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A SCATTERING THEORY FOR AUTOMORPHIC FUNCTIONS

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L. D. Faddeev and B. S. Pavlov [2] have recently found an interesting connection between automorphic functions and scattering theory. They consider the initial value problem for the wave equation :

(1) $$u_{tt} = y^2 \Delta u + \frac{1}{4} u$$

$$u(x,u) = f_1(x), \quad u_t(x,u) = f_2(x)$$

for functions defined in the Poincaré plane and automorphic with respect to a discrete subgroup Γ of fractional linear transformations with real coefficients of finite type. They show that the approach to scattering theory developed by the authors [3] is applicable to this problem.

Techniquely the Faddeev-Pavlov development is based on an earlier paper by Faddeev which derives the spectral theory for the Laplacian using the method of integral operators. Our aim is to give an independent and more direct proof of these results within the framework of our scattering theory.

The core of our theory deals with a one-parameter group of unitary operators $U(t)$ with infinitesimal generator A acting on a separable Hilbert space H in which two subspaces D_- and D_+, called incoming and outgoing spaces are distinguished by the following properties :

i) $U(t)D_+ \subset D_+$ for $t \geq 0$, $U(t)D_- \subset D_-$ for $t \leq 0$

ii) $\wedge\, U(t)D_+ = 0 = \wedge\, U(t)D_-$

iii) $\vee\, U(t)D_+ = H_c = \vee\, U(t)D_-$;

here H_c is the subspace orthogonal to the eigenfunctions of A. Corresponding to D_- [or D_+] there is an incoming [outgoing] unitary translation representation mapping H_c onto $L_2(R,N)$, N some auxiliary Hilbert space, for which the action of $U(t)$ is translation to the right by t units and D_- [D_+] corresponds to $L_2(R_-,N)$ [$L_2(R_+,N)$].

If f is an element of H and k_- and k_+ are its incoming and outgoing representers, then the scattering operator S is defined as

(2) $$S : k_- \to k_+ .$$

It is clear that S is unitary on $L_2(R,N)$ and commutes with translation.

We now introduce a further assumption :

iv) D_- and D_+ are orthogonal .

In this case S is causal in the sense that

$$S\, L_2(R_-) \subset L_2(R_-) \quad . \tag{3}$$

A translation representation is turned into a spectral representation by the Fourier transform :

$$\tilde{k}_\pm(\sigma) = \frac{1}{\sqrt{2\pi}} \int e^{i\sigma s} k_\pm(s)\, ds = \mathcal{F} k_\pm \tag{4}$$

In the spectral representation S becomes

$$\mathcal{S} = \mathcal{F} S \mathcal{F}^{-1} \quad . \tag{5}$$

If properties (i) - (iv) hold then $\mathcal{S}$ is multiplication by a function $\mathcal{S}(\sigma)$ called the scattering matrix whose values are unitary operators on N. Furthermore $\mathcal{S}(\sigma)$ is the boundary value of an analytic function defined in the lower half plane whose values are contraction operators on N.

Next let P_+ and P_- denote the projections into $D_+^\perp$ and $D_-^\perp$, respectively. Then the operators

$$Z(t) = P_+ U(t) P_- \ , \qquad t \geq 0 \tag{6}$$

play a central roles in the theory. They define a semigroup of operators on $K = H \ominus (D_- \oplus D_+)$.

Let B denote the infinitesimal generators of Z . If the resolvent of B that is $(\lambda - B)^{-1}$, is meromorphic then the scattering matrix $\mathcal{S}(\zeta)$ can be continued analytically from the lower half plane to be meromorphic in the whole plane. Further the spectrum of B consists of $\sqrt{-1}$ times the position of the poles of $\mathcal{S}$.

