SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

J. J. KOHN

Boundary regularity of solutions of the inhomogeneous Cauchy-Riemann equations

Séminaire Équations aux dérivées partielles (Polytechnique) (1972-1973), exp. nº 24, p. 1-9

<http://www.numdam.org/item?id=SEDP_1972-1973____A25_0>

ECOLE POLYTECHNIQUE
CENTRE DE MATHEMATIQUES
17, rue Descartes
75230 Paris Cedex 05

S E M I N A I R E   G O U L A O U I C - S C H W A R T Z   1 9 7 2 - 1 9 7 3

BOUNDARY REGULARITY OF SOLUTIONS OF THE INHOMOGENEOUS CAUCHY-RIEMANN EQUATIONS

by J. J. KOHN

Exposé N° XXIV

4 Avril 1973

Given an open, relatively compact domain M in a complex manifold M' such that :bM, the boundary of M, is smooth. We are given a form $\alpha \in L_2(M)$ of degree (0,1), i.e. in terms of local holomorphic coordinates :

(1) $$\alpha = \Sigma\, \alpha_j\, d\bar{z}_j\ ,$$

where $\alpha_j \in L_2(M)$. We are interested in finding a solution u of the equation

(2) $$\bar{\partial} u = \alpha$$

which is as "smooth as possible". More precisely, we seek a function u satisfying (2) such that

(3) $$\text{sing supp } (u) \subset \text{sing supp } (\alpha).$$

This means that if Ω is an open subset of $\overline{M}$ on which α is of class C^∞ then u restricted to Ω is of class C^∞. Since the system (2) is elliptic the condition (3) is satisfied in the interior for every solution u of (2). At the boundary, however, the problem is more delicate ; for if h is any holomorphic function on M and if u satisfies (2) then $u + h$ also satisfies (2), so that there are many solutions of (2) which do not satisfy (3) at the boundary.

The assumption that the boundary bM is smooth means that there is a real-valued function r, of class C^∞, defined in a neighborhood of bM such that $dr \neq 0$ and $r(P) = 0$ if and only if $P \in bM$. We will fix the sign of r so that $r > 0$ outside of $\overline{M}$ and $r < 0$ inside of M. For each $P \in bM$ we denote by $T_P^{1,0}(bM)$ the subspace of the complex tangent vectors $\mathbf{C}T_P(bM)$ of the form

(4) $$L = \Sigma\, \zeta_j \frac{\partial}{\partial z_j} \qquad \text{with } L(r) = \Sigma\, \zeta_j\, r_{z_j}(P) = 0.$$

The <u>Levi form</u> at $P \in bM$ is a hermitian form on $T_P^{1,0}(bM)$ defined by :

(5) $$\langle \partial\bar{\partial} r, L_\wedge \bar{L}\rangle = \Sigma\, r_{z_i \bar{z}_j}(P)\, \zeta_i \bar{\zeta}_j .$$

If this form is non-negative for each $P \in bM$, we say that M is <u>pseudo-convex.</u> From now on we will assume that M is pseudo-convex.

If $M \subset\subset \mathbb{C}^2$ is a pseudo-convex domain such that in a neighborhood U of (0,0) the function $r = \mathrm{Re}(z_2)$; then, we set $\alpha = \frac{\bar{\partial}P}{z_2}$ with $\rho \in C_0^\infty(U)$ and $\rho \equiv 1$ in a neighborhood U' of (0,0). Now we will show that there is no solution of (2) which satisfies (3). For if there were a function u satisfying (2) and (3) then the function $h = u - \frac{\rho}{z_2}$ would be holomorphic.

Restricting h to the line $z_2 = -\delta$ we obtain a function on a disc in z_1 which on the boundary of the disc is bounded independently of δ and at the origin behaves like $\frac{1}{\delta}$, this is a contradiction. Nevertheless we do have the following positive result.

