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TRODUCTION.

Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary Γ. The motion of an incompressible perfect fluid filling Ω is governed by the Euler equations

$$(0.1) \qquad \frac{\partial u}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \text{grad}\, \pi = f \quad \text{in} \quad \Omega \times (0,T) ,$$

$$(0.2) \qquad \text{div}\, u = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$(0.3) \qquad u.n = 0 \quad \text{on} \quad \Gamma \times (0,T)$$

$$(0.4) \qquad u(x,0) = u_o(x) \quad \text{in} \quad \Omega ,$$

where $f = f(x,t)$, $u_o = u_o(x)$ are given, $u(x,t) = u = (u_1,u_2,u_3)$ and $\pi = \pi(x,t)$ are the unknowns, the velocity vector and the pressure ; n is the unit outward normal on Γ.

The problem of existence and uniqueness of solutions of the Euler equations has been considered by several authors and most recently by T. Kato [4], [5], D. Ebin and J. Marsden [3], J.P. Bourguignon and H. Brezis [2]. In [4] T. Kato proves the existence of a global solution in the two dimensional case and in [5] the existence of a local solution in the three dimensional case, for $\Omega = \mathbb{R}^3$. The existence of a local solution in the general case, i.e. Ω a domain of $\mathbb{R}^3$ with a boundary, was then proved by D. Ebin and J. Marsden [3] using technics of Riemanian Geometry on infinite dimensional manifolds, and by J.P. Bourguignon and H. Brezis [2] who give an alternate proof of the local existence, more analytical but relying still on geometrical technics.

Our purpose here is to give a new short proof of this result, using a new local a priori estimate and standard technics in partial differential equations. Our proof is essentially an extension of that of T. Kato [5] to bounded domain, with a suitable treatment of the boundary terms which do not appear in [5].

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PLAN.

1. A priori estimates of the solutions of the Euler Equations.

2. The existence and uniqueness result.

1. A PRIORI ESTIMATE OF THE SOLUTIONS OF THE EULER EQUATIONS.

1.1. Notations.

We will use classical notations and results concerning the Sobolev spaces : $W^{s,p}(\Omega)$, s integer, $1 \leq p < \infty$, is the Sobolev spaces of real valued L^p functions on Ω , such that all their derivatives up to order s belong to $L^p(\Omega)$. If $p = 2$, we write $H^s(\Omega) = W^{s,2}(\Omega)$.

We write (f,g) , $|f|$, the scalar product and the norm in $L^2(\Omega)$, $((f,g))_m$ and $\|f\|_m$, the scalar product and the norm in $H^m(\Omega)$,

$$((f,g))_m = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g) ,$$

where D^α is a multi-index derivation, $\alpha = \{\alpha_1,\alpha_2,\alpha_3\}$. The norm in $L^p(\Omega)$ is denoted $|f|_p$ and $\|f\|_{m,p}$ denotes that of $W^{m,p}(\Omega)$. The same notations will be used also for the norms and scalar products in $L^2(\Omega)^3$, $H^m(\Omega)^3$, ...

We assume that the boundary of Ω is a two dimensional manifold of class $\mathscr{C}^r$ with r sufficently large so that the usual embedding theorems hold. In particular : $W^{m,p}(\Omega) \subset L^r(\Omega)$ where $\frac{1}{r} = \frac{1}{p} - \frac{m}{3}$ if $m < \frac{3}{p}$, $1 \leq r < \infty$ is arbitrary if $m = \frac{3}{p}$, $r = \infty$ if $m > \frac{3}{p}$ (in this case $W^{m,p}(\Omega)$ is even a space of Hölderian functions).

We recall also that if $m > \frac{3}{p}$, (and Ω is smooth), $W^{m,p}(\Omega)$ is an algebra for the pointwise multiplication of functions (see [2], [3]).

