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PROPAGATION OF SINGULARITIES FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS AND REFLECTIONS AT A BOUNDARY

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♦ Le présent exposé nous est parvenu trop tard pour paraître dans le séminaire 1974-75 ; il est donc inséré en premier dans le présent séminaire 1975-76.

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Propagation of Singularities for Linear Partial Differential Equations and Reflections at a Boundary

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§1. There has been much study of propagation of singularities of solutions of partial differential equations, particularly hyperbolic equations. For the pure initial value problem for hyperbolic equations this is well understood — at least if only simple characteristics occur. In case of higher multiplicity things can be very complicated (an interesting example is presented in J. Ralston [9]) and one has results only in certain generic situations.

For the wave equation, and pure initial value problem, the singularities are propagated along the characteristics — straight lines inclined at 45° to the t-axis. If there is a boundary, the characteristics are reflected by the boundary according to the rules of geometrical optics. One has, for instance, the following result of Povzner, Sukharevskii [8]. Consider the wave equation inside a strictly convex cylinder Ω in R^n and consider the solution u satisfying (here $x_o \in \Omega$)

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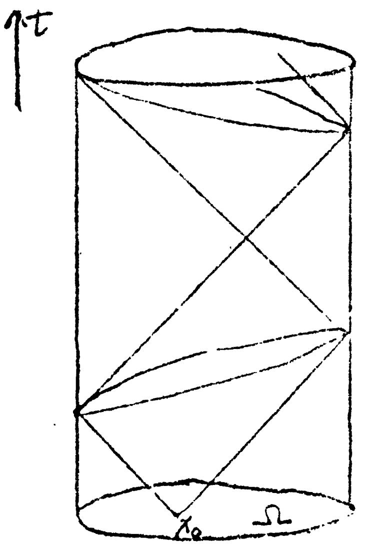
$$u_{tt} - \Delta u = 0 ,$$

$$u(0,x) = 0 , \quad u_t(0,x) = \delta(x-x_0) ,$$

$$u = 0 \quad \text{on } \partial\Omega .$$



For small time the characteristics from x_o carrying the singularities fill up a right circular cone. Consider the multiply reflected sheet obtained by extending each characteristic, when it hits the boundary, by its geometrical reflection.

<u>Theorem</u>: Outside of the sheet of surfaces so obtained the solution u belongs to C^∞ in $\bar{\Omega} \times [0, \infty)$.

The strict convexity of $\partial\Omega$ is essential here; it ensures that every characteristic from the interior of Ω strikes the boundary transversally, i.e. there are no "glancing" (or "grazing") rays. When glancing rays occur things are still very unclear — a variety of phenomena can occur — though simple generic cases are now being worked out. After the early, basic, work of J. Keller and D. Ludwig recent developments are due to F. G. Friedlander [2], R. B. Melrose [6] and M. Taylor [10], [11].

How singularities are propagated in the case of non-glancing rays is now rather well understood, and I will describe a general result in this direction together with the ideas involved in the proof. The first result on reflection of singularities in a rather general framework is due to P. Lax and the author (see §9 of [7]); it operates under boundary conditions which prescribe the function u and a number of its normal derivatives at the boundary — as in the preceding example. Recently A. Majda and S. Osher [5] extended this result to much more general boundary conditions. M. Taylor [10] has given an elegant derivation of this result and we shall present his argument here. A week before this lecture J. Duistermaat gave a talk at IHES in Bures in which he presented arguments very similar to those of Taylor [10] and also some results on glancing rays.

§2. We start by recalling a basic result of Hörmander on propagation of singularities of a distribution scalar solution u(x) of a (pseudo) differential equation

$$Pu = f \in C^{\infty} \;,$$

$$P = p_m + p_{m-1} + \dots \text{ a sum of homogeneous terms,}$$

and the symbol of the leading part

$$p_m(x,\xi) = p(x,\xi) \text{ is real.}$$

Theorem*([3]):[1] If $(x_o,\xi^o) \in WFu$ (so that necessarily $p(x_o,\xi^o) = 0$), then the entire null bicharacteristic Γ of p through (x_o,ξ^o) belongs to WFu.