<u>Theorem</u> : If M is pseudo-convex and if there exists a strongly pluri-sub harmonic non-negative function λ in a neighborhood of bM (for example if $M \subset \mathbb{C}^n$ we can set $\lambda = |z|^2$) and if α is a (0,1)-form in L^2 such that $\bar{\partial}\alpha = 0$ and such that α is orthogonal to the null space of $\bar{\partial}^*$ (the L_2-adjoint of $\bar{\partial}$), then there exists $u \in L_2(M)$ such that $\bar{\partial}u = \alpha$. If furthermore sing supp $(\alpha) = \emptyset$ (i.e. $\alpha \in C^\infty(\bar{M})$) then for each m there exists $u_m \in C^m(\bar{M})$ such that $\bar{\partial}u_m = \alpha$.

<u>Outline of proof</u> : The existence of a solution u has been proved by Hörmander (see [4]). His proof is based on an estimate with weight functio which we also use here. For $t \geq 0$ set

(6) $$(\varphi,\psi)_{(t)} = (\varphi, e^{-t\lambda}\psi) \text{ and } \|\varphi\|^2_{(t)} = (\varphi,\varphi)_{(t)}.$$

Denote by $\bar{\partial}^*_t$ the adjoint of $\bar{\partial}$ with respect to the norm $\| \ \|_{(t)}$. The smooth forms in the domain of $\bar{\partial}^*_t$ are given by

(7) $$\mathcal{D} = C^\infty(\bar{M}) \cap \mathcal{D}om(\bar{\partial}^*_t) = \{\varphi \mid \Sigma\, r_{z_j}\varphi_j = 0 \quad \text{on } bM\}.$$

Let Q_t be a quadratic form on $\mathcal{D}$, defined by :

(8) $$Q_t(\varphi,\psi) = (\bar\partial\varphi,\bar\partial\psi)_{(t)} + (\bar\partial_t^*\varphi,\bar\partial_t^*\psi)_{(t)} + (\varphi,\psi)_{(t)}$$

and let $\tilde{\mathcal{D}}_t$ be the completion of $\mathcal{D}$ under Q_t. Now the estimate referred to above (and proved in [4]) is the following : there exists a function $f \in C_o^\infty(M)$, a constant $C > 0$ independent of t and for each t a $C_t > 0$ such that :

(9) $$t\|\varphi\|^2_{(t)} \le C\,Q_t(\varphi,\varphi) + C_t\|f\,\varphi\|_1^2 \quad ,$$

where $\| \;\|_1$ denotes the Sobolev one-norm. Given α there exists a unique $\varphi_t \in \tilde{\mathcal{D}}_t$ such that :

(10) $$Q_t(\varphi_t,\psi) = (\alpha,\psi)_{(t)},$$

for all $\psi \in \mathcal{D}$. Using the methods of [8] one can establish the following estimate for $\varphi_t \in C^\infty(\bar M)$. For each s there exists T_s and $C_{s,t}$

(11) $$\|\varphi_t\|_s \le C_{s,t}\,\|\alpha\|_s, \quad \text{whenever } t \ge T_s.$$

Here $\| \;\|_s$ denotes the Sobolev s-norm. We can also show that, if $\mathcal{H}_t$ is defined by

(12) $$\mathcal{H}_t = \{\varphi \in \tilde{\mathcal{D}}_t \mid Q_t(\varphi,\varphi) = \|\varphi\|^2_{(t)}\},$$

then for t sufficiently large there exists $C > 0$ such that for all $\varphi \in \tilde{\mathcal{D}}_t$ with $\varphi \perp \mathcal{H}_t$ we have

(13) $$\|\varphi\|^2_{(t)} \le C(\|\bar\partial\varphi\|^2_{(t)} + \|\bar\partial_t^*\varphi\|^2_{(t)}) \;.$$

From (9) and interior ellipticity, it follows that $\mathcal{H}_t$ is finite dimensional if t is sufficiently large ; again using the methods of [8] it can be shown that $\mathcal{H}_t \subset H_s$ when $t \ge T_s$, where H_s denotes the Sobolev space. It then follows the unique solution v_t of $\bar\partial v_t = \alpha$ which is orthogonal to the holomorphic functions under the $(\ ,\ )_{(t)}$ inner product has the property that $v_t \in H_s$ if $t \geq T_s$. The assertion then follows by the Sobolev imbedd theorem.

The details of this proof will appear in [6].

We remark that in [2] Grauert gives examples of pseudo convex domains for which the above conclusions do not hold, in his examp the function λ does not exist. It would be desirable to improve the abov theorem and to establish the existence of a solution $u \in C^{\infty}(\overline{M})$.