Let

$$X_m = \{v \in H^m(\Omega)^3 , \operatorname{div} v = 0, v.n = 0 \text{ on } \Gamma\}$$

$$X_{m,p} = \{v \in W^{m,p}(\Omega)^3 , \operatorname{div} v = 0, v.n = 0 \text{ on } \Gamma\} .$$

For $m = 0$, X_0 is a closed subspace of $L^2(\Omega)^3$ and we denote P the orthogonal projection in $L^2(\Omega)^3$ on X_0 . We recall that P is also a linear continuous operator from $W^{m,p}(\Omega)^3$ into itself $(m\geq 1)$. Indeed if $v \in W^{m,p}(\Omega)^3$, then $(I-P)v = \text{grad}\,\pi$, where π is solution of the Neuman problem

$$(1.1)\quad \begin{cases} \Delta\pi = \text{div } v & (\in W^{m-1,p}(\Omega)) \\ \dfrac{\partial\pi}{\partial n} = v.n & (\in W^{m-\frac{1}{p},p}(\Gamma)) \end{cases}$$

and $\pi \in W^{m,p}(\Omega)$ by the classical results of regularity for the Neuman problem (Agmon Douglis Nirenberg [1]).

1.2. Representation of π as a functional of u .

We will now assume that u and π are solutions of (0.1)-(0.4) and we will establish an energy inequality satisfied by u . We assume at present that u and π are classical solutions of (0.1),(0.4) as smooth as necessary for the subsequent calculations to make sense.

The following result will be usefull

Lemma 1.1. If u and π satisfy (0.1)-(0.3), then

$$(1.2)\qquad \Delta\pi = \text{div } f - \sum_{i,j} D_j u_i . D_i u_j \ , \text{ in } \Omega$$

$$(1.3)\qquad \frac{\partial\pi}{\partial n} = f.n + \sum_{i,j} u_i u_j \ \phi_{ij} \quad \text{on } \Gamma ,$$

the functions ϕ_{ij} depending only on Γ , $D_i = \frac{\partial}{\partial x_i}$, $n = \{n_1,n_2,n_3\}$.

Proof. We get (1.2) by applying the divergence operator on both sides of (0.1). Taking then the scalar product of each side of (0.1) with n , we get on Γ :

$$(1.4)\qquad \frac{\partial\pi}{\partial n} = f.n - \sum_{i,j} u_i (D_i u_j)\ n_j \ .$$

Since Γ is a smooth manifold, we can locally represent it by an equation

$$\phi(x) = 0 ,$$

and on the corresponding part of Γ (say Γ_0) ,

$$n(x) = \frac{\text{grad } \phi(x)}{|\text{grad } \phi(x)|}$$

(ϕ is a smooth function in some neighborhood Ω_o of Γ_o).

Then

$$|\text{grad } \phi(x)| \; u_i(x)(D_i \; u_j(x)).n_j(x) = u_i(x)(D_i \; u_j(x)) \; D_j\phi(x) \; .$$

Since

$$u(x) \; . \; n(x) = 0 \quad \text{on} \quad \Gamma \; ,$$

we have

$$u(x) \; . \; \text{grad } \phi(x) = 0$$

when $\phi(x) = 0$ and the gradients of these two functions are therefore parallel on Γ_o:

$$D_i(u.\text{grad } \phi) = k \; D_i\phi \quad \text{on} \quad \Gamma_o \; .$$

$$\sum_{j=1}^{3} D_i \; u_j \; . \; D_j\phi = - \sum_{j=1}^{3} u_j \; . \; D_{ij}\phi + kD_i\phi \quad .$$

Whence with (0.3)

$$\sum_{i,j=1}^{3} u_i \; . \; D_iu_j \; . \; D_j\phi = - \sum_{i,j=1}^{3} u_i \; . \; u_j \; . \; D_{ij}\phi$$

and (1.3) follows with

(1.5) $$\phi_{ij}(x) = \frac{D_{ij} \; \phi(x)}{|\text{grad } \phi(x)|}$$

1.3. Quadratic estimation of π in term of u .

Lemma 1.2. If u and π satisfies (0.1)-(0.3) then for each $t>0$, for $m>\frac{5}{2}$,

(1.6) $$\|\text{grad } \pi(t)\|_m \leqslant c_1 \; \{\|f(t)\|_m + \|u(t)\|_m^2\}$$

and for $m > 1 + \frac{3}{p}$,

(1.7) $$\|\text{grad } \pi(t)\|_{m,p} \leqslant c_2 \; \{\|f(t)\|_{m,p} + \|u(t)\|_{m,p}^2\} \; ,$$

the constant c_1 depending only m and Ω , c_2 depending on p, m, and Ω .