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* We assume familiarity with the notion of wave front set of a distribution u in the cotangent space. Its projection down in physical space is the singular support of u.

The null bicharacteristic Γ is the curve in the cotangent space given by

$$\begin{aligned} \dot{x} &= p_\xi \\ \dot{\xi} &= -p_x \end{aligned} \qquad \text{through } (x_o, \xi^o)$$

(as long as the curve exists).

We now explain what is meant by reflection of bicharacteristics at a boundary. Since the discussion is local, consider a domain Ω with locally flat boundary in (for convenience) R^{n+1}, given by coordinates $x \in R^n$, $y \in R^1$ and with $y > 0$ in Ω, $y = 0$ in $\partial\Omega$. Denote the dual variables by $\xi \in R^n$, $\eta \in R^1$ so that (y, x, η, ξ) denotes a point in the cotangent space T^*R^{n+1}. The leading symbol of our differential operator P is

$$p(y, x, \eta, \xi) \text{ real of order } m .$$

For $(x_o, 0)$ a fixed point on $\partial\Omega$, and fixed $\xi^o \in R^n \setminus 0$, consider the real roots η of the polynomial

$$p(0, x_o, \eta, \xi^o) = 0 .$$

Assume that there are exactly k real roots $\eta_1(\xi^o), \ldots, \eta_k(\xi^o)$ all of which are simple.

Definition: We say that the k null bicharacteristic curves $\gamma_1, \ldots, \gamma_k$ of p in T^*R^{n+1} through the points $(0, x_o, \eta_j(\xi^o), \xi^o)$, $j = 1, \ldots, k$ belong to the same reflected family (associated with $(0, x_o, \xi^o) \in T^*\partial\Omega$).

Note: The assumption that the roots $\eta_i(\xi^o)$ are simple is equivalent to the assertion that each of these bicharacteristics (or rather its projection in physical (y,x) space) is non-glancing at $(0,0)$. For on γ_j we have at

$$(y,x,\eta,\xi) = (0,x_o\eta_j(\xi^o),\xi^o) \ ,$$

$$\dot{y} = p_\eta(0,x_o,\eta_j(\xi^o),\xi^o) \neq 0 \ .$$

Lax and Nirenberg proved (under certain boundary conditions) that if ℓ of these null bicharacteristics $\gamma_1,\dots,\gamma_k$ are not in WFu for $y > 0$ then the same is true of the rest (i.e. they formulate the result as propagation of singularity-free rays).

This result is micro-local but somewhat primitive in that if some of the γ_j are in WFu, it does not say which others are; this is because it does not study their interaction. Indeed if a boundary is present in a hyperbolic problem, knowledge of the wave front set of a solution for $t \leq t_o$ is not sufficient to enable one to predict the wave front set for later time — because of the interaction of the bicharacteristics at the boundary.

Taylor's method of treating reflection of singularities under more general boundary conditions uses the Calderón reduction of the equation $Pu = f$ to a first order system (with which we assume the reader is familiar (see for instance [7], §6). So we will simply start with that, and consider a general first order system for an N-vector $u(y,x) = (u^1,\dots,u^N)$:

$$u_y = Gu + f$$

where

$$G = G(y,x,D_x) = G_1 + G_o + \dots$$

is a first order pseudo-differential operator in the x variables varying smoothly with y in some neighborhood of y = 0. $G_1(y,x,\xi)$ is its leading symbol homogeneous of degree one in ξ, and the remaining terms are homogeneous of degrees $0,-1,-2,\ldots$. We assume that $f \in C^{\infty}$ in some neighborhood of $(y = 0,\ x_0)$.

Consider a solution in $y \geq 0$ satisfying boundary conditions on y = 0

$$Bu\Big|_{y=0} = \phi \text{ given}$$

where B is a (classical) pseudo-differential operator in the x variables of order zero. Assume

$$\det\left(\eta I - \frac{1}{i} G_1(y,\eta,\xi)\right) = p(y,x,\eta,\xi) \text{ is real}$$

and that at $(0,x_0,\xi^0)$, p has exactly k real roots $\lambda_1,\ldots,\lambda_k$ all of which are simple (i.e. the corresponding null-bicharacteristics $\gamma_1,\ldots,\gamma_k$ of p are non-glancing).