Returning to our general question, we wish to find conditions on M such that whenever (2) has a solution it also has a solu tion satisfying (3). Examples such as the one above lead to the followin conjecture :

<u>Conjecture</u> : If bM contains a connected non-trivial analytic variety then there exists a form $\alpha = \bar{\partial} v$ with the property that no solution of (2) satisfies (3).

If $P \in bM$ and P is a regular point of a non-trivial con ted analytic variety $V \subset bM$ then there exists a vectorfield L of degree (1,0) defined in a neighborhood U of P with the property that L restric to V is tangent to V.

Denoting by $T_P^{0,1}(bM)$ the space of vectors conjugate to $T_P^{1,0}(bM)$; we observe that all vectors tangent to V are contained in $T_P^{1,0}(bM) + T_P^{0,1}(bM)$. In particular, since all elements of the Lie algeb generated by L and $\overline{L}$ are tangent to V they are all contained in $T_P^{1,0}(bM) + T_P^{0,1}(bM)$. This motivates the following definition :

<u>Definition</u> : If $P \in bM$ and L is a vectorfield of type (1,0) defined o a neighborhood U of P such that for each $Q \in U \cap bM$, $L_Q \in T_Q^{1,0}(bM)$ th we let $\mathcal{L}^0(L)$ be the space spanned by L and $\overline{L}$ and for each integer $k > 0$ we let

(14) $$\mathcal{L}^k(L) = \mathcal{L}^{k-1}(L) + [\mathcal{L}^{k-1}(L), \mathcal{L}^0(L)] \ .$$

We denote by $\mathcal{L}_P^k(L)$ the space of vectors obtained by evaluating all the vector fields in $\mathcal{L}^k(L)$ at P. We say that L is of finite order at P if for some k :

$$\mathcal{L}_p^k(L) \not\subset T_P^{1,0}(bM) + T_P^{0,1}(bM) , \tag{15}$$

We say L is of order k at P if k is the lowest integer for which (15) holds and we say that L is infinite order at P if (15) does not hold for any k.

The following are properties of the above definitions.

(a) The order of L at P depends only on the value of L at P, i.e. if L and L' are two vectorfields which on bM are in $T^{1,0}(bM)$ and if $L_p = L'_p$ then the order of L at P is equal to the order of L' at P. Thus we can speak of the order of a vector in $T_p^{1,0}(bM)$.

(b) If M is pseudo-convex $L \in T_P^{1,0}(bM)$ is of order k then k is odd.

(c) If M is pseudo-convex, then M is strongly pseudo-convex (i.e. the Levi form (5) is positive definite) if and only if each non-zero $L \in T_p(bM)$ for all $P \in bM$ is of order one.

(d) All vectors in $T_P^{1,0}(bM)$ are of infinite order if and only if the Levi form applied to every vectorfield L has a zero of infinite order at P.

These properties show that, in some sense , the notion of order measures the convexity of bM at P. However, an example given in a joint paper with L. Nirenberg (see [9]) shows that this convexity does not imply the existence of separating holomorphic functions.

Definition : We say that subellipticity holds for the domain M if there exists $\varepsilon > 0$ and $C > 0$ such that

$$\|\varphi\|_\varepsilon^2 \le C\,Q(\varphi,\varphi) \qquad \text{for all } \varphi \in \mathcal{D}, \tag{16}$$

where $Q = Q_o$ defined by (8), $\mathcal{D}$ is defined by (7) and $\| \ \|_\varepsilon$ is the Sobolev ε-norm.

An important consequence of this concept is that if subellipticity holds then the unique solution u of (2), which is orthogonal to the holomorphic functions, satisfies (3) (see [1] and [8]). We will now discuss under what circumstances this condition is satisfied.

The estimate (16) can never hold with $\varepsilon > 1$. This estimate holds with $\varepsilon > \frac{1}{2}$ if and only if the dimension of M is one, in this case $\varepsilon = 1$ and Q is basically the classical Dirichlet integral. The estimate holds with $\varepsilon = \frac{1}{2}$ if and only if M is strongly pseudo-convex.

