Séminaire Équations aux dérivées partielles - École Polytechnique

M. Sato

Microlocal structure of a single linear pseudodifferential equation

Séminaire Équations aux dérivées partielles (Polytechnique) (1972-1973), exp. n° 18, p. 1-9

<http://www.numdam.org/item?id=SEDP_1972-1973____A19_0>

© Séminaire Équations aux dérivées partielles (Polytechnique)
(École Polytechnique), 1972-1973, tous droits réservés.

L'accès aux archives du séminaire Équations aux dérivées partielles (http://sedp.cedram.org) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

ECOLE POLYTECHNIQUE
CENTRE DE MATHEMATIQUES
17, rue Descartes
75230 Paris Cedex 05

S E M I N A I R E G O U L A O U I C - S C H W A R T Z 1 9 7 2 - 1 9 7 3

MICROLOCAL STRUCTURE OF A SINGLE LINEAR PSEUDODIFFERENTIAL EQUATION

by M. SATO

Exposé N^{o} XVIII

14 Février 1973

§ 1. Let $P(x,D)u = 0$ be a single pseudodifferential equation of finite order m defined in a neighborhood of $(x_o, i\eta_o\infty)$, a point in the cosphere bundle $\sqrt{-1}\, S^*M$ of a real analytic manifold M of dimension n, and denote with V and $\overline{V}$ its characteristic variety and the complex conjugate thereof, namely the complex hypersurfaces in a complex neighborhood U of $(x_o, i\eta_o\infty)$ defined by $P_m(z,\zeta) = 0$ and $\overline{P}_m(z,\zeta)(= \overline{P_m(\overline{z},\overline{\zeta})}) = 0$, respectively, P_m denoting the principal symbol of P. If $f(z,\zeta) = 0$ be a reduced local equation for V, one can write $P_m(z,\zeta) = a(z,\zeta)(f(z,\zeta))^l$ with some integer $l > 0$ and non vanishing factor $a(z,\zeta)$.

Assumption 1 : $(x_o, i\eta_o\infty)$ is a non singular point of V as well as of $V \cap \overline{V}$.

Assumption 2 : The restriction onto $V \cap \overline{V}$ of the canonical 1-form $\omega = \zeta_1\, dz_1 + \dots + \zeta_n\, dz_n$ does not vanish at $(x_o, i\eta_o)$.

The codimension of $V \cap \overline{V}$ in U is either 1 or 2 according as $V = \overline{V}$ (the "real characteristics" case) or not. In the latter case, the degree of osculation of V and $\overline{V}$ is a constant integer, say $k(\geq 1)$, along $V \cap \overline{V}$ in a neighborhood of $(x_o, i\eta_o\infty)$. This case we classify further into two, according as $V \cap \overline{V}$ is involutory or not. Here $V \cap \overline{V}$ is said to be involutory if, together with the (reduced) local defining equations $f_1 = f_2 = 0$ of $V \cap \overline{V}$, their Poisson bracket $\{f_1,f_2\}$ vanishes on $V \cap \overline{V}$. (Of course, similar definition applies to a subvariety of an arbitrary codimension). In the opposite case of non-involutory $V \cap \overline{V}$, $(x_o, i\eta_o\infty)$ is a non degenerate point if $\{f_1,f_2\}(x_o,i\eta_o) \neq 0$.

Assumption 3 : In the case of non real V and non involutory $V \cap \overline{V}$, our $(x_o, i\eta_o\infty)$ be a non degenerate point of $V \cap \overline{V}$.

Note that in this case assuption 3 plus the first part of Assumption 1 implies Assumption 2 and the second part of Assumption 1.

Theorem 1 : Under the Assumptions 1, 2 (and 3, in the case (iii) below), the equation $P(x,D)u = 0$ is microlocally equivalent to one of the following equations, considered at $x = 0$, $\eta = (1, 0, 0, \dots, 0)$. (Note that our assumptions implies $n \geq 2$ in the cases (i), (iii) and $n \geq 3$ in the case (ii).)