Supposing that

$$\text{on } y = 0 : (x_0,\xi^0) \notin WFBu\ ,$$

and that for $y > 0$, $\gamma_1,\ldots,\gamma_\ell$ are not in WFu, we will find conditions to ensure that

(1) $\gamma_{\ell+1},\ldots,\gamma_k$ are not in WFu for $y > 0$.

(2) In addition, $(x_0,\xi^0) \notin WF(\partial_y^j u)\Big|_{y=0}$ for $j = 0,1,\ldots$.

As in [7] the result is obtained by reduction to classical results for hyperbolic, elliptic and parabolic equations, however here the argument is simpler.

§3. To begin, in some neighborhood of $(y = 0, x_0, \xi^0)$ we may find a nonsingular matrix $U(y,x,\xi)$, homogeneous of degree zero in ξ with which we may make G_1 essentially diagonal:

$$G_1(y,x,\xi) = U^{-1}(y,x,\xi)\begin{pmatrix} \begin{matrix} i\lambda_1 & & 0 \\ & \ddots & \\ 0 & & i\lambda_k \end{matrix} & 0 \\ 0 & \begin{matrix} E_+ & \\ & E_- \end{matrix} \end{pmatrix} U(y,x,\xi)$$

Here E_+ (E_-) is a square matrix with eigenvalues all having positive (negative) real parts.

Since $U(y,x,\xi)$ is nonsingular the corresponding pseudo-differential operator $U(y)$ is elliptic and so has a parametrix which we call $U(y)^{-1}$. Set $v = U(y)u$; then

$$v_y = UGU^{-1}v + U_yU^{-1} + Rv + Uf$$

where R is an infinitely smoothing operator in the x variables, i.e. is an integral operator in the x variables (depending on y) with C^∞ kernel. Thus

$$v_y = \begin{pmatrix} i\lambda_1 & & & & 0 \\ & \ddots & & & \\ & & i\lambda_k & & \\ & & & E_+ & \\ 0 & & & & E_- \end{pmatrix} v + Av + Uf$$

where $A(y)$ is a pseudo-differential operator in the x variables of order zero.

Suppose $A \equiv 0$. Then this is an uncoupled system, and the result on reflection of singularities is an almost immediate consequence of known results. Namely, write v in the form

$$v = \begin{pmatrix} v^{I} \\ v^{II} \\ v^{III} \\ v^{IV} \end{pmatrix} \begin{matrix} \leftarrow \text{first } \ell \text{ components} \\ \leftarrow \text{next } k-\ell \text{ components} \\ \leftarrow \text{next } \frac{N-k}{2} \text{ components} \\ \leftarrow \text{last } \frac{N-k}{2} \text{ components} \end{matrix}$$

(Note that since p is real and homogeneous of degree N in ξ it follows that $N-k$ is even.) For each v^{J} we may remark:

(i) v^{I} satisfies a diagonal hyperbolic system for which both forward and backward initial value problems are well posed (thinking of y as time). Hence v^{I} can be extended also to negative values of y as a solution. By hypothesis the null bicharacteristics $\gamma_1,\ldots,\gamma_\ell$ are not in WFv^{I} for $y > 0$ and hence, by Hörmander's theorem above, they are also not in WFv^{I} for $y < 0$. It follows then that

$$(3.1) \qquad (x_o,\xi^o) \notin WF(\partial_y^j v^{I})\Big|_{y=0} \qquad \text{for } j = 0,1,\ldots .$$

(In applying Hörmander's result some care should be taken; $\partial_y - i\lambda_j(y,x,D_x)$ is not a genuine pseudo-differential operator in the (y,x) variables. However we are just interested in a micro-local analysis in a neighborhood of the zero of the symbol, and the operator may be modified outside to be pseudo-differential.)

