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APPLICATIONS OF FOURIER INTEGRAL OPERATORS

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§ 0. INTRODUCTION

The calculus of Fourier integral operators, which can be traced back to the work of Lax [11], Ludwig [13], Maslov [14] and was brought into a final form by Hörmander [9], has been applied in at least three different forms.

Firstly we have the idea of Egorov [3] to use Fourier integral operators as similarity transformations : if A is a Fourier integral operator defined by a canonical transformation χ and if P, Q are pseudo-differential operators with principal symbols p, q respectively, then $p = q \circ \chi$ if $PA = AQ$. In this way the operator P can be transformed (at least locally in the cotangent bundle) to an operator which has a principal symbol in some standard form. Transforming back the results for the simple operator Q, one can in this way obtain results for P. This procedure has been used by Egorov [4, 6] and Nirenberg-Trèves [15] in the study of subellipticity and local solvability for general operators P with complex principal symbol p, and by Duistermaat and Hörmander [1] in the special case that p is real or $\{p,\bar{p}\} = 0$ when $p = 0$ for (semi-)global regularity and existence theorems. See also the review article of Hörmander [10] for examples of the idea of Egorov. Finally work of Sjöstrand [17, 18] and Sato C. S. [16] gives hope that the same procedure will be very fruitful in the study of general overdetermined systems.

Whereas some results obtained by Egorov's procedure also can be proved without using Fourier integral operators (see for instance Duistermaat [2]), this no longer holds when the solution operators (partly) are Fourier integral operators defined by a canonical relation differnt from the identity in $T^*(X)\backslash 0$, making them very much different from pseudo-differential operators. For instance in the case of the Cauchy problem for strictly hyperbolic operators a characterization of the solution operators as certain Fourier integral operators was already given by Lax [11] and Ludwig [13], and Maslov [14] applied this to the Schrödinger equation as $h \to 0$. The global parametrices E found by

Duistermaat-Hörmander [2] for more general operators P also contain such a part. (In fact they used Egorov's idea in the construction of E).

Finally it may happen that Fourier integral operators already occur in the formulation of the problem at hand. For instance if X_0 is a submanifold of X, then the restriction operator : $C^\infty(X) \to C^\infty(X_0)$ is a Fourier integral operator defined by the normal bundle of the diagonal $\Delta_{X_0 \times X_0} \subset X_0 \times X$. So interior boundary value problems can be formulated in terms of Fourier integral operators. As an example we shall treat a simple Cauchy-type problem from this point of view. Hopefully more complicated mixed boundery value problems can be understood better in this way. One can even imagine that in the future entirely new interesting problems will be formulated in terms of Fourier integral operators.

§ 1. REVIEW OF THE CALCULUS

Before giving a more detailed description of the applications mentioned above we give a very brief review of the calculus of Fourier integral operators.

Let u be a distribution in $X = \mathbb{R}^n$. According the Pauley-Wiener theorem we have $x_0 \notin$ Sing supp u if and only if $\widehat{vu}(\xi) = O(|\xi|^{-N})$ for $|\xi| \to \infty$, for all N and for some $v \in C_0^\infty$ with $v(x_0) \neq 0$. In other words :

$$(1.1) \qquad < u, e^{-it<x,\xi>} v(x) > = O(t^{-N}) \text{ for } t \to \infty$$

uniformly on $|\xi| = 1$, for all N. This means that u is tested by rapidly oscillating test functions $v(x)\, e^{-it<x,\xi>}$, and the asymptotic behaviour is investigated as the frequency t goes to infinity. Localizing this with respect to the normal ξ on the wave fronts $<x,\xi> =$ constant, this leads to the following definition of the wave front set WF(u) of the distribution u :

For each (x_0,ξ_0), $\xi_0 \neq 0$ we have $(x_0,\xi_0) \notin WF(u) \Leftrightarrow$ (1.1) is valid for some $v \in C_0^\infty$ with $v(x_0) \neq 0$ and uniformly for all ξ in a neighborhood of ξ_0.