The following conjecture has been proved for very large classes of domains and the proof of the sufficiency in the general case is almost complete.

<u>Conjecture</u> : Subellipticity holds for some $\varepsilon > 0$ in a domain M if and only if for each $P \in bM$ and each $L \in T_P^{1,0}(bM)$, $L \neq 0$, is of finite type.

<u>Outline of proof of sufficiency</u> : First we remark that the estimate (16) is localizable, i.e. it suffices to show that for each $P \in bM$ there exists a neighborhood U of P, such that (16) holds for all $\varphi \in \mathcal{D} \cap C_0^\infty(U \cap \overline{M})$. Next, subellipticity holds independently of the hermitian metric (this is proved in great generality in [10]). The proof involves choosing an appropriate basis for the vectorfield in $T^{1,0}(bM)$ and the hermitian metric is defined by requiring that basis be orthonormal. Let $L_1,\dots,L_n$ be a basis for the vectorfields of degree (1,0) on a neighborhood U of $P \in bM$, such that :

$$L_j(r) = 0 \quad \text{for } j = 1,..,n-1 \text{ and } L_n(r) = 1, \tag{17}$$

and define N by

$$N = L_n - \overline{L}_n \quad . \tag{18}$$

Then for each $P \in U \cap bM$ the vectors L_j, $\overline{L}_j$ for $1 \le j \le n-1$ and N, evaluated at P, are a basis of $\mathbb{C}T_P(bM)$. Let $\omega^1,\dots,\omega^n$ be the dual basis of $L_1,\dots,L_n$; thus if φ is a (0,1)-form on U it can be expressed as :

$$\varphi = \sum_{j=1}^{n} \varphi_j \overline{\omega}^j \ . \tag{19}$$

The condition that $\varphi \in \mathcal{D}$ is equivalent to

(20) $$\varphi_n = 0 \quad \text{on} \quad bM.$$

In terms of the above basis for $\mathbb{C}T_p(bM)$ the Levi form can be expressed as follows :

(21) $$[L_i, L_j] = c_{ij} N \quad (\text{mod} \sum_{j=1}^{n-1} \mathcal{L}^o(L_j)),$$

c_{ij} is then the Levi form.

Now, if M is pseudo-convex we have the following estimate (see [1]).

(22) $$\sum_{i,j=1}^{n-1} \int_{bM} c_{ij}\varphi_i \overline{\varphi}_j dS + \sum_{i,j=1}^{n} \|\overline{L}_i \varphi_j\|^2 + \|\sum_{i=1}^{n-1} L_i \varphi_i\|^2 + \|\varphi_n\|_1^2$$

$$\leq \; CQ(\varphi,\varphi), \quad \text{for all } \varphi \in \mathcal{D} \cap C_o^\infty(U \cap \overline{M}).$$

Let $X_1, \ldots, X_{2n-1}$ be C^∞ functions such that $X_1, \ldots, X_{2n-1}$, r form a local real C^∞ coordinate system in a neighborhood U of P. If $u \in C_o^\infty(U \cap \overline{M})$ we define the tangential Fourier transform by :

(23) $$\tilde{u}(\xi, r) = \int_{\mathbb{R}^{2n-1}} e^{-ix.\xi} u(x,r)\, dx,$$

where

$$\xi = (\xi_1, \ldots, \xi_{2n-1}), \quad x = (x_1, \ldots, x_{2n-1}), \; x.\xi = \sum_1^{2n-1} x_j \xi_j$$

and $dx = dx_1 \ldots dx_{2n-1}$.

For each $s \in \mathbb{R}$ we define the tangential s-norm of u by :

(24) $$|||u|||_s^2 = \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^{o} (1 + |\xi|^2)^s \, |\tilde{u}(\xi, r)|^2 \, d\xi \; dr \; .$$

The following estimate is equivalent to (16) :

(25) $$\sum_{k=1}^{n} \sum_{j=1}^{2n-1} |||\frac{\partial \varphi_k}{\partial x_j}|||_{\varepsilon-1}^2 + \sum_{k=1}^{n} |||\frac{\partial \varphi_k}{\partial r}|||_{\varepsilon-1}^2 \leq CQ(\varphi,\varphi) \quad \text{for } \varphi \in \mathcal{D} \cap C_o^\infty(U \cap \overline{M}).$$