(i) (The real characteristics case)

$$D_2^1 u = 0 \qquad (\text{or } x_2^1 u = 0, \text{ if one prefers}),$$

(ii) (The non real characteristics case, with involutory $V \cap \overline{V}$)

$$(D_1^{k-1} D_2 + i D_3^k)^1 u = 0$$

$$\left(\begin{array}{ll} \text{or } (D_2 + i x_3^k D_1)^1 u & = 0 \\ \text{or } (x_2 + i x_3^k)^1 u & = 0 \end{array} \right),$$

(iii) (The non real characteristics case, with non involutory $V \cap \overline{V}$)

$$(D_2 \pm i x_2^k D_1)^1 u = 0.$$

By virtue of the principles of microlocal analysis developed in [1], this theorem is readily reduced to the corresponding geometrical statement, namely to the following.

Theorem 2 : By a real contact transformation any hypersurface V satisfying assumptions 1, 2, 3 reduces microlocally to one of the following

(i) $\zeta_2 = 0$ (or $z_2 = 0$),

(ii) $\zeta_1^{k-1}\zeta_2 + i\zeta_3^k = 0$ (or $\zeta_2 + i z_3^k \zeta_1 = 0$ or $z_2 + i z_3^k = 0$),

(iii) $\zeta_2 \pm i z_2^k \zeta_1 = 0$.

The case (i) is a classical result since Lagrange-Hamilton-Jacobi (see [1]). The case (iii) is proved in [2]. Here we shall supply a proof for the case (ii), by slightly modifying the proof of theorem 2.2.1 of [1] (which says that an involutory manifold V of an arbitrary codimension r which intersects transversally with its complex conjugate $\overline{V}$ at an involutory submanifold (of codimension 2r) and satisfies the Assumptions 1 and 2 above at $(x_0, i\eta_0 \infty)$, can always be contact-transformed microlocally to $\zeta_2 + i\zeta_3 = 0, \ldots, \zeta_{2r} + i\zeta_{2r+1} = 0$ considered at $x = 0$, $\eta = (1, 0, \ldots, 0)$. We always have $2r+1 \leq n$).

Namely, we first prove Lemma 3 below, and thence our statement above (as well as theorem 2.2.1 of [1] cited above) will follow .

§ 2. Let V denote an involutory submanifold of codimension r in U, and V_o a submanifold of codimension 1 in V, both of them passing through $(x_o, i\eta_o)$. Their local defining equations will be given by $f_1 = \dots = f_r = 0$ and $f_1 = \dots = f_r = q = 0$, respectively. (Hence $q = 0$ defines a non singular hypersurface U_o in U passing through $(x_o, i\eta_o)$ which intersects transversally with V at V_o.) Here and in what follows, all functions to be considered on U are holomorphic functions in $(z,\zeta) = (z_1,\dots,z_n; \zeta_1,\dots,\zeta_n)$ which are homogeneous in variables ζ_j.

Let Λ denote an open set in $\mathbb{C}^r$ containing the origin whose point we denote by $\lambda = (\lambda_1,\dots,\lambda_r)$. Let $\Phi(\lambda) = \Phi(z,\zeta;\lambda)$ and $\psi(\lambda) = \psi(z,\zeta;\lambda)$ be holomorphic functions in $U\times\Lambda$ which vanish on $V\times\Lambda$. Hence we can write

$$\Phi(\lambda) = \Phi_1(\lambda)f_1 + \dots + \Phi_r(\lambda)f_r, \quad \psi(\lambda) = \psi_1(\lambda)f_1 + \dots + \psi_r(\lambda)f_r$$

with $\Phi_j(\lambda)$ and $\psi_j(\lambda)$ holomorphic in a neighborhood of $(x_o, i\eta_o; 0)$ in $U\times\Lambda$. Finally, we denote with $\Delta(\lambda)$ the determinant of the following $r\times r$-matrix

$$\{q,\psi(\lambda)\}\left(\frac{\partial\Phi_j(\lambda)}{\partial\lambda_k}\right)_{j,k=1,\dots,r} - \{q,\Phi(\lambda)\}\left(\frac{\partial\psi_j(\lambda)}{\partial\lambda_k}\right)_{j,k=1,\dots,r} .$$