(iii) v^{III} satisfies

$$v_y^{III} = E_+ v^{III} + (Uf)^{III} .$$

This is a backward parabolic system (i.e. with y as time the backward initial value problem is well posed) and the solution is automatically C^∞ (in micro-local sense). Thus in particular

$$(3.2) \qquad (x_o, \xi^o) \notin WF(\partial_y^j v^{III})\Big|_{y=0} \quad \text{for} \quad j = 0, 1, \ldots .$$

(iv) v^{IV} satisfies a forward parabolic system and so is automatically in C^∞ for $y > 0$ (again micro-locally).

(ii) v^{II} satisfies a diagonal hyperbolic system and so by regularity theory for such equations (at the micro-local level) the null bicharacteristics $\gamma_{\ell+1}, \ldots, \gamma_k$ in $y > 0$ will not be in WFv^{II} provided

$$(3.3) \qquad (x_o, \xi^o) \notin WFv^{II}\Big|_{y=0} .$$

Thus we are led to the following

<u>Theorem I</u>: Consider the pseudo-differential system on $y = 0$ for

$$v = \begin{pmatrix} v^I \\ \vdots \\ v^{IV} \end{pmatrix}$$

$$(3.4) \qquad v^I = g^I , \quad v^{III} = g^{III} , \quad BU^{-1}(0)v = h .$$

Suppose this system is elliptic (or merely hypoelliptic) at (x^o, ξ^o), i.e.

(3.5) $(x_o, \xi^o) \notin WFg^I, g^{III}$ and h implies $(x_o, \xi^o) \notin WFv$.

Then (1) and (2) hold. Furthermore, if we just assume (on $y = 0$) that

(3.6) $(x_o, \xi^o) \notin WFg^I, g^{III}$ and h implies $(x_o, \xi^o) \notin WFv^{II}$,

then (1) holds.

Both assertions follow by application of the remarks (i)-(iv) above.

In general the condition (3.5) of hypoellipticity is difficult to verify, however to check if the system (3.4) is elliptic at (x_o, ξ^o) is done by a direct, well-known, computation. It means that if we set $v^I \equiv 0$ then $v(y) = 0$ is only bounded solution on $(0 \leq y < \infty)$ of the system of ordinary differential equations

$$v_y = \begin{pmatrix} i\lambda_1(y, x_o, \xi^o) & & & & \\ & \ddots & & & 0 \\ & & i\lambda_k & & \\ & & & E_+(y, x_o, \xi^o) & \\ & 0 & & & E_-(y, x_o, \xi^o) \end{pmatrix} v$$

satisfying $B_o v\Big|_{y=0} = 0$ where B_o is the principal part of B.

We shall give some examples later for which (3.4) is hypoelliptic at (x_o, ξ^o) but not elliptic there.

§4. We have presented our result in case of a totally uncoupled system. There is a simple procedure to accomplish this which we now describe. For simplicity we shall carry it out in a case of two blocks (in place of four). Consider

$$v_y = \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} v + Av + F \tag{4.1}$$

where $H = \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix}$ is a pseudo-differential operator in the x variables, depending smoothly on y, with symbol homogeneous of degree one in ξ; A is a classical pseudo-differential operator (smooth in y) of order zero, $A = A_0 + A_{-1} + \dots$. Assume that the square matrices $F(y,x,\xi)$, $E(y,x,\xi)$ of orders $r \times r$ and $s \times s$ have disjoint sets of eigenvalues for each (y,x,ξ) in a neighborhood of (y_0, x_0, ξ^0). We will make a sequence of transformations of v which, in succession, decouple the terms $A_0, A_{-1}, \dots$. In fact we will just carry out the first step — for A_0.