Proof. We infer from (1.2), (1.3) and [1] that

$$\|\operatorname{grad} \pi\|_{m,p} \leq c_o \{\|\operatorname{div} f - \sum_{i,j} D_j u_i . D_i u_j\|_{m-1,p} + \|f.n + \sum_{i,j} u_i u_j \phi_{ij}\|_{W^{m-\frac{1}{p},p}(\Gamma)}\} .$$

By the triangle inequality and obvious majorations for f , it remains to estimate

$$\|\sum_{i,j} D_j u_i . D_i u_j\|_{m-1,p} \quad \text{and} \quad \|\sum_{i,j} u_i u_j . \phi_{ij}\|_{W^{m-\frac{1}{p},p}(\Gamma)} .$$

Since $m > 1 + \frac{3}{p}$, $W^{m-1,p}(\Omega)$ is an algebra and

$$\|D_j u_i . D_i u_j\|_{m-1,p} \leq c_3 \|D_j u_i\|_{m-1,p} \|D_i u_j\|_{m-1,p}$$

(c_o, c_3 depend on m, p and Ω).

For the boundary term we write

$$\|\sum_{i,j} u_i u_j \phi_{ij}\|_{W^{m-\frac{1}{p},p}(\Gamma)} \leq c_4 \|\sum_{i,j} u_i u_j\|_{W^{m-\frac{1}{p},p}(\Gamma)}$$

where c_4 depends only on m, p, and the ϕ_{ij} i.e. Γ . Observing that $m - \frac{1}{p} > \frac{2}{p}$, we see that $W^{m-\frac{1}{p},p}(\Gamma)$ is an algebra and hence

$$\|\sum_{i,j} u_i u_j\|_{W^{m-\frac{1}{p},p}(\Gamma)} \leq c_5 \|u_{|\Gamma}\|^2_{W^{m-\frac{1}{p},p}(\Gamma)^3}$$

$$\leq \text{(by the trace theorems)}$$

$$\leq c_6 \|u\|^2_{W^{m,p}(\Omega)^3} .$$

1.4 A priori estimate for $p = 2$.

Let α be a multi index, $|\alpha| \leq m$. We apply the operator D^α on each side of (0.1). We then multiply by $D^\alpha u$, integrate over Ω and add these equalities for $|\alpha| \leq m$. We obtain

$$\frac{1}{2} \left(\frac{d}{dt}\right) \|u\|_m^2 = - \sum_{j=1}^{3} ((u_j \frac{\partial u}{\partial x_j}, u))_m - ((\operatorname{grad} \pi, u))_m + ((f,u))_m .$$

The first term on the right can be majorized using T. Kato [4] {(2.2) p.298[1]} and we find

$$\left|\sum_{j=1}^{3} ((u_j \frac{\partial u}{\partial x_j}, u))_m\right| \leq c' \|u\|_m^3 ,$$

where c' depends only on m.

For the other terms, we clearly have

$$((f,u))_m \leq \|f\|_m \|u\|_m ,$$

$$- ((\text{grad } \pi, u))_m \leq \|\text{grad } \pi\|_m \|u\|_m$$

$$\leq \text{(by Lemma 1.2)}$$

$$\leq c_1 \{\|f\|_m + \|u\|_m^2\} \|u\|_m .$$

Whence

$$\frac{1}{2} (\frac{d}{dt}) \|u\|_m^2 \leq c_1' \|u\|_m^3 + c_2' \|f\|_m \|u\|_m$$