We note that the equation $\Delta(\lambda) = 0$ as well as the condition that $\Delta(\lambda)$ should be non vanishing for a generic vector λ, depends only on V, V_o, $\Phi(\lambda)$ and $\psi(\lambda)$ and is not affected by the ambiguity of the choice of f_j, q, $\Phi_j(\lambda)$ and $\psi_j(\lambda)$. We now state.

Lemma 3 : Let holomorphic functions $h_{o1},\dots,h_{or}$ which vanish at $(x_o, i\eta_o)$ be given on U_o so that $\Delta(h_{o1},\dots,h_{or}) \not\equiv 0$ on V_o. Then they can be prolonged to holomorphic functions $h_1,\dots,h_r$ in a neighborhood of U_o in U so that $\{\psi(h_1,\dots,h_r),\Phi(h_1,\dots,h_r)\} = 0$ holds identically.

And indeed, one can construct such $h_1,\dots,h_r$ by solving a Kowalewskian system of (non-linear) first order partial differential equations, as will be seen in the below.

We remark that, if $h_j^*(z,\zeta)$ denote <u>any</u> holomorphic extension of h_{oj} into a neighborhood of U_o in U, the restriction onto V: $\{q,\Phi(h^*)\}|_V$ coincides with $\{q,\Phi(\lambda)\}|_{\lambda\mapsto h_o}$ because one has

$$\{q,\Phi(h^*)\} = \{q,\Phi(\lambda)\}|_{\lambda\to h^*} + \sum_j \{q,h_j^*\} \frac{\partial\Phi(\lambda)}{\partial\lambda_j}|_{\lambda\to h^*}$$

and $\frac{\partial\Phi(\lambda)}{\partial\lambda_j} \equiv 0$ mod. $f_1,\dots,f_r$.

<u>Proof of lemma 3</u> : Along with the ordinary Poisson bracket $\{\psi,\Phi\} = \sum_k \left(\frac{\partial\psi}{\partial\zeta_k}\frac{\partial\Phi}{\partial z_k} - \frac{\partial\psi}{\partial z_k}\frac{\partial\Phi}{\partial\zeta_k}\right)$, we have the following "prolonged" expression for the bracket of functions $\Phi(w) = \Phi(z,\zeta;w)$ and $\psi(w) = \psi(z,\zeta;w)$ involving functions $w_j = w_j(z,\zeta)$:

$$\{\psi(w),\Phi(w)\} = \sum_k \Big(\Big(\frac{\partial\psi}{\partial\zeta_k} + \sum_l (w_\zeta)_{l,k}\frac{\partial\psi}{\partial w_l}\Big)\Big(\frac{\partial\Phi}{\partial z_k} + \sum_l (w_z)_{l,k}\frac{\partial\Phi}{\partial w_l}\Big)$$

$$- \Big(\frac{\partial\psi}{\partial z_k} + \sum_l (w_z)_{l,k}\frac{\partial\psi}{\partial w_l}\Big)\Big(\frac{\partial\Phi}{\partial\zeta_k} + \sum_l (w_\zeta)_{l,k}\frac{\partial\Phi}{\partial w_l}\Big)\Big),$$

