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COMMENTS TO ENFLO'S CONSTRUCTION OF BANACH SPACE WITHOUT THE APPROXIMATION PROPERTY

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The Enflo's construction of Banach spaces without the approximation property consists of three parts : the criterion for a Banach space to fail to poses the approximation property, decomposition of finite dimensional spaces into two "bad subspaces" and the final construction which is a "convolution" of constructed in the second step finite dimensional subspaces. Each of these steps is interesting i its own. We shall discuss them separately. Also we shall give some other comments and we shall pose some problems.

I - Let E be a Banach space and E' its dual. Let $\{e_i , e'_i\}_{i \in I}$ be a family of elements of $E \times E'$. Given a finite subset A of I for each $u \in L(E)$ let us define

$$tr_A u = \frac{1}{|A|} \sum_{i \in A} < u(e_i), e'_i > .$$

Proposition 1 : Assume that there exists a sequence (α_n) at positive numbers with $\sum_{n=1}^{\infty} \alpha_n < \infty$ and a sequence (A_n) of finite subsets of N such that

$$|tr_{A_{n+1}} u - tr_{A_n} u| \le \alpha_n \|u\| \quad \text{for each } n \text{ and } u \in L(E)$$

and assume that

1^o $\lim_{n \to \infty} tr_{A_n} I_d \neq 0$

2^o $\lim_{n \to \infty} tr_{A_n} u = 0$ for each $u \in L_o(E)$.

The E has not the approximation property.
The condition 1^o is fulfiled if $< e_i , e'_i > = 1$ for each $i \in I$, and the condition 2^o is fulfiled if one of the following is true:

a) for some constant K $\|e_i\|$, $\|e'_i\| \le K$ for each $i \in I$ and $\lim_{i \to \infty} e_i = 0$ in $\sigma(E,E')$ or $\lim_{i \to \infty} e'_i = 0$ in $\sigma(E,E')$.

b) $\overline{\text{span } \{e_i | i \in I\}} = E$, $< e_i , e'_j >$ for $i \neq j$ of mutaly disjoint subsets of I.

Proof : It is known that E has the approximation property if and only if the canonical mapping $i : E' \hat{\otimes} E \to L(E)$ is an injection. But $z_n = tr_{A_n}(\ .\ ) \in E' \otimes E$ and by the assumption $\|z_{n+1} - z_n\|_{\wedge} \le \alpha_n$. Since $\sum_{n=1}^{\infty} \alpha_n < \infty$ the sequence (z_n) is convergent in $E' \hat{\otimes} E$ to some z_0.

But then 1^0 implies that $z_0 \neq 0$ and 2^0 implies that $i(z_0) = 0$, and thus E has not the approximation property.

Remark 1 : Proposition 1 enables us to avoid the use of Grothendieck result that for reflexive spaces the approximation property and the bounded approximation property coincide.

Remark 2 : It is not known yet if there exists a Banach space with the approximation property and which has not the bounded approximation property.

II - Let X be a finite dimensional Banach space. Let $\{e_i, e'_i\}_{i\in I}$ be a biorthogonal complete system in E, and let $A \subset I$. Assume that for each $u \in L(E)$ there holds

$$|tr_A u - tr_{I\setminus A} u| \le \alpha \|u\|$$

then in particulary this implies that if P is any projection of X onto $E^A = \text{spen}\{e_i \mid i \in A\}$ then

$$1 \le \alpha \|P\| \text{ and hence } \|P\| \ge \frac{1}{\alpha} .$$

Thus X may be decomposed into $X^A \oplus X^B$ in such way that each projection from X onto X^A and each projection of X onto X^B is of norm greater than $\frac{1}{\alpha}$.

If X and Y are Banach spaces of the same dimension let us define

$$d(X,Y) = \inf\{\|T\|\ \|T^{-1}\| \mid T \text{ is an isomorphisme of } X \text{ onto } Y\}$$

and let $h(X) = d(X,H)$ where H is a Hilbert space of the same dimension as X.

<u>Conjecture</u> : If X is a finite dimensional Banach space then there exist a biorthogonal complete system $\{e_i , e'_i\}_{i\in I}$ in X and a subset $A\subset I$ such that

$$|tr_A u - tr_{I\setminus A} u| \le \frac{C}{h(X)} \|u\| \text{ for each } u\in L(E)$$

(C is a universal constant).

Let $2\le p<\infty$. Exactly in the same method as in Lemma 1 and Lemma 2 of the preceeding note[♦] we can find a subset $A\subset[1,n]$ such that for each $u\in L(L_p^{[1,n]})$

$$|tr_A u - tr_{I\setminus A} u| \le C_p n^{\frac{1}{p}-\frac{1}{2}} .$$

It is known that $d(L_p^{[1,n]}) \le C'_p$ (cf. [1], chapt. X, Theorem 7.10) and it is easy to see that $h(l_p^n) = n^{|\frac{1}{p}-\frac{1}{2}|}$ for $1<p<\infty$. Combining all these we arrive at :

<u>Proposition 2</u> : Let $2<p<\infty$. There exist a constant $\overline{C}_p$, a biorthogonal system $\{e_i , e'_i\}_{i\in I}$ in l_p^n and a subset $A\subset I$ such that

$$|tr_A u - tr_{I\setminus A} u| \le \overline{C}_p \frac{\|u\|}{h(l_p^n)} .$$

By duality arguments we can extend this result on l_p $1<p<2$.

In fact this is true for $1\le p\le\infty$. It was observed by A. Pelczynski that the Sobczyk decomposition of l_p^n gives the desired property. Also we can obtain it in a similar method to the one used in Lemma 1 and Lemma 2 of [4], but instead of the unite circle T the Cantor group $K=\{0,1\}^N$ is taken and the trigonometrical system is replaced by the Walsh system. This approach was developed by Figiel [2] and by Figiel and Pelczynski [3]. The advantage of this approach is that it allows to construct subspaces of l_p , $2<p\le\infty$, without the approximation property.

♦ Ref. [4].

Problem 1 : If E is not isomorphic with Hilbert space is it true that E contains a subspace without the approximation property ?

Problem 2 : Let $1 \leq p < 2$. Does L_p contain a subspace without the approximation property ?

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