Establishing (25) is equivalent to bounding

$$(N\varphi,\ T^{2\varepsilon-1}\varphi) \tag{26}$$

by the left hand side of (22), here $T^{2\varepsilon-1}$ is a pseudo differential operator of order $2\varepsilon - 1$ on the hyperplanes r = const. which depends in a C^∞ manner on r. The condition of finite order can be expressed as follows,

$$L = \sum_{j=1}^{n-1} \zeta_j L_j \quad , \tag{27}$$

L is of order k at P if and only if k is the lowest integer such that

$$(L\overline{L})^{\frac{k-1}{2}} (\Sigma\, c_{ij}\zeta_i\overline{\zeta}_j) \neq 0. \tag{28}$$

In case there exists a basis $L_1,\ldots,L_n$ such that $c_{ij} = \delta_{ij}$ the conjecture is proved in [7] (such a basis always exists if there is at most one eige value that vanishes) . . We can also prove the conjecture if there exists a basis $L_1,\ldots,L_n$, non-negative functions $f_1,\ldots,f_{n-1}$ and integers $m_1,\ldots,m_{n-1}$ such that

$$f_k|\zeta_k|^2 \leq \sum_{i,j=k}^{n-1} c_{ij}\, \zeta_i\overline{\zeta}_j \tag{29}$$

with

$$[(L_k\overline{L}_k)^{m_k} f_k]_P > 0 \tag{30}$$

and

$$L_j L_k^{m_k-p}\, \overline{L}_k^{m_k-p+1} f_k \sim 0 \tag{31}$$

$$\overline{L}_j L_k^{m_k-p+1}\, \overline{L}_k^{m_k-p} f_k \sim 0$$

for $j > k$ and $p = 1,\ldots,m_k$. Here ~ 0 indicates that the quantity can be estimated by lower derivatives of the f.

As yet we do not know whether such a basis exists in gener However, it is possible to construct one satisfying (29), (30) and whic satisfies (31) only for $p = 1$ by use of the following lemma.

<u>Lemma</u> : Let $L_1,\ldots,L_k$ be independent vectorfields of degree (1,0) with values in $T_P^{1,0}(bM)$ for $P \in bM$, let c_{ij}, $i,j=1,\ldots,k$ be defined by (21) and $f = \det(c_{ij})$.

Then if M is pseudo-convex and if all non zero vector fields which are combinations of $L_1,\ldots,L_k$ are of finite order there exists $L = \sum_{j=1}^{n} a_j L_j$ such that $[(L\bar{L})^{m_k} f] > 0$.

In case of complex dimension 2 the necessity was proved by Greiner (see [3]) and we expect that the same methods will give the necessity as soon as sufficiency is proved in general.

REFERENCES

[1] Folland G. B. and Kohn J. J. : The Neumann problem for the Cauchy-Riemann complex. Annals of Math. Study, Vol. 75, Princeton Univ. Press (1972).

[2] Grauert H. : Bemerksverte pseudokonvexe mannifaltigkeiten, Math. Z. 81 (1963), 377-391.

[3] Greiner P. : On subelliptic estimates for the $\bar{\partial}$-Neumann problem in $\mathbb{C}^2$, J. Diff. Geom. (to appear).

[4] Hörmander L. : L^2 estimates and existence theorems for the $\bar{\partial}$ operator, Acta Math. 113 (1965), 89-152.

[5] Hörmander L. : An introduction to complex analysis in several variables, Van Nostrand (1966).

[6] Kohn J. J. : Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds, Trans. A. M. S. (to appear).

[7] Kohn J. J. : Boundary behaviour of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension 2, J. Diff. Geom. 6 (1972), 523-542.

[8] Kohn J. J. and Nirenberg L. : Non-coercive boundary value problems, Comm. P. App. Math. 18 (1965), 451-472.

[9] Kohn J. J. and Nirenberg L. : A pseudo-convex domain not admitting a holomorphic support function, Math. Annalen (to appear).

[10] Sweeney W. J. : Coerciveness in the Neumann problem, J. Diff. Geom. 6 (1972), 375-393.