with $(w_z)_{l,k}$ and $(w_\zeta)_{l,k}$ denoting $\frac{\partial w_l}{\partial z_k}$ and $\frac{\partial w_l}{\partial\zeta_k}$, respectively. The right hand side expression will be denoted by $\Theta(w,w_z,w_\zeta) = \Theta(z,\zeta;w,w_z,w_\zeta)$. Since V is involutory, there exist holomorphic functions $\Theta_{oj}(\lambda)$ in a neighborhood of $(x_o,i\eta_o;0)$ in $U\times\Lambda$ so that we have $\{\psi(\lambda),\Phi(\lambda)\}$ $\left(= \sum_k \left(\frac{\partial\psi(\lambda)}{\partial\zeta_k}\frac{\partial\Phi(\lambda)}{\partial z_k} - \frac{\partial\psi(\lambda)}{\partial z_k}\frac{\partial\Phi(\lambda)}{\partial\zeta_k}\right)\right) = \Theta_{01}(\lambda)f_1 + \dots + \Theta_{0r}(\lambda)f_r$,

whence we obtain

$$\Theta(w,w_z,w_\zeta) = \Theta_1(w,w_z,w_\zeta)f_1 + \dots + \Theta_r(w,w_z,w_\zeta)f_r$$

by setting

$$\Theta_j(w,w_z,w_\zeta) = \Theta_{0j}(w) +$$

$$+ \sum_{k,1}((w_z)_{1,k}(\frac{\partial\psi(w)}{\partial\zeta_k} + \frac{1}{2}\sum_p(w_\zeta)_{p,k}\frac{\partial\psi(w)}{\partial w_p}) - (w_\zeta)_{1,k}(\frac{\partial\psi(w)}{\partial z_k} + \frac{1}{2}\sum_p(w_z)_{p,k}\frac{\partial\psi(w)}{\partial w_p}))\frac{\partial\Phi_j(w)}{\partial w_1}$$

$$-\sum_{k,1} \text{(the similar expression with } \Phi(w) \text{ and } \psi(w) \text{ interchanged).}$$

Let us further consider the case where $w_j = w_j(t) = w_j(z,\zeta;t)$ involve a parameter t and are holomorphic in $(z,\zeta;t) \in U\times\mathbb{C}$ in a neighborhood of $(x_o,i\eta_o;0)$. Of course we have $\{\psi(w(t)),\Phi(w(t))\} = \Theta(w(t),w_z(t),w_\zeta(t))$ as long as t is an independent parameter, while we obtain , when t is substituted by $q(z,\zeta)$, the following identity :

$$\{\psi(w(q)),\Phi(w(q))\} = (\Theta(w(t),w_z(t),w_\zeta(q)) + \frac{\partial\psi(w(t))}{\partial t}\{q,\Phi(w(t))\} - \frac{\partial\Phi(w(t))}{\partial t}\{q,\psi(w(t))\})_{t\mapsto q} .$$

The expression inside the bracket on the right hand side is again a linear form of $f_1,\dots, f_r$, and, by equating to 0 each of the coefficients we form a system of equations.

$$\Theta_j(z,\zeta;w,w_z,w_\zeta) + \frac{\partial\psi_j(w)}{\partial t}\{q,\Phi(w)\} - \frac{\partial\Phi_j(w)}{\partial t}\{q,\psi(w)\} = 0 ,$$

or equivalently

$$\Theta_j(w,w_z,w_\zeta) + \sum_k(\{q,\Phi(w)\}\frac{\partial\psi_j(w)}{\partial w_k} - \{q,\psi(w)\}\frac{\partial\Phi_j(w)}{\partial w_k})\frac{\partial w_k}{\partial t} = 0 \quad , (j = 1,\dots,r)$$

This is a determined system of first order differential equations for unknown functions $w_1,\dots,w_r$ in $(z,\zeta;t)$, and, under the assumptions of the lemma, one has a well-posed Cauchy problem if one assigns to $w_j(t)$ initial data at $t = 0$ such that $\Delta(w(0)) \neq 0$. Therefore, existence of prolongations h_j of h_{oj} with the properties claimed in the lemma is implied if one first choose an arbitrary holomorphic extension h_j^* of h_{oj} to a neighborhood of U_o in U, then solves the above system of equations by assigning h_j^* as initial data (see the remark following the lemma) to obtain the local solutions $w_j(z,\zeta;t)$ and finally, defines h_j by $h_j(z,\zeta) = w_j(z,\zeta;q(z,\zeta))$. Note that h_j and h_j^* coincide on U_o because we have $w_j(q) \equiv w_j(0) \bmod q$. (q.e.d.)