$c_1' = c' + c_1$, $c_2' = 1 + c_1$,

(1.8) $$(\frac{d}{dt}) \|u\|_m \leq c_1' \|u\|_m^2 + c_2' \|f\|_m .$$

so that

(1.9) $$\|u(t)\|_m \leq y(t) , \qquad 0<t<T_o ,$$

where y is the solution of the differential equation

(1.10) $$\begin{cases} \dfrac{dy(t)}{dt} = c_1' y(t)^2 + c_2' \|f(t)\|_m , \\ y(0) = \|u_o\|_m , \end{cases}$$

and $(0,T_o)$, $0<T_o\leq+\infty$, is the interval of existence of y ; T_o depends only on c_1' c_2', and the H^m-norms of the datas f, u_o.

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(1) See (1.12) below giving a more general result using L^p norms, $p \neq 2$.

In conclusion if Ω is a smooth bounded domain, if u and π are smooth solutions of (0.1)-(0.4) and $m > \frac{5}{2}$, then the estimate (1.9) holds.

1.5. A priori estimate for $p \neq 2$.

We rapidly establish an estimate similar to (1.9), involving the norms in $W^{m,p}(\Omega)$, $m > 1 + \frac{3}{p}$.

We apply the operator D^α on each side of (0.1), we multiply by $|D^\alpha u|^{p-2} D^\alpha u$, integrate over Ω and add these equalities for $|\alpha| \leq m$. This leads to

$$(1.11) \qquad \frac{1}{p}\left(\frac{d}{dt}\right) \|u\|_{m,p}^p = - \sum_{|\alpha|\leq m} (D^\alpha(\psi + \operatorname{grad} \pi - f), |D^\alpha u|^{p-2} D^\alpha u) ,$$

where $\psi = \sum_j u_j \frac{\partial u}{\partial x_j}$. We prove hereafter that

$$(1.12) \qquad \left| \sum_{|\alpha|\leq m} (D^\alpha \psi, |D^\alpha u|^{p-2} D^\alpha u) \right| \leq c_7 \|u\|_{m,p}^3 .$$

From (1.7) and Holder inequality we see then that the right hand side of (1.11) is less than

$$c_7 \|u\|_{m,p}^3 + c_2\{\|f\|_{m,p} + \|u\|_{m,p}^2\} \|u\|_{m,p} + \|f\|_{m,p} \|u\|_{m,p} .$$

Whence

$$(1.13) \qquad \frac{d}{dt} \|u\|_{m,p} \leq c_3' \|u\|_{m,p}^2 + c_4' \|f\|_{m,p}$$

with

$$c_3' = c_2 + c_7 , \quad c_4' = 1 + c_2 .$$

We conclude from (1.11) that

$$(1.14) \qquad \|u(t)\|_{m,p} \leq z(t) , \quad 0<t<T_1 ,$$

where z is the solution of

$$(1.15) \qquad \begin{cases} \frac{dz}{dt}(t) \leq c_3' \, z(t)^2 + c_4' \|f(t)\|_{m,p} \\ z(0) = \|u_0\|_{m,p} , \end{cases}$$

and $(0,T_1)$ is the interval of existence of z.

There remains to establish (1.12).

Proof of (1.12). Application of the Leibnitz rule gives

(1.16) $$D^\alpha \psi = (u.\text{grad})\, D^\alpha u + \sum_{0<\beta\leq\alpha} c_{\alpha,\beta} (D^\beta u.\text{grad})\, D^{\alpha-\beta} u$$

Because of (0.2), (0.3), the contribution of the first term of (1.16) is zero, for each α. The contribution of the subsequent terms is less than

$$\sum_{0<\beta\leq\alpha} |c_{\alpha,\beta}|\, |(D^\beta u.\text{grad})\, D^{\alpha-\beta} u|_p |D^\alpha u|_p \ .$$

In order to prove (1.12) it is then sufficient to show that

(1.17) $$|(D^\beta u_i)(D_i\, D^{\alpha-\beta} u_j)|_p \leq c\, \|u\|^2_{m,p} \ ,$$

for each i, j, α, β, $1\leq i,j\leq 3$, $1\leq|\alpha|\leq m$, $0<\beta\leq\alpha$.