If u is a distribution as a manifold X then $WF(u) \subset T^*(X)\backslash 0$ can be invariantly defined by $(x_0,\xi_0) \notin WF(u) \Leftrightarrow$

(1.2) $$< u, e^{-it\psi(x)} v(x) > = O(t^{-N}), \text{ for } t \to \infty, \text{ for all } N.$$

Here $v \in C_0^\infty(X)$, $v(x_0) \neq 0$, $d\psi(x_0) = \xi_0$ (ψ is real). Moreover the estimates must be locally uniform in the additional parameters on which ψ may depend.

In terms of pseudo-differential operators an equivalent characterization can be given by : $(x_0,\xi_0) \notin WF(u) \Leftrightarrow Au \in C^\infty$ for some $A \in L^0(X)$ with a principal symbol $a(x,\xi)$ which is invertible in a conic neighborhood of (x_0,ξ_0).

Because $\text{sing supp}\, u = \pi(WF(u))$ if π is the projection : $T^*(X) \to X$, WF(u) gives more information than sing supp u, and in fact (1.1), (1.2) can be regarded as a <u>spectral analysis</u> of the singularities of u.

A <u>Fourier integral distribution</u> is a distribution A which is defined by an integral of the form

(1.3) $$< A, v > = \iint e^{i\varphi(x,\theta)} a(x,\theta)\, v(x)\, dx\, d\theta \quad , \; v \in C_0^\infty .$$

Here $\theta = (\theta_1, \dots, \theta_n)$ are auxiliary variables, called <u>the frequency variables</u>. The <u>phase function</u> φ is supposed to be a real C^∞ function on $X \times (\mathbb{R}^N \backslash 0)$, homogeneous of degree 1 in θ. Moreover it is assumed that φ is <u>regular</u>, that is $d_{(x,\theta)}\, d_\theta\, \varphi$ has maximal rank N if $d_\theta\, \varphi = 0$. This implies that

(1.4) $$C_\varphi = \{(x,\theta) \in X \times (\mathbb{R}^N \backslash 0) \; : \; d_\theta\, \varphi(x,\theta) = 0\}$$

in an n-dimensional conic C^∞ submanifold of $X \times \mathbb{R}^N \backslash 0$ and that the mapping

(1.5) $$C_\varphi \ni (x,\theta) \mapsto (x, d_x\varphi(x,\theta)) \in T^*(X)\setminus 0$$

is an immersion : $C_\varphi \to T^*(X)\setminus 0$. The image is called Λ_φ and is an n-dimensional conic C^∞ submanifold of $T^*(X)\setminus 0$. It appears moreover that Λ_φ is Lagrangian, that is $i^*\sigma = 0$ if σ is the canonical 2-form on $T^*(X)$ and i denotes the identity : $\Lambda_\varphi \to T^*(X)\setminus 0$.
Conversely it can be shown that every conic Lagrangian submanifold Λ of $T^*(X)\setminus 0$ is locally equal to Λ_φ for some regular phase function φ.

For the amplitude function a we assume that it belongs to the symbol class $S^\mu_\rho(X\times \mathbb{R}^N)$, $0<\rho\le 1$, that is we have estimates

(1.6) $$|D^\beta_x D^\alpha_\theta a(x,\theta)| \le C(1+|\theta|)^{\mu-\rho|\alpha|+(1-\rho)|\beta|} .$$

The integral (1.2), which needs not to be asbsolutely convergent, can be interpreted as the limit of the same integrals with a replaced by a_j , a_j rapidly decreasing for $|\theta|\to\infty$ for each j, and finally $a_j \to a$ in $S^{\mu'}_\rho(X\times\mathbb{R}^N)$ for all $\mu'>\mu$ as $j\to\infty$. An equivalent interpretation can be given using partial integrations.

Now applying the method of stationnary phase to the integral

(1.7) $$\langle A, e^{-it\phi(x)} v\rangle , \quad t\to\infty$$

one obtains immediately that

(1.8) $$WF(A) \subset \Lambda_\varphi .$$

Moreover, if the graph $(x, d\phi_{(x)})$ of $d\phi$ intersects Λ_φ transversally, then the method of stationnary phase leads to an asymptotic expansion for (1.7). The leading term of this asymptotic expansion then gives rise to an invariantly defined principal symbol of A, being a density of order 1/2 with values in a complex line bundle on Λ, called the Maslov line bundle L . Here "invariant" means that the principal symbol at $(x_0,\xi_0)\in\Lambda$ does not depend on the choice of the "test phase function" ϕ

with $(x_0, d\phi_{(x_0)}) = (x_0, \xi_0)$ and $(x, d\phi_{(x)})$ intersects Λ_φ transversally at (x_0, ξ_0).

