k - \frac{1}{3}$. The function $a_k^{\pm} = g_k \pm \sqrt{\rho}\, h_k$, $0 \leq k \leq \infty$ can be found in the region $\rho \geq 0$ by successive solution of the transport equation (see [4] section 3 and 4] and it is possible to choose the initial data for $a_k^{\pm}$ in such a way that

$$g_o(x',0,\xi') \neq 0, \; h_k(x',0,\xi') = O(\alpha^\infty)|\xi'|^{-k-\frac{1}{3}} \quad \forall\, k \geq 0 \qquad (4.2)$$

We take C^∞ continuation of $g_k(x,\xi')$ and $h_k(x,\xi')$, $o \leq k < \infty$ on the region $\rho \leq 0$, $x \in U_o$ and then we take almost analytic continuation of $\theta(x,\xi')$, $\rho(x,\xi')$, $g_k(x,\xi')$, $h_k(x,\xi')$ with respect to $\alpha = \frac{\xi_o}{|\xi'|}$.

We shall denote the almost analytic continuation of some function f by $\check{f}$. Thus

$$\check{\theta}(x,(\alpha + i\alpha')|\xi'|,\xi'') = \sum_{k=o}^{\infty} \frac{\partial^k \theta(x,\alpha|\xi'|,\xi'')}{\partial\alpha^k} \frac{(i\alpha')}{k!} \chi_o(N_k\alpha') \tag{4.3}$$

where the sequence $\{N_k\}$ is increased sufficiently fast .

It can be shown that

$$A(x,D)\check{G}(x,(\alpha + i\alpha')|\xi'|,\xi'') = \begin{cases} O(\frac{1}{|\xi'|^\infty}) & \text{for } \rho \geq 0 \\ (O(\frac{1}{|\xi'|^\infty}) + O(\frac{\rho}{|\xi'|^{\frac{2}{3}}})|\xi'|^{2+\frac{1}{6}})e^{-\frac{2}{3}|\rho|^{\frac{3}{2}}} & \text{for } \rho \leq 0 \end{cases} \tag{4.4}$$

where $\alpha' = \frac{T}{|\xi'|}$, $T > 0$ is fixed.

For $\alpha < -\frac{C}{|\xi'|^\varepsilon}$ we shall choose an asymptotic solution in the following form (see [4], section 5) :

$$G(x,\xi') = d(x,\xi')e^{-i\theta(x,\xi') - i\theta_1(x,\xi')} \tag{4.5}$$

where

$$\begin{aligned}\theta_1(x,\xi') = & -x_n(\theta_{x_n}(x',0,\xi') - \lambda(x',0,\theta_{x'}(x',0,\xi') + \\ & + i\sqrt{-\mu(x',0,\theta_{x'}(x',0,\xi'))})\end{aligned} \tag{4.6}$$

and

$$d(x,\xi') \sim d_o(x,\xi') + x_n \sum_{k=1}^{\infty} d_k(x,\xi') \tag{4.7}$$

where $d_o(x,\xi')$ is arbitrary and $d_k(x,\xi')$, $1 \le k < \infty$, can be found succesively by simple formulas (see [4], section 5). We note that $\mu(x',0,\theta_x,(x',0,\xi')) < 0$ for $\alpha < 0$ and we take

$$d_o(x',0,\xi') = \check{g}_o(x',0,\zeta')e^{-i(\check{\theta}(x',0,\zeta') - \theta(x',0,\xi'))} \tag{4.8}$$

where $\zeta' = ((\alpha + \frac{iT}{|\zeta'|})|\xi'|,\xi'')$.

We shall look for a parametrix of the equation $A(x,D)u = 0$ for $x_n \ge 0$ in the following form

$$Gv = G_1v + G_ov + G_{-1}v \tag{4.9}$$

where

$$G_1v = \int (\check{g}A_o(\check{p}) + i\check{h}A'_o(\check{p}))\frac{e^{-i\check{\theta}}}{A((\alpha+i\alpha')|\xi'|^{\frac{2}{3}})}\chi_1(\alpha|\xi'|^{\varepsilon})\tilde{v}(\xi')d\xi' \tag{4.10}$$

$$G_ov = \int (\check{g}A_o(\check{p}) + i\check{h}A_o(\check{p}))\frac{e^{-i\check{\theta}}}{A_o((\alpha+i\alpha')|\xi'|^{\frac{2}{3}})}\chi_1(\alpha|\xi'|^{\varepsilon})\tilde{v}(\xi')d\xi' \tag{4.11}$$

$$G_{-1}v = \int d(x,\xi')e^{-i\theta - i\theta_1(x,\xi)}\chi_{-1}(+\alpha|\xi'|^{\varepsilon})\tilde{v}(\xi')d\xi' \tag{4.12}$$

and where $\varepsilon < \frac{1}{2}$, $\alpha' = \frac{T}{|\xi'|}$.