Let us show (1.17). We set $g = D^\beta u_i$, $h = D_i\, D^{\alpha-\beta} u_j$ and we observe that

$$g \in W^{m-|\beta|,p}(\Omega) \subset L^\rho(\Omega) \ ,$$

$$h \in W^{m-|\alpha|+|\beta|-1}(\Omega) \subset L^\sigma(\Omega) \ ,$$

$$|g|_\rho \leq c|g|_{m-|\beta|,p} \leq c\|u\|_{m,p} \ ,$$

$$|h|_\sigma \leq c|h|_{m-|\alpha|+|\beta|-1} \leq c\|u\|_{m,p} \ ,$$

for the values of ρ and σ given by Sobolev inclusion theorems $(m-|\beta| \geq 0$, $m-|\alpha|+|\beta|-1 \geq 0$ as $|\alpha| \geq |\beta| \geq 1)$. If ρ or σ is infinite then we just write

$$|gh|_p \leq |g|_\infty |h|_p \leq c|g|_{m-|\beta|,p}\, |h|_p \leq c\|u\|^2_{m,p}$$

or

$$|gh|_p \leq |g|_p |h|_\infty \leq c\,|g|_p |h|_{m-|\alpha|+|\beta|-1} \leq c\,\|u\|^2_{m,p} \ .$$

If $|\beta| = m - \frac{3}{p}$, $\rho\geq 1$ is arbitrary, but in this case

$m-|\alpha|+|\beta|-1 = 2m-|\alpha|-\frac{3}{p}-1 \geq m-\frac{3}{p}-1 > 0$ by assumption. Hence $\sigma>p\geq 1$ and setting $\rho = \frac{\sigma}{\sigma-1}$, we write

$$|gh|_p \leq |g|_\rho \ |h|_\sigma \leq c \quad \|u\|^2_{m,p} .$$

Similarly if $m-|\alpha|+|\beta|-1 = \frac{3}{p}$, then $\sigma\geq 1$ is arbitrary but in this case $m-|\beta| = 2m-|\alpha|-1-\frac{3}{p} \geq m-1-\frac{3}{p} > 0$. Hence $\rho>p\geq 1$, we choose $\sigma = \frac{\rho}{\rho-1}$ and we write

$$|gh|_p \leq |g|_\sigma \ |h|_\rho \quad .$$

The last case to consider is the case where ρ and σ are finite and given by

$$\frac{1}{\rho} = \frac{1}{p} - \frac{m-|\beta|}{3} , \quad \frac{1}{\sigma} = \frac{1}{p} - \frac{m-|\alpha|+|\beta|-1}{3} .$$

By Holder inequality (1.17) is satisfied in this case provided that

$$\frac{1}{\rho} + \frac{1}{\sigma} \leq \frac{1}{p}$$

i.e.

$$\frac{3}{p}-2m+|\alpha|-1 \leq 0$$

and this is true as $|\alpha|\leq m$ and $m > 1 + \frac{3}{p}$.

2. THE EXISTENCE AND UNIQUENESS RESULT.

Theorem. *Assume that Ω is a regular bounded open set of $\mathbb{R}^3$ ([1]); let m and p be given, $p\geq 1$, $m>1+\frac{3}{p}$. Then for each u_o and f,*

(2.1) $\quad u_o \in W^{m,p}(\Omega)^3$, $\operatorname{div} u_o = 0$, $u_o.n = 0$ on $\partial\Omega$,

(2.2) $\quad f \in L^1(0,T;W^{m,p}(\Omega)^3)$,

there exists a unique function u and π, defined on $(0,T_*)$,

([1]) It is sufficient to assume that $\partial\Omega$ is a two dimensional manifold of class $\mathscr{C}^{m+2}$ and Ω is locally situated on one side of $\partial\Omega$.