Remark 1 : If Φ_j, ψ_j, h_{oj} are all of real coefficients (i.e. $\overline{\Phi_j(\bar{z},\bar{\zeta};\bar{\lambda})} = \Phi_j(z,\zeta;\lambda)$, etc.) h_j can also be chosen real-coefficiented.

Remark 2 : If W is another involutory submanifold of codimension $s(\le r)$ in U containing V as submanifold (i.e. $V \subset W \subset U$), if our defining equation $f_1 = 0,\ldots,f_r = 0$ of V is so chosen that the first s equations define W, and if $\Phi(\lambda)$ vanishes on $W \times \Lambda$ so that it has the form $\Phi(\lambda) = \Phi_1(\lambda)f_1 + \ldots + \Phi_s(\lambda)f_s$, then we have

$$\psi(w(t))|_W = \psi(w(0))|_W \text{ and hence, } \psi(h)|_W = \psi(h^*)|_W ,$$

provided that $\{q,\Phi(w(0))\} \neq 0$ at $(x_o, i\eta_o)$. In particular, if initial data h^* are so chosen that $\{f_j,\psi(h^*)\}|_W = 0$ holds for $j = 1,\ldots,s$, then one has $\{f_j,\psi(h)\}|_W = 0$ for $j = 1,\ldots,s$, because for a holomorphic function g on U, $\{f_j,g\}|_W$, $j = 1,\ldots,s$ is completely determined by $g|_W$ (and hence one can naturally talk about $\{f_j,g_o\}|_W$ for a holomorphic function g_o on U_o).

Proof : Combining the equations

$$\{\psi(w),\Phi(w)\} = \Theta(w,w_z,w_\zeta)$$

$$\Theta(w,w_z,w_\zeta) + \frac{\partial\psi(w)}{\partial t}\{q,\Phi(w)\} - \frac{\partial\Phi(w)}{\partial t}\{q,\psi(w)\} = 0$$

and taking into account the congruence $\Phi(w) \equiv 0 \pmod{f_1,\ldots,f_s}$ we have

$$\{q,\Phi(w)\}\frac{\partial\psi(w)}{\partial t} + \{\psi(w),\Phi(w)\} \equiv 0 \pmod{f_1,\ldots,f_s}$$

and this we regard as a differential equation on W, satisfied by an unknown function $\psi(w) = \psi(z,\zeta;w(z,\zeta;t))$ of $(z,\zeta;t)$ modulo $f_1,\ldots,f_s$. (Φ is regarded as known). Then the given $\psi(w(t))$ as well as t independent $\psi(w(0))$ both constitute holomorphic solutions to this equation corresponding to the same initial data $\psi(w(0)) (\text{mod.} f_1,\ldots,f_s)$. Therefore by uniqueness of holomorphic solutions they coincide. (q.e.d.)

§ 3. Proof of theorem 2

We can assume without loss of generality that the reduced principal symbol $f(z,\zeta)$ be of the form $f = f_1 + if_2^k$ (cf. [2]). The involutory $V \cap \overline{V}$ is defined by $f_1 = f_2 = 0$. Letting a homogeneous polynomial A of u, v be given by

$$(u+v)^k = u^k + A(u,v).v \quad (\text{i.e. } A(u,v) \underset{\text{def}}{=} \Sigma_{\nu=1}^{k} \binom{k}{\nu} u^{k-\nu} v^{\nu-1})$$

we define Φ, Φ_j, ψ, ψ_j as follows :

$$\Phi(\lambda) = \Phi(z,\zeta;\lambda) = \lambda_1^k f_1 - A(\lambda_1 f_2, \lambda_2 f_1)\lambda_2 f_2^k$$