Suppose now that Λ is an arbitrary closed conic Lagrangian manifold in $T^*(X)\setminus 0$. Then a global Fourier integral A of order n, defined by Λ, is a locally finite sum of distribution A_j as in (1.3) with $\varphi = \varphi_j$, $a = a_j$, the Λ_{φ_j} forming a locally finite system of open cones in Λ and with $a_j \in S_\rho^{m+n/4 - N_j/2}$. (The number $-N_j/2$ in the growth order is necessary to get an order m of A which is independent of the number of frequency variables used, and the number $n/4$ is introduced in order to obtain additivity of the orders when Fourier integral operators, to be defined later, are multiplied). The mapping :

$$(1.9) \quad I_\rho^m(X,\Lambda)/I_\rho^{m+1-2\rho}(X,\Lambda) \to S_\rho^{m+n/4}(\Lambda,\Omega_{1/2}\otimes L)/S_\rho^{m+n/4+1-2\rho}(\Lambda,\Omega_{1/2}\otimes L)$$

which assigns to A its principal symbol (defined as the locally finite sum of the principal symbols of the A_j) is an isomorphism. This is of course only useful when $\rho > 1/2$.

If X and Y are manifolds and K is a distribution in $X \times Y$, then

$$(1.10) \qquad \langle Av, u \rangle = \langle K, u \otimes v \rangle, \quad u \in C_0^\infty(X), \ v \in C_0^\infty(Y)$$

defines a continuous linear mapping : $C_0^\infty(Y) \to \mathcal{D}'(X)$. Conversely Schwartz's kernel theorem states that every continuous linear mapping A : $C_0^\infty(Y) \to \mathcal{D}'(X)$ can be obtained in this way. The distribution K is called the <u>distribution kernel</u> K_A of the operator A. The formulation is automatically coordinate invariant if all functions, resp. distributions are taken to be densities of order 1/2, as we shall do throughout in this lecture.

From the calculus of wave front sets it follows that if $WF(K_A)$ does not contain points of the form $(x,\xi,y,0)$, $\xi \neq 0$ or $(x,0,y,\eta)$, $\eta \neq 0$, then A can be extended to a continuous linear mapping :

$\mathcal{E}'(Y) \to \mathcal{D}'(X)$ and

(1.11) $$WF(Au) \subset WF'(K_A) \circ WF(u) .$$

Here $WF'(K_A) = \{(x,\xi,y,-\eta) \in T^*(X) \times T^*(Y) \ ; \ (x,\xi,y,\eta) \in WF(K_A)\}$. We have identified $T^*(X\times Y)$ with $T^*(X) \times T^*(Y)$ and regarded $WF'(K_A)$ as a relation between $T^*(X)$ and $T^*(Y)$, in (1.1) acting on $T^*(Y)$. A can be extended to a continuous linear mapping : $\mathcal{D}'(Y) \to \mathcal{D}'(X)$ still satisfying (1.10) if in addition the projection of $\operatorname{supp} K_A$ onto X is a proper mapping.

Now a Fourier integral operator of order m defined by the closed conic Lagrangian submanifold Λ of $T^*(X)\backslash 0 \times T^*(Y)\backslash 0$ is an operator A : $C_0^\infty(Y) \to \mathcal{D}'(X)$ such that $K_A \in I_\rho^m(X\times Y, \Lambda)$. That Λ is Lagrangian means that $\sigma_{T^*(X\times Y)} = \sigma_{T^*(X)} \oplus \sigma_{T^*(Y)}$ vanishes on Λ.

Because of (1.11) and (1.8) we prefer to work with the relation $C = \Lambda'$, and we get that $\sigma_{T^*(X)} - \sigma_{T^*(Y)}$ vanishes on C. If C is the graph of a mapping $\Phi : T^*(Y) \to T^*(X)$ then this condition means that $\Phi^* \sigma_{T^*(X)} = \sigma_{T^*(Y)}$, that is Φ is a canonical transformation : $T^*(Y) \to T^*(X)$.
Because C is conic, Φ is homogeneous of degree 1.
For a general conic Lagrangian manifold Λ, the relation C therefore will be called a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$.
The operator A will be called a Fourier integral operator of order m defined by the canonical relation C, notation $A \in I_\rho^m(X,Y \,;\, C)$.