(2.3) $$u \in L^\infty(0,T_*;W^{m,p}(\Omega)^3)$$

(2.4) $$\pi \in L^\infty(0,T_*;W^{m+1,p}(\Omega))$$

where $T_* <$ *inf* (T,T_1) ([1]), *and satisfying* (0.1)-(0.4) *on* $(0,T_*)$.

Remarks. (i) The Theorem is also valid in higher dimensions, with the natural modification on the assumption on m $(m > 1+\frac{N}{p})$;

(ii) Because of the boundary layer effects we can not expect to prove as in Kato [5] the existence on $(0,T_*)$, for each $\nu>0$ of a solution of the Navier Stokes equations belonging to $H^m(\Omega)^3$

The proof of uniqueness is standard. We will just show the existence of u and π, considering successively the case $p = 2$ and $p \neq 2$.

Case $p = 2$.

We apply the Galerkin method with a special basis $\{w_k\}$ which we first describe

(i) For m fixed as before, we consider the space $X_m \subset H^m(\Omega)^3$, endowed with the Hilbert scalar product $((.,.))_m$, and the space X_o which is a closed subspace of $L^2(\Omega)^3$. It is clear that $X_m \subset X_o$ and X_m is dense in X_o. By the Lax-Milgram theorem, for each $g \in X_o$, there exists a unique $w \in X_m$ such that

(2.5) $$((w,v))_m = (g,v) , \qquad \forall v \in X_m .$$

The linear mapping $g \longmapsto w(g)$ is a compact self adjoint operator in X_o and it possesses an orthonormal complete family of eigenvectors w_k :

(2.6) $$\begin{cases} w_k \in X_m \text{ and} \\ ((w_k,v))_m = \lambda_k(w_k,v) , \qquad \forall v \in X_m . \end{cases}$$

(ii) Let us use the Galerkin method with this basis. For $\mu>0$ fixed we look for

(2.7) $$u_\mu = \sum_{j=1}^{\mu} g_{j\mu}(t)\, w_j$$

([1]) See (1.10) and (1.15).

satisfying

(2.8) $\frac{d}{dt}(u_\mu, w_k) + ((u_\mu . \text{grad})u_\mu, w_k) = (f, w_k)$, $1 \leq k \leq \mu$,

(2.9) $u_\mu(0) = u_{o\mu} = P_\mu u_o$,

P_μ = the orthogonal projection in X_o (or as well in X_m) on the space spanned by $w_1, \ldots, w_k$.

The equations (2.8), (2.9) are equivalent to a system of ordinary differential equations for the $g_{j\mu}$, and the existence of a solution on some interval $(0, T_\mu)$ is standard. The following a priori estimates on u_μ show that $T_\mu = T_*$ is independant of μ .

(iii) The first a priori estimate is obtained by multiplying (2.8) by $g_{k\mu}(t)$ and adding in k . It is well known (see also §.1) that

$$((u_\mu . \text{grad})u_\mu, u_\mu) = 0$$

and there remains

$$\frac{1}{2}\left(\frac{d}{dt}\right) |u_\mu|^2 = (f, u_\mu) \leq |f| . |u_\mu| .$$

This shows that $T_\mu = T$ and that

(2.10) u_μ remains bounded in $L^\infty(0,T;L^2(\Omega)^3)$ as $\mu \longrightarrow \infty$.

We can also write (2.8) as

(2.11) $\left(\frac{du_\mu}{dt}, w_k\right) + (P[(u_\mu . \text{grad})u_\mu], w_k) = (Pf, w_k)$

($w_k \in X_o$) . Now $P[(u_\mu(t) . \text{grad})u_\mu(t)] \in X_m$, $Pf(t) \in X_m$, $\forall\, t$, (see (1.11)) and we can use (2.6). We multiply (2.11) by $\lambda_k g_k$ and dd in k , $k = 1, \ldots, \mu$. We obtain

(2.12) $\frac{1}{2}\left(\frac{d}{dt}\right) \|u_\mu\|_m^2 = ((P(f - (u_\mu . \text{grad})u_\mu, u_\mu))_m$.