$$\Phi_1(\lambda) = \lambda_1^k, \qquad \Phi_2(\lambda) = -A(\lambda_1 f_2, \lambda_2 f_1)\lambda_2 f_2^{k-1}$$

$$\psi(\lambda) = \lambda_1 f_2 + \lambda_2 f_1, \qquad \psi_1(\lambda) = \lambda_2, \qquad \psi_2(\lambda) = \lambda_1,$$

so that we have

$$(\lambda_1^k + i\,A(\lambda_1 f_2, \lambda_2 f_1)\lambda_2)(f_1 + if_2^k) = \Phi(\lambda) + i(\psi(\lambda))^k,$$

$$\Phi(\lambda) = \Phi_1(\lambda) f_1 + \Phi_2(\lambda) f_2, \qquad \psi(\lambda) = \psi_1(\lambda) f_1 + \psi_2(\lambda) f_2,$$

and apply lemma 3 to it. The matrix $(\partial \psi_j/\partial\lambda_k)_{j,k}$ is equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while $(\partial\Phi_j/\partial\lambda_k)_{j,k}$ is congruent to $\begin{pmatrix} k\lambda_1^{k-1} & 0 \\ 0 & 0 \end{pmatrix}$ (resp. to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) modulo f_1 and f_2 if $k \geq 2$ (resp. $k = 1$). Also we have $\{q,\Phi(\lambda)\} \equiv \lambda_1^k\{q,f_1\}$ (mod. f_1,f_2).

Hence $\Delta(\lambda)|_V$, which is the determinant of

$$\{q,\psi(\lambda)\}\begin{pmatrix} k\lambda_1^{k-1} & 0 \\ 0 & 0 \end{pmatrix} - \{q,\Phi(\lambda)\}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is given by $-(\lambda_1^k\{q,f_1\})^2$ for $k \geq 2$. (Similarly we have $\Delta(\lambda) = -(\lambda_1^2 + \lambda_2^2)(\{q,f_1\}^2 + \{q,f_2\}^2)$ for $k = 1$).

So, in the case of $k \geq 2$, by choosing a real-coefficiented $q(z,\zeta)$ such that $q(x_o, i\eta_o) = 0$, $\{q, f_1\}(x_o, i\eta_o) \neq 0$ which of course exists, and initial data h_{oj}, $j = 1,2$, such that $h_{01}(x_o, i\eta_o) \neq 0$ (e.g. $h_{o1} = 1$, $h_{o2} = 0$), the condition $\Delta\ (h_{01}, h_{02}) \neq 0$ holds at $(x_o, i\eta_o)$ and h_{oj} are prolonged to such h_j that satisfy $\{\psi(h_1,h_2), \Phi(h_1,h_2)\} = 0$. The homogeneous degree of $\Phi(h_1,h_2)$, and $\psi(h_1,h_2)$ in ζ-variables can be adjusted (to 0, for example) by a corresponding adjustment to the initial data h_{oj}. The property that $h_{o1} \neq 0$ at $(x_o, i\eta_o)$ also implies that $\Phi(h_1,h_2) + i(\psi(h_1,h_2))^k = 0$ is equivalent to $f_1 + if_2^k = 0$ as a reduced defining equation of V, and $\Phi(h_1,h_2) = \psi(h_1,h_2) = 0$ to $f_1 = f_2 = 0$ as reduced defining equations of $V \cap \overline{V}$. Consequently $d\Phi$, $d\psi$ and ω are linearly independent at $(x_o, i\eta_o)$. The classical Jacobi theory now tells that $\Phi(h_1,h_2)$ and $\psi(h_1,h_2)$ go to z_2 and z_3 by a suitable contact transformation which is real coefficiented and sends $(x_o, i\eta_o)$ to $(0, i(1,0,\ldots,0))$. Then the defining equation of V assumes the form $z_2 + iz_3^k = 0$ and our theorem is proved. In place of (z_2, z_3) one may as well choose $(\zeta_2/\zeta_1, z_3)$ or $(\zeta_2/\zeta_1, \zeta_3/\zeta_1)$ to result $\zeta_2 + iz_3^k\,\zeta_1 = 0$ or $\zeta_1^{k-1}\zeta_2 + i\zeta_3^k = 0$ as the standard form of defining equation of V. (q. e. d.)