It is not difficult to see that

$$A(x,D)Gv = 0 \ (\text{mod } C^\infty) \quad \text{for} \quad x_n \ge 0, \ x \in U_o$$

4.3 Solution of the equation on the boundary

We shall choose $v(x')$ such that

$$Gv\Big|_{x_n = o} = g(x')(\text{mod } C^\infty) \tag{4.13}$$

We assume that $g(x') = 0$ for $x_o < 0$ and that the wave front set of

$g(x')$ is contained in a small neighbourhood of $(x'_o,\xi'_o) \in N_o$. Denote $\rho_1(x',\xi') = \frac{2}{3}\rho^{\frac{3}{2}}(x',0,\xi') - \frac{2}{3}\alpha^{\frac{3}{2}}|\xi'|$ for $\alpha \geq 0$, $\rho_1 = 0$ for $\alpha \leq 0$. Then $\rho_1 \in C^\infty$ for $\xi' \neq 0$. It follows from (4.10), (4.11), (4.12) and from the asymptotic properties of the Airy functions that

$$Gv\Big|_{x_n=o} = \Phi_o v + \Phi_1 v \tag{4.14}$$

where Φ is the Fourier integral operator with the phase function $\varphi_o(x',\xi') = \theta(x',0,\xi') - \rho_1(x',\xi')$ and with the symbol $g^{(o)}(x',\xi') \in S^o_{1,0}$

$$\Phi_o v = \frac{1}{(2\pi)^n}\int g^{(o)}(x',\xi')e^{-i\varphi_o(x',\xi')}\tilde{v}(\xi')d\xi' \tag{4.15}$$

Also Φ_1 is the Fourier integral operator with the phase function $\varphi_1(x',\xi') = \theta(x',0,\xi') + \rho_1(x',\xi') - \frac{4}{3}\alpha^{\frac{3}{2}}|\xi'|$ and with the symbol $g^{(1)}(x',\xi')e^{-2\sqrt{\alpha}\,T}\chi_1(\alpha|\xi'|^\varepsilon) \in S^o_{1-\varepsilon,o}$:

$$\Phi_1 v = \frac{1}{(2\pi)^n}\int g^{(1)}(x',\xi')\,e^{-2\sqrt{\alpha}\,T}\chi_1(\alpha|\xi'|^\varepsilon)e^{-i\varphi_1(x',\xi')}\tilde{v}(\xi')d\xi' \tag{4.16}$$

We note that $\alpha \geq \frac{C}{|\xi'|^\varepsilon}$ on the support of the symbol of Φ_1, $g^{(o)}(x',\xi') - g^{(1)}(x',\xi') = O(\alpha)$ and Φ_o is an elliptic Fourier integral operator .

Thus there exists a Fourier integral operator R_o such that

$$R_o\,\Phi_o = I\,(\mathrm{mod}\ C^\infty)$$

Therefore if we apply R_o to the equation

$$\Phi_o v + \Phi_1 v = g$$

we shall obtain

$$v - Bv = R_o g \tag{4.17}$$

where $B = -R_{\omega}\Phi_1$ is also a Fourier integral operator. It can be shown by the stationary phase method

$$Bv = \frac{1}{(2\pi)^n} \int b_o(x',\xi')e^{-2\sqrt{\alpha}\,T} \chi_1(\alpha|\xi'|^{\varepsilon})e^{-\frac{i\varphi_2(x',\xi')}{2}} \tilde{v}(\xi')d\xi' \qquad (4.18)$$

where

$$\varphi_2(x',\xi') = (x',\xi') - \frac{4}{3}\alpha^{\frac{3}{2}}|\xi'| + \rho_2(x',\xi'), \quad \rho_2 = O(\alpha^{\infty})|\xi'| \qquad (4.19)$$

and

$$b_o(x',\xi') = b_1(x',\xi') + b_2(x',\xi') \;,\; b_i(x',\xi') \in S^o_{1-\varepsilon,o} \;,\; i = 1,2$$

$$|b_1(x',\xi')| \leq 1 \quad , \quad b_2(x',\xi') = O(\alpha) \qquad (4.20)$$

Now by using the sharp Gårding inequality we can prove the following lemma :

<u>Lemma 4.1</u> : Let $w \in H_{s+\frac{\varepsilon}{4}}(\mathbb{R}^n)$ where $\varepsilon < \frac{1}{2}$ and let $v \in H_s(R^n)$ be a solution of the equation

$$v - Bv = w \qquad (4.21)$$

Then the following estimate holds

$$\|v\|_{s-\frac{\varepsilon}{4}} \leq C\|w\|_{s+\frac{\varepsilon}{4}} \qquad (4.22)$$

If we apply the same arguments to the conjugate equation $p - B^*p = q$ then we can obtain the existence theorem for the equation (4.21) : for every $w \in H_{s+\frac{\varepsilon}{4}}(\mathbb{R}^4)$ there exists the solution $v \in H_{s-\frac{\varepsilon}{4}}(\mathbb{R}^n)$ of the equation (4.21) and the estimate (4.22) holds.