Theorem 1.1 : Let C_1 and C_2 be homogeneous canonical relations from $T^*(Y)$ to $T^*(X)$ and from $T^*(Z)$ to $T^*(Y)$ respectively. Assume that $C_1 \times C_2$ intersects the diagonal in $T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$ transversally and that the projection from the intersection to $T^*(X) \times T^*(Z)$ is proper, thus giving a homogeneous canonical relation $C_1 \circ C_2$ from $T^*(Z)$ to $T^*(X)$.

If $A_1 \in I_\rho^{m_1}(X,Y \,;\, C_1)$, $A_2 \in I_\rho^{m_2}(Y,Z \,;\, C_2)$, and the projection to $X\times Z$ of the intersection of $\operatorname{supp} K_{A_1} \times \operatorname{supp} K_{A_2}$ with the diagonal in

$X \times Y \times Y \times Z$ is proper, then $A_1 \circ A_2 \in I_\rho^{m_1+m_2}(X, Z ; C_1 \circ C_2)$ and the principal symbol of $A_1 \circ A_2$ is equal to the product of the principal symbols A_1 and A_2.

Here the last sentence means that if $a_1 = (x, \xi, y, \eta) \in C_1$, $a_2 = (y, \eta, z, \zeta) \in C_2$, then there is a bilinear mapping $(\sigma_1, \sigma_2) \mapsto \sigma_1 \times \sigma_2$, $\sigma_i \in$ the fiber at a_i of the line bundle $\Omega_{1/2} \otimes L$ over C_i, $i = 1, 2$, $\sigma_1 \times \sigma_2 \in$ the fiber at $a = (x, \xi, z, \zeta) \in C_1 \circ C_2$ of $\Omega_{1/2} \otimes L$ over $C_1 \circ C_2$. This bilinear mapping is canonically defined in terms of C_1 and C_2. Then the principal symbol of $A_1 \circ A_2$ at (x, ξ, z, ζ) is equal to the finite sum $\sum_j \sigma_1^{(j)} \times \sigma_2^{(j)}$, where $\sigma_1^{(j)}$ and $\sigma_2^{(j)}$ are the principal symbols of A_1 and A_2 at the finitely many points (x, ξ, y_j, η_j) and (y_j, η_j, z, ζ) such that

$$(x, \xi, y_j, \eta_j) \in C_1 \ , \ (y_j, \eta_j, z, \zeta) \in C_2 \ .$$

If C = identity from $T^*(X)\backslash 0$ to $T^*(X)\backslash 0$, then $I_\rho^m(X, X; C) = L_\rho^X$ = space of pseudo-differential operators of order m or X. There is a standard trivialisation of the line bundle $\Omega_{1/2} \otimes L$ over the identity, leading to an identification of the principal symbol with the classical principal part of the symbol of a pseudo-differential operator. If $A \in L_\rho^m(X)$ and $B \in I_\rho^m(X, Y ; C)$ then the principal symbol of $A \circ B$ at $(x, \xi, y, \eta) \in C$ is equal to the ordinary product of the principal symbol of A at (x, ξ) (a complex number) and the principal symbol of B at (x, ξ, y, η).

§ 2. OPERATORS WITH REAL PRINCIPAL SYMBOL

In this section we give a sketch of the results obtained in [1] for operators $P \in L_1^m(X)$, with a real principal symbol $p(x, \xi)$, being a homogeneous C^∞ function of degree m on $T^*(X)\backslash 0$. We assume that the Hamilton field $H_p(x, \xi) = \sum \frac{\partial p}{\partial \xi_j}(x, \xi) \frac{\partial}{\partial x_j} - \sum \frac{\partial p}{d x_j}(x, \xi) \frac{\partial}{\partial \xi_j}$ is not tangent to the cone axis when $p(x, \xi) = 0$, that is not $d_\xi\, p(x, \xi) = 0$ and $d_x\, p(x, \xi)$

is a multiple of ξ. In particular $dp \neq 0$ when $p=0$, so $p=0$ defines a C^{∞} conic submanifold N of $T^*(X)\setminus 0$, of codimension 1. The vector field H_p is tangent to N and its solution curves in N are calles the <u>bicharacteristic strips</u> for the operator P.