Try $w_1 = (I + K_1)v$ with K_1 of order -1 and of the form

$$K_1 = \begin{pmatrix} 0 & K_{12} \\ K_{21} & 0 \end{pmatrix}.$$

Inserting this in (4.1) we find (ignoring F)

$$w_{1y} = Hw + (K_1 H - HK_1 + A)w_1 + \text{lower order terms}$$

and we wish to choose K_{12}, K_{21} so that

$$K_1H - HK_1 + A \text{ has the form } \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} .$$

If

$$A_o = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

we find by a direct calculation that K_{12}, K_{21} are to satisfy

$$K_{12}F - EK_{12} = -A_{12}$$

$$K_{21}F - EK_{21} = -A_{21} \quad .$$

There is a simple exercise in linear algebra (see Malgrange [5]) which enables one to solve uniquely for the matrices $K_{12}(y,x,\xi)$, K_{21}:

<u>Exercise:</u> If F, E are $r \times r$ and $s \times s$ matrices with disjoint eigenvalues then for any given $s \times r$ matrix M there is a unique $s \times r$ matrix T satisfying

$$TF - ET = M .$$

Thus K_1 is determined so as to decouple A_o in the equation for w_1. Next one decouples the terms of order -1 in this equation by a similar transformation $w_2 = (I + K_2)w_1$ with K_2 of order -2 by the same kind of argument. Repeating this process one finally sets

$$w = (I + K)v = \ldots (I + K_2)(I + K_1)v ;$$

the equation for $w = \begin{pmatrix} w^I \\ \vdots \\ w^{IV} \end{pmatrix}$ is totally decoupled (modulo a smoothing term).

If we combine this procedure with Theorem I we obtain the general result:

Theorem II: Suppose $\gamma_1, \ldots, \gamma_\ell$ associated with $\lambda_1, \ldots, \lambda_\ell$ are not in WFu for $y > 0$. On $y = 0$ set

$$P^J u = U^{-1}(I+K)^{-1} w^J , \quad J = I, \ldots, IV .$$

If the system of pseudo-differential operators in x

$$P^I u , \quad P^{III} u , \quad Bu$$

is hypoelliptic at (x_o, ξ^o) then (1) and (2) hold. If we merely assume

$$(x_o, \xi^o) \notin WFP^I u,\ P^{III} u \text{ and } Bu \text{ implies } (x_o, \xi^o) \notin WFP^{II} u$$

then (1) holds.

The principal symbols of P^I, P^{III} are the projections onto linear span of the eigenspaces associated with the eigenvalues $i\lambda_1, \ldots, i\lambda_\ell$ of $G_1(0,x,\xi)$ and the generalized eigenspace corresponding to the spectrum of $G_1(0,x,\xi)$ with positive real part.

Remark: There is a more quantitative version of Hörmander's theorem referring to solutions in H^s. Theorem 2 also extends to this case [10].

Using Theorem 2, Taylor [10] proves a generalization of the result of Povzner and Sukharevskii for systems satisfying the well posed initial boundary value conditions of Kreiss-Sakamoto — assuming that the singular support of the initial data does not touch the boundary. This is done with the aid of an approximate solution having the expected singularities. The energy estimates of Kreiss-Sakamoto are used to show that this differs from the solution by a smooth error. J. Chazarain [1] at a conference in Rennes in June 1975 presented a similar result.

§5. We now present some examples in which Theorem 2 applies. These are due to Majda and Osher [4].

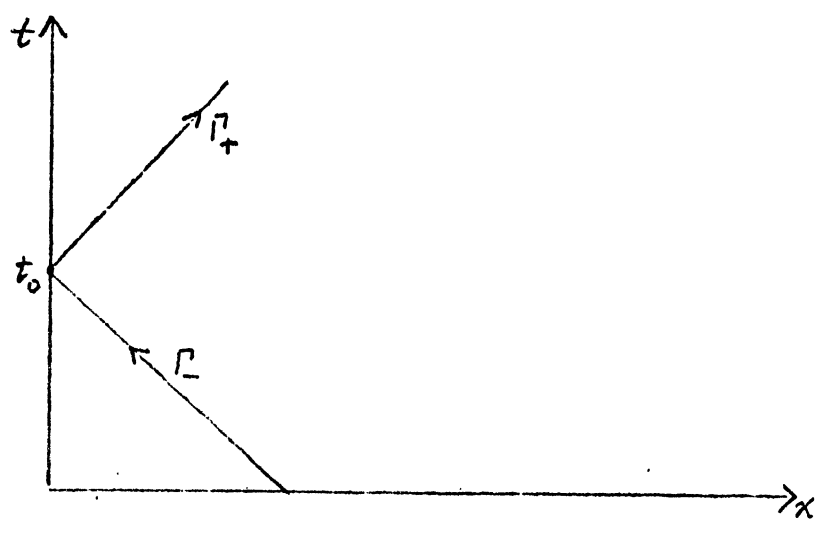
Exercise 1: Consider a simple second order hyperbolic equation with one space variable x

