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CLASSES OF SPATIALLY INHOMOGENEOUS PSEUDODIFFERENTIAL OPERATORS

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Abstract. One can obtain sharp information on a pseudodifferential operator $p(x, D)$ by embedding the symbol p in a symbolic calculus specially designed to reflect the behavior of p. We sketch the development of symbolic calculi arising in this connection, and use our results to give simple proofs of the sharp Gårding inequality and of sufficiency of Nirenberg-Treves' condition (P) for local solvability of equations of principal type.

I. A Calculus of Symbols.

Let $\emptyset$ and φ be positive smooth functions on $R^n \times R^n$ satisfying the inequalities

(1) $$c \leq \emptyset(x, \xi) \leq C(1 + |\xi|^2)^{1/2}, \quad c(1 + |\xi|^2)^{-1/2} \leq \varphi(x, \xi) \leq C .$$

(2) $$\emptyset\varphi \geq c .$$

(3) $$|D_x \varphi| \leq C , \quad |D_\xi \emptyset| \leq C , \quad |D_x \emptyset(x, \xi)| \leq C \frac{\emptyset(x, \xi)}{\varphi(x, \xi)} ,$$

$$|D_\xi \varphi(x, \xi)| \leq C \frac{\varphi(x, \xi)}{\emptyset(x, \xi)} .$$

(4) $$c \frac{\emptyset(x, \xi)}{\varphi(x, \xi)} \leq \frac{\emptyset(y, \eta)}{\varphi(y, \eta)} \leq C \frac{\emptyset(x, \xi)}{\varphi(x, \xi)} \quad \text{if} \quad |\xi - \eta| < \frac{1}{2}|\xi| .$$

For real numbers M and m we define the class $S^{M,m}_{\phi,\varphi}$ of symbols to consist of all smooth functions $a(x,\xi)$ on $R^n \times R^n$ which satisfy the conditions $|D^\alpha_x D^\beta_\xi a(x,\xi)| \leq C_{\alpha\beta}\, \phi^{M-|\beta|}(x,\xi)\, \varphi^{m-|\alpha|}(x,\xi)$ for all multi-indices α and β.

Example: Take $\phi = (1+|\xi|^2)^{\rho/2}$ and $\varphi = (1+|\xi|^2)^{-\delta/2}$ with $0 \leq \delta \leq \rho \leq 1$ $(\delta \neq 1)$. Then $S^{M,m}_{\phi,\varphi}$ is essentially Hörmander's class $S^{M\rho - m\delta}_{\rho,\delta}$ (see [1]).

Corresponding to $a(x,\xi) \in S^{M,m}_{\phi,\varphi}$ we define an operator $a(x,D)$ from the Schwartz class $\mathcal{S}$ to itself by the standard formula

$$a(x,D)\, u(x) = \int_{R^n} e^{ix\cdot\xi} a(x,\xi)\, \hat{u}(\xi) d\xi ,$$

where $\wedge$ denotes the Fourier transform.

Theorem 1. Let $a \in S^{M,n}_{\phi,\varphi}$ and $b \in S^{M',m'}_{\phi,\varphi}$ be symbols. Then

(A) The operator $a(x,D)\circ b(x,D)$ defined on $\mathcal{S}$ is the pseudodifferential operator arising from a symbol $a\circ b \in S^{M+M',m+m'}_{\phi,\varphi}$. Moreover, $a\circ b$ has the asymptotic expansion $a\circ b \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi a(x,\xi)\cdot D^\alpha_x b(x,\xi)$ in the sense that for any $N > 0$,

$$a\circ b - \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial^\alpha_\xi a \cdot D^\alpha_x \in S^{M+M'-N,\, m+m'-N}_{\phi,\varphi} .$$

(B) Similarly, the adjoint operator $a^*(x,D)$ restricted to $\mathcal{S}$ is the pseudo-**differential operator arising** from a symbol $a^{\#}(x,\xi) \in S^{M,m}_{\phi,\varphi}$. Moreover $a^{\#}$ has the asymptotic expansion $a^{\#} \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_x \bar{a}(x,\xi)$, i.e.,

$$a^{\#} - \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_x \bar{a} \in S^{M-N,\,m-N}_{\phi,\varphi} .$$

Theorem 2. If $a(x, \xi)$ is a symbol in $S^{0,0}_{\phi,\varphi}$, then $a(x, D)$ is bounded on L^2.

Remarks: 1. Theorem 2 leads to an extension of itself in which one defines analogues of the Sobolev spaces H^s relative to ϕ and φ.