Finally we show how the key Lemma 2.2.2 to the theorem 2.2.1 of [1] is derived from lemma 3. Let again V be an involutory manifold of codimension s whose local defining equations $f_1 = \ldots = f_s = 0$ have the property that $df_1, \ldots, df_s$, df_1^c, $\ldots$, df_s^c, ω are linearly independent in the neighborhood of $(x_o, i\eta_o)$. (Whence V intersects with its complex conjugate <u>transversally</u>), and assume <u>$V \cap \overline{V}$ is also</u> involutory (of codimension 2s). Here f_j^c is defined by $f_j^c(z,\zeta) \underset{\text{def}}{=} \overline{f_j(\overline{z},\overline{\zeta})}$.

Choose first a $G(z,\zeta)$ such that $\{G, f_j\}|_V = 0$ (i.e. $\{G, f_j\} \equiv 0$ mod. $f_1, \ldots, f_s$) for $j = 1, \ldots, s$ and such that dG, $df_1, \ldots, df_s$, ω are linearly independent at $(x_o, i\eta_o)$. Choose then a real coefficiented function $q(z,\zeta)$ so that $q(x_o, i\eta_o) = 0$ and $\{G, q\}(x_o, \eta_o) \neq 0$ hold. Define $\Phi(\lambda)$ and $\Phi^c(\overline{\lambda})$ by $\Phi(\lambda) = \lambda_1 f_1 + \ldots + \lambda_s f_s$ and $\Phi^c(\overline{\lambda}) = \overline{\lambda}_1 f_1^c + \ldots + \overline{\lambda}_s f_s^c$, respectively. This means in particular that V, r, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $f = (f_1, \ldots, f_r)$ and (Φ, ψ) in lemma 3 are now replaced by $V \cap \overline{V}$, 2s

$(\lambda,\bar{\lambda}) = (\lambda_1,\ldots,\lambda_s; \bar{\lambda}_1,\ldots,\bar{\lambda}_s)$, $(f,f^c) = (f_1,\ldots,f_s; f_1^c,\ldots,f_s^c)$ and (Φ,Φ^c), respectively. Under these circumstances $\Delta(\lambda)$ in lemma 3, as the determined of the matrix

$$\{q,\Phi^c(\bar{\lambda})\}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} - \{q,\Phi(\lambda)\}\begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix},$$

takes the form $\Delta(\lambda,\bar{\lambda}) = (-\{q,\Phi(\lambda)\}\{q,\Phi^c(\bar{\lambda})\})^s = (-1)^s|\{q,\Phi(\lambda)\}|^{2s}$.

Hence, by lemma 3 and remark 2 to lemma 3, we can conclude that by a suitable choice of $h_j(t)$ we have

$$\{\Phi^c(h^c(q)),\ \Phi(h(q))\} = 0, \text{ and } \{\Phi^c(h^c(q)),f_j\} \equiv 0 \pmod{f_1,\ldots,f_s}$$

while $d\Phi(h(q))$, $d\Phi^c(h^c(q))$ and ω are linearly independent at $(x_0,i\eta_0)$. This is lemma 2.2.2 of [1].

REFERENCES

[1] M. Sato, T. Kawai and M. Kashiwara : Microfunctions and pseudo-differential equations, Proceedings of Katata Conference 1971, Springer, lecture notes in mathematics, 287 (1973) pp. 265-529.

[2] M. Sato, T. Kawai and M. Kashiwara : On the structure of single linear pseudo-differential equations, Proc. Japan Acad. , 48 (1972) pp. 643-646.