$$(\partial_x + \partial_t + \alpha)(\partial_x - \partial_t + \beta)u = 0 \quad \text{in} \quad x > 0$$

with boundary conditions:

$$\text{On } x = 0, \ (\partial_x - \gamma\partial_t + \sigma)u = \phi \in C^\infty .$$

Here α, β, γ are constants.



Setting $w = u_x - u_t + \beta u$ we shall give an entirely elementary analysis

Suppose u is smooth on the outgoing characteristic Γ^+ (with slope 1) for $t > t_o$. Since

$$(\partial_x + \partial_t + \alpha)w = 0$$

we see that $w\big|_{x_o}$ is smooth near $t = x_o$ and hence

$$(1-\gamma)u_t + (\sigma-\beta)u \text{ is smooth there.}$$

Thus if $\gamma \neq 1$ we find $u\big|_{x=0}$ is smooth there. Since the Cauchy data of u is smooth there it follows that u is smooth on Γ_-.

The same conclusion still maintains in case $\gamma = 1$ and $\sigma-\beta \neq 0$. This corresponds to the hypoelliptic, but not elliptic case in Theorem 2.

Suppose $\gamma = 1$, $\beta = \sigma = 0$. Then any function of the form $u = w(x+t)$ satisfies the equation and the boundary condition. For suitable w, u is smooth on Γ_+ but not on Γ_-.

<u>Exercise 2</u>: Consider the wave equation

$$u_{tt} - u_{yy} - \sum_1^{n-1} u_{x_i x_i} = f \in C^\infty \quad \text{in } y > 0 .$$

Under boundary conditions

$$u_y - (\gamma_1 + i\gamma_2)u_t = 0 \quad \text{on } y = 0 , \quad \gamma_1, \gamma_2 \text{ real} .$$

Let τ be the dual variable to t. Associated with (τ, ξ), $\xi \in R^{n-1}$ for $\tau^2 > |\xi|^2$, are two bicharacteristic curves corresponding to the roots $\eta = \pm\sqrt{\tau^2 - |\xi|^2}$; consider them parametrized by time t. These are $\Gamma^+(\tau,\xi)$, $\Gamma^-(\tau,\xi)$ outgoing and incoming, i.e. $\frac{dy}{dt} > 0$, $\frac{dy}{dt} < 0$

$$\Gamma^{\pm} \text{ is } \eta = \mp\sqrt{\tau^2 - |\xi|^2} .$$

We suppose $\Gamma^+ \notin$ WFu and wish to conclude, using Theorem 2, that $\Gamma^- \notin$ WFu.

One verifies that the condition for ellipticity means

$$\sqrt{\tau^2 - |\xi|^2} - (\gamma_1 + i\gamma_2)\tau \neq 0 \quad \text{for} \quad \tau^2 - |\xi|^2 > 0 .$$

Thus it is elliptic if $\gamma_1\gamma_2 \neq 0$. If $\gamma_2 = 0$ then it is elliptic $\Longleftrightarrow \gamma_1 \leq 0$ or $\gamma_1 > 1$. In case $\gamma_1 = 1$ one can construct solutions of the form

$$u = w(y+t)$$

so that, in fact, the solution is identically zero after a finite time — showing that the backward problem is not well posed.

In Theorem 2 the condition of hypoellipticity is satisfied (one has, in fact, subellipticity) in case $\gamma_1 = \text{constant} \neq 0$ and, wherever γ_2 vanishes,

$$\frac{\partial}{\partial t}\gamma_2 > \left[\frac{\sum_i \left(\frac{\partial\gamma_2}{\partial x_i}\right)^2}{1-\gamma_1^2}\right]^{1/2} .$$

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