Returning now to the automorphic wave equation, we note that the energy is invariant under the action of the wave equation. Denote the initial data $\{f_1, f_2\}$ by f; then the energy form is

$$E(f) = \int_F \{\partial_x f_1|^2 + |\partial_y f_1|^2 - \frac{|f_1|^2}{4y^2} + \frac{|f_2|^2}{y^2}\}dx\,dy \tag{6}$$

where F is the fundamental domain for the subgroup Γ. It can be shown that U(t) is unitary with respect to E. However the fact that E is indefinite requires that some modifications be made in the theory outlined above. To simplify the exposition we consider automorphic functions defined by the modular group. In this case F consists of that portion of the strip $\{-1/2 < x < 1/2\}$ which lies above the unit circle.

We split F into two parts :

$$F_1 = F \cap \{y > a > 1\} \quad \text{and} \quad F_o = F - F_1$$

an integration by parts shows that E can be rewritten as

$$E(f) = E_o(f) + \int_{F_1} \{|\partial_x f_1|^2 + y|\partial_y(\frac{f_1}{\sqrt{y}})|^2 + \frac{|f_2|^2}{y^2}\}dxdy - \int_{y=a} \frac{|f_1|^2}{2y}dx \tag{7}$$

where E_o is defined as E integrated over F_o instead of F.

We define the Hilbert space H by means of the related form

$$Q(f) = \int_{F_o} \{\partial_x f_1)^2 + |\partial_y f_1)^2 + \frac{|f_1|^2}{4y^2} + \frac{|f_2|^2}{y^2}\}dxdy \tag{8}$$

$$+ \int_{F_1} \{|\partial_x f_1)^2 + y\,|\,\partial_y(\frac{f_1}{\sqrt{y}})\,|^2 + \frac{|f_2|^2}{y^2}\}dxdy + \int_{y=a} \frac{|f_1|^2}{2y}dx\ .$$

It can be shown that the solution operators U(t) of (1) form a group of operators on H which are unitary respect to E. Moreover there exist eigenfunctions $\{f_j^{\pm}, j = 1,\dots,m\}$ of A :

$$Af_j^{\pm} = \pm\lambda_j f_j^{\pm} \tag{9}$$

satisfying the following biorthogonality relations :

$$E(f_j^+, f_k^+) = 0 = E(f_j^-, f_k^-) \quad \text{for all } j,\ k, \tag{10}$$

$$E(f_j^+, f_k^-) = 0 \quad \text{for all } j \neq k, \text{ and} \tag{11}$$

(12) $$E(f_j^+, f_j^-) < 0 \quad \text{for all } j.$$

Furthermore if

(13) $$P = \text{span of } \{f_j^\pm\}$$

and

(14) $$H' = \text{E-orthogonal complement of } P$$

Then E is non negative on H'.

To simplify the discussion further, we assume that E is positive on H'_j ∧ otherwise we proceed as in [4]. In this case it can be shown that E and Q are equivalent on H', in what follows we use E for the inner product on H'.

It is easy to verifythat the functions

$$u_-(z,t) = y^{1/2}\varphi(y\,e^t) \text{ and } u_+(z,t) = y^{1/2}\varphi(y\,e^{-t})$$

are incoming and outgoing traveling wave solutions of (1). We use the corresponding initial data to define incoming and outgoing subspaces for H :

(15) $$D_\pm = \text{closure } \{y^{1/2}\varphi, \mp\, y^{3/2}\varphi'\}$$

where the φ are smooth functions with compact support in (a,∞). These subspaces satisfy properties (i) (ii) and (iv). Unfortunately they are not contained in H'. However there are two alternatives which suggest themselves :

(16)
$$D'_\pm = \text{E-projections of } D_\pm \text{ into } H'$$
$$D''_\pm = D_\pm \cap H'.$$

It can be shown that $D'_\pm$ satisfy properties (i) - (iii) in H'_c whereas $D''_\pm$ satisfy (i) - (iv) in H'_c. For data f in H' we denote the corresponding translation reprenters by $R'_\pm$ and $R''_\pm$, respectively. The scattering operator S' is then defined as the unitary mapping :

(17) $$S' : k'_- \to k'_+ .$$

In general D'_- and D'_+ are not orthogonal in which case S' is not causal. However by factoring S' as follows :