2. If, in addition to (1)-(4) one assumes the estimate

$$\phi(x, \xi) \geq c|\xi|\varphi(x, \xi) , \tag{5}$$

then one can prove a theorem on the behavior of operators in $S^{M,m}_{\phi,\varphi}$ under smooth changes of co-ordinates.

We forgo writing down the explicit results.

Our proofs of theorems 1 and 2 rely heavily on ideas from a recent paper of A.P. Calderón and R. Vaillancourt [2].

II. Applications. We begin with a simple proof of the sharp Gårding inequality [3], which asserts that $\mathrm{Re}(a(x, D)u, u) \geq -C\|u\|^2$ for all $u \in L^2$, whenever $a(x, \xi)$ is a non-negative classical first-order symbol. Clearly, the estimate is unaffected (only the constant C changes) if we replace a by $a+1$, so we may and shall assume that $a(x, \xi) \geq 1$. Now set $\phi(x, \xi) = (1 + |\xi|^2)^{1/4} a^{1/2}(x, \xi)$ and $\varphi(x, \xi) = (1 + |\xi|^2)^{-1/4} a^{1/2}(x, \xi)$ and observe that inequalities (1)-(4) hold, and that $a^{1/2}(x, \xi)$ belongs to $S^{1/2, 1/2}_{\phi,\varphi}$. Thus for $T = a^{1/2}(x, D)$ we have $T^*T = a(x, D)$ + a pseudo-differential operator with symbol in $S^{0,0}_{\phi,\varphi}$ by Theorem 1; and we know that the error term is bounded, by Theorem 2. That is, $a(x,D) = T^*T$ + a bounded error, from which the sharp Gårding inequalit is obvious. Note that we could have replaced T by $T' = (a^{1/4}(x,D))^* \circ (a^{1/4}(x,D))$ to obtain a positive self-adjoint pseudo-differential oper tor whose sc are differs from $a(x,D)$ only by a bounded error. A slig t generalization of this procedure produces approximate square roots of operators with non-negative symbols in $S^{1,1}_{\phi,\varphi}$ classes; the squar roots have symbols in $S^{1/2,1/2}_{\Psi,\psi}$ for suitable Ψ, ψ.

Next we sketch a new proof of sufficien y of Nirenberg-Treves' condition (P) for local solvability of partial dif erential equations of principal type. In their paper [4], Nirenberg nd Treves reduced the whole problem to proving the following result: Lemma NT: Let $p_t(x,\xi)$ be a first-order classical symbol depending sn oothly on the real parameter t; and suppose that for each fixed (x,ξ), the function $t \to p_t(x,\xi)$ does not change sign. Then the corresponding operator $p_t(x,D)$ may be written in the form $p_t(x,D) = A_t B + C_t$, where B is a fixed (unbounded) self-adjoint operator, A_t is self-adjoint, bour led and non-negative, and C_t is a bounded error.

Ideally, one should prove the lemma si nply by writing the symbol $p_t(x,\xi)$ in the form

(*) $p_t = a_t \cdot b + c_t$, where $a_t \geq 0$ and a_t, b, c_t are classical symbols of order $0, 1, 0$ respectively.

If p_t could be so expressed, the conclusion of Lemma NT would follow instantly from the classical symbolic calculus and sharp Gårding inequality. Nirenberg and Treves carried this out in [6] for the case of real-analytic symbols p_t, but unfortunately p_t cannot in general be written in the form (*) using classical symbols. (The unpublished counterexample is due to Mather.) In our original solution [5] to the sufficiency problem, the authors circumvented direct use of Lemma NT by a complicated argument. Now, however, we can use $S^{M,m}_{\phi,\varphi}$ classes to give a rather simple direct proof of Lemma NT: we simply pick the proper ϕ and φ to enable us to write $p_t(x,\xi)$ in the form (*) with a_t, b, c_t in $S^{0,0}_{\phi,\varphi}$, $S^{1,1}_{\phi,\varphi}$, $S^{0,0}_{\phi,\varphi}$ respectively. Essentially, we can take

$$\varphi(x,\xi) = (1+|\xi|^2)^{-1/4} + \sup_t\{|D_\xi p_t(x,\xi)| + \frac{1}{|\xi|}|D_x p_t(x,\xi)|\} + \sup\{\delta>0 \,|\, p_t(y,\eta) \text{ does not change sign as a function of } (t,y,\eta) \text{ in the region } |y-x|<\delta,\ |\xi-\eta|<\delta|\xi|\}, \text{ and } \phi(x,\xi) = (1+|\xi|^2)^{1/2}\varphi(x,\xi).$$

The construction of a_t, b, c_t is not hard, and can be made without mentioning the elaborate Whitney decompositions of [5], though the latter still lurk in the background.

Detailed proofs of the results mentioned in this note will appear in [6].

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