We have simply

$$P[f - (u_\mu . \text{grad})u_\mu] = f - (u_\mu . \text{grad})u_\mu - \text{grad}\ \pi_\mu ,$$

where π_μ is defined in term of u_μ and f by relations similar to (1.2), (1.3) (replacing u by u_μ). The relation similar to (1.6) is satisfyed and we get exactly the same relation as (1.8)

$$(\frac{d}{dt}) \|u_\mu\|_m^2 \leq c_1'\|u_\mu\|_m^2 + c_2'\|f\|_m .$$

We recall also that

$$\|u_\mu(0)\|_m = \|u_{o\mu}\|_m \leq \|u_o\|_m .$$

Whence,

$$\|u_\mu(t)\|_m \leq y(t) , \quad \forall\ t < \inf(T,T_o)$$

and

(2.13) $\quad$ As $\mu \longrightarrow \infty$, u_μ remains bounded in $L^\infty(0,T_*,\mathbb{H}^m(\Omega)^3)$ $\forall\ T_* < \inf(T,T_o)$

(iv) In order to pass to the limit in the non linear term using a compactness theorem, we need an estimate on $\frac{du_\mu}{dt}$.

Since the w_k are orthogonal in X_o, we deduce from (2.11) that

$$\frac{du_\mu}{dt} = P_\mu\ P(f-(u_\mu.\text{grad})u_\mu) .$$

Hence

$$|\frac{du_\mu}{dt}(t)| \leq |f(t) - (u_\mu(t).\text{grad})u_\mu(t)|$$

and with (2.13) it is easily found that

(2.14) $\quad \frac{du_\mu}{dt}$ remains bounded in $L^\infty(0,T_*;L^2(\Omega)^3)$ as $\mu \longrightarrow \infty$.

(v) The passage to the limit using (2.13), (2.14) and a compactness theorem (as in Lions [7]) is standard. We obtain at the limit the existence of $u \in L^\infty(0,T_*;X_m)$ such that

(2.15) $\quad \frac{d}{dt}(u(t),v) + ((u(t).\text{grad})u(t),v) = (f(t),v) \quad \forall\ v \in X_o , \quad 0<t<T_*$.

(2.16) $$u(0) = u_o$$

u satisfies all the properties announced, i.e. (0.2)-(0.4) and (2.3). Because of (2.15) the existence of π such that (0.1) is satisfied is standard (see Ladyzhenskaya [6]).

Case $p \neq 2$.

We proceed by regularization. We approximate u_o and f by $u_{o\varepsilon}$ and f_ε ,

$$u_{o\varepsilon} \in X_s$$

$$f_\varepsilon \in L^1(0,T;H^s(\Omega)^3)$$

with s sufficiently large so that

$$H^s(\Omega) \subset W^{m,p}(\Omega)$$

and $X_s \subset X_{m,p}$. We solve (0.1)-(0.4) with u_o and f replaced by $u_{o\varepsilon}$ and f_ε . The estimate analog to (1.14) and an easy estimate on $\frac{\partial u_\varepsilon}{\partial t}$ allow us to pass to the limit as $\varepsilon \longrightarrow 0$ and we obtain (0.1)-(0.4) on $(0,T_*)$.

REFERENCES.

[1] AGMON, DOUGLIS and NIRENBERG - Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I , Comm. Pure Appl. Math., 17, 1959, p.623-727.

[2] J.P. BOURGUIGNON and H. BREZIS - Remarks on the Euler equations, J. of Funct. Anal,

[3] D. EBIN and J. MARSDEN - Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math. 92, 1970, p.102-163.

[4] T. KATO - On classical solutions of two dimensional nonstationary Euler equations, Arch. Rat. Mech. Anal., 25, 1967, p.188-200.

[5] T. KATO - Non stationary flows of viscous and ideal fluids in $\mathbb{R}^3$, J. of Funct. Anal., 9, 1972, p.296-305.

[6] O.A. LADYZHENSKAYA - The mathematical theory of viscous incompressible flow, Gordon and Breach, New-York, 1969.

[7] J.L. LIONS - Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod-Gauthier-Villars, Paris, 1969.