(18) $$k'_- \to k''_- \to k''_+ \to k'_+$$

we get around this difficulty. In fact since $D'_- \supset D''_-$ and $D'_+ \supset D''_+$ it is easy to show that the following operators which appear in this decomposition are causal scattering-like operators :

(19) $$\begin{aligned} S_- &: k''_- \to k'_- \\ S_+ &: k'_+ \to k''_+ \\ S'' &: k''_- \to k''_+ . \end{aligned}$$

In terms of these we can write

(20) $$S' = S_+^{-1} S'' S_-^{-1} .$$

Corresponding to this factorization we have the following decomposition of the scattering matrix :

(21) $$\mathcal{S}' = \mathcal{S}_+^{-1} \mathcal{S}'' \mathcal{S}_-^{-1} .$$

By studying the corresponding semigroups $Z_\pm$ and Z'' it can be shown that $\mathcal{S}_\pm$ and $\mathcal{S}''$ have meromorphic extensions into the upper half plane. In fact the poles of $\mathcal{S}'_\pm$ are just the points $\{i\lambda_j ; j = 1,\ldots,m\}$, and, by the Schwarz reflection principle, the zeros are at $\{-i\lambda_j ; j = 1,\ldots,m\}$. Hence by (21) the pole of $\mathcal{S}'$ occur at the points $\{-i\lambda_j ; j = 1,\ldots,m\}$ and at the poles of $\mathcal{S}''$.

The scattering matrix $\mathcal{S}'(\sigma)$ can be obtained from the Eisenstein functions. In fact one of consequences of the above theory is that the Eisenstein function can be continued to be meromorphic in the whole complex plane. In the case of the modular group we find that

(22) $$\mathcal{S}'(\sigma) = \frac{\Gamma(\frac{1}{2})\Gamma(i\sigma)}{\Gamma(i\sigma + \frac{1}{2})} \frac{\zeta(2i\sigma)}{\zeta(2i\sigma + 1)}$$

where ζ is the Riemann zeta function and Γ is the gamma function. Further it can be shown that

$$(23) \qquad \mathcal{S}''(\sigma) = \left(\frac{\sigma + i/2}{\sigma - i/2}\right)^2 \mathcal{S}'(\sigma)$$

If the Riemann hypothesis were true then the poles of $\mathcal{S}''$ would lie in the line $\operatorname{Im}\sigma = 1/4$ and the spectrum of B'' on the line $\operatorname{Re}\lambda = -1/4$. A necessary condition for the latter to occur is

$$(24) \qquad \limsup_{t \to \infty} t^{-1} \log \| Z''(t) f \| \leq -1/4$$

for a dense set of f in K. Faddeev and Pavlov have found a similar condition which is both necessary and sufficient for the validity of the Riemann hypothesis .

To apply either these criteria it is necessary to have a useful characterization of K'' and for this you need a useful characterization of H'_c. It can be shown that A has an infinite set of eigenfunction in H'. Because of this fact the only useful characterization of H'_c which we have been able to find is

$$\text{iii)} \qquad \vee\, U(t) D''_- = H'_c$$

Starting with data in D''_- , say $f = \{y^{1/2}\varphi, y^{3/2}\varphi^1\}$ with $\operatorname{supp}\varphi \subset (1,a)$, the corresponding solution of (1) in the halfspace is

$$(25) \qquad v(z,t) = y^{1/2}\varphi(y\, e^t) .$$

From this we obtain the automorphic solution by averaging over Γ-modulo the translations Γ_∞ :

$$(26) \qquad u(z,t) = \sum_{\Gamma/\Gamma_\infty} \frac{y^{1/2}}{|cz+d|} \varphi\left(\frac{y\, e^t}{|cz+d|^2}\right)$$

here the elements of Γ are denoted by

$$\gamma z = (az+b)(cz+d)^{-1}$$

where a, b, c, d, are integers and $ad - bc = 1$. Now (26) essentially arithmetizes the problem and throws the exponential decay of $Z''(t)f$ back onto number theoretical considerations. It is for this reason that we do not believe that this approach will be helpful in verifying the Riemann hypothesis.

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