We note that the equation (4.17) is very similar to the equation (3.15). The phase function $\varphi_2(x',\xi')$ generated the c.t.

$$\begin{aligned} \eta' &= \varphi_{2x'} = \xi' + \rho_{2x'}(x',\xi') \\ y_o &= \varphi_{2\xi_o} = x_o - 2\sqrt{\alpha} + \frac{2}{3}\alpha^{\frac{5}{2}} + \rho_{2\xi_o}(x',\xi') \\ y'' &= \varphi_{2\xi''} = x'' + \frac{2}{3}\alpha^{\frac{3}{2}}\frac{\xi''}{|\xi'|} + \rho_{2\xi''}(x',\xi') \end{aligned} \tag{4.23}$$

which is closed to the c.t. (3.16) since $\rho_2 = O(\alpha^\infty)|\xi'|$. So that the remark 3.1 can be also applied to the equation (4.17) and a proof similar to the proof of lemmas 3.1, 3.2 gives that

$$WF(v) \subset WF(R\circ g) \cup \bigcup_{p=1}^{\infty} \varphi_2^{(k)} \circ (WF(R\circ g) \cap \hat{N}_+) \cup \hat{M}_o \tag{4.24}$$

Let φ_o and φ_1 be the c.t. generated by the phase function $\varphi_o(x',\xi')$ and $\varphi_1(x',\xi')$. We note that φ_1 is defined for $\alpha \in \operatorname{supp}\chi_1(\alpha|\xi'|^\varepsilon)$. It may be shown that $\varphi_2 = \varphi_o^{-1}\varphi_1$, $WF(R\circ g) \subset \varphi_o^{-1} WF(g)$ and $N_o = \varphi_o^{-1}\circ\hat{N}_o$, $N_+ = \varphi_o^{-1}\circ\hat{N}_+$, $M_o = \varphi_o^{-1}\circ\hat{M}_o$, $\varphi = \varphi_1\circ\varphi^{-1}$ where N_o, N_+, M_o, φ are the same as in the section 2. Thus

$$WF(v) \subset \varphi_o^{-1}\,(WF(g) \cup \bigcup_{p=1}^{\infty} \varphi^p\,(WF(g) \cap N^+ \cup M_o) \tag{4.25}$$

Now we can prove the Theorem 2.1. We have

$$\frac{\partial u(x',0)}{\partial x_n} = \frac{\partial}{\partial x_n}(G_1 v + G_o v + G_{-1} v)\Big|_{x_n=o} \tag{4.26}$$

It follows from (4.10), (4.11), (4.12) that

$$u_1(x') = \frac{\partial}{\partial x_n}(G_1 v + G_{-1} v)\Big|_{x_n=o} = \Phi_3 v + \Phi_4 \chi_1 v$$

where Φ_3 and Φ_4 are the Fourier integral operators with the phase function $\varphi_o(x',\xi')$ and $\varphi_1(x',\xi')$ and χ_1 is the Ψdo with the symbol $\chi_1(\alpha|\xi'|^\varepsilon)$. So that

$$WF(u_1) \subset \varphi_o \circ WF(v) \cup \varphi_1 \circ WF(\chi_1 v) \subset WF(g) \cup \bigcup_{k=1}^{\infty} \varphi^{(k)}\,(WF(g) \cap N_+) \cup M_o \tag{4.27}$$

We note that $\varphi_o = \varphi_1$ for $\alpha = 0$. Now

$$u_2(x') = \frac{\partial}{\partial x_n} G_o v|_{x_n=o} = \Phi_5 \chi_2 v + \Phi_6 K_o \chi_2 v, \tag{4.28}$$

where Φ_5, Φ_6 are Fourier integral operators with the phase function $\theta(x',0,\xi')$ and $K_o(\zeta_o') = \frac{A_o'(\zeta_o')}{A_o(\zeta}$, $\zeta_o' = (\alpha + \frac{iT}{|\xi'|})|\xi'|^{\frac{2}{3}}$. We note that

$|\alpha| < \frac{C}{|\xi'|^\varepsilon}$ on the supp $\chi_2(\alpha|\xi'|^\varepsilon)$ so that $O(\alpha^\infty) = O(\frac{1}{|\xi'|^\infty})$.

Since $\theta(x',0,\xi') = \varphi_o(x',\xi')$ for $\alpha = 0$ and $WF(\chi_2 v) \subset \hat{M}_o$ we have

$$WF(\Phi_5 \chi_2 v) \subset \theta \circ WF(\chi_2 v) \subset \theta \circ \hat{M}_o = M_o \tag{4.29}$$

Now the proof similar to the proof of the lemma 3.3 gives

$$WF(K_o \chi_2 v) \subset \hat{M}_o \tag{4.30}$$

So that

$$WF(\Phi_6 K_o \chi_2 v) \subset \theta \circ WF(K_o \chi_2 v) \subset M_o \tag{4.31}$$

Therefore the theorem 2.1 follows from (4.27), (4.29) and (4.31).

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