Because we only shall consider results which are invariant under multiplication by an elliptic pseudo-differential operator, we may change to $q = ap$ where $a \neq 0$ is some homogeneous C^{∞} function. Because $p=0 \Leftrightarrow q=0$ and $H_q = a H_p$ on $p=0$ we see that the same assumptions are satisfied by q. Taking a homogeneous of degree 1-m we therefore see that we may assume $m=1$.

The condition that H_p is not tangent to the cone axis at (x_o, ξ_o) is necessary and sufficient for the existence (in a conic neighborhood of (x_o, ξ_o)) of a canonical transformation

$$\chi : (x,\xi) \mapsto (X_1(x,\xi), \ldots, X_n(x,\xi), \Xi_1(x,\xi), \ldots, \Xi_n(x,\xi))$$

from $T^*(X)$ to $T^*(\mathbb{R}^n)$ such that χ is homogeneous of degree 1 and $\Xi_n(x,\xi) = p(x,\xi)$.

This follows from classical Jacobi theory : the conditions for χ are $\{X_j, X_k\} = 0$, $\{\Xi_j, \Xi_k\} = 0$, $\{\Xi_j, X_k\} = \delta_{jk}$. X_j and Ξ_k homogeneous of degree 0 and 1 respectively. ($\{f,g\} = H_f g$ denotes the classical Poisson brackets). These first order differential equations can recurrently be solved by solving a suitable Cauchy problem. In order to obtain the desired homogeneity we need that the initial manifold is conic and because H_p must be transversal to the initial manifold say for the equations $\{\Xi_n, X_k\} = \delta_{nk}$, $\{\Xi_n, \Xi_k\} = 0$, we see how the transversality condition is used.

So if $A \in I^o(X, \mathbb{R}^n; \chi)$ with invertible principal symbol near $((x_o,\xi_o), \chi(x_o,\xi_o))$ then $((x_o,\xi_o), \chi(x_o,\xi_o)) \notin WF'(PA - AQ)$, where $Q \in L^1(\mathbb{R}^n)$ has principal symbol equal to ξ_n = principal symbol of $D_n = \frac{1}{i}\frac{\partial}{\partial x_n}$. (This is the idea of Egorov). By recurrently solving equations of the form

(2.1) $$D_n b_j - b_j r = S_j \in S^{-j}$$

for the principal symbols of $B_j \in L^{-j}(\mathbb{R}^n)$ we obtain an elliptic operator $B = B_0 + B_1 + \ldots \in L^0(\mathbb{R}^n)$ such that $QB - B_n D_n = [Q,B] - B(Q-D_n) \equiv 0$ ($\equiv$ means equality modulo an integral operator with C^∞ kernel). Here $r \in S^0$ is the principal symbol of $R = Q - D_n$. It follows that $(PA - AQ)B \equiv PAB - ABD_n$, so

(2.2) $$((x_0,\xi_0), \chi(x_0,\xi)) \notin WF'(PC - CD_n)$$

if $C = AB$. The formula (2.2) expresses that not only the principal part, but the <u>whole</u> operator can be transformed to the simple operator D_n. (Locally in $T^*(X)\setminus 0$ and modulo integral operators with C^∞ kernel of course).

For the operator D_n we have the forward and backward solution $E_n^+ = i\,.\,\delta(x'-y').H(x_n-y_n)$ and $E_n^- = -i\delta(x'-y').H(y_n-x_n)$. Here H is the Heaviside function, $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$. We have used the notation $x = (x',x_n)$, $y = (y',y_n)$ for points in $\mathbb{R}^n$. It is easy to show that if $v \in \mathcal{E}'(\mathbb{R}^n)$, $v \in H_{(s)}$ before (χ,ξ) on a bicharacteristic strip of D_n, then $E_n^+ v \in H_{(s)}$ before (x,ξ) on the bicharacteristic strip. Similarly with "before" replaced by "after" for E_n^-. Here for any distribution u, $u \in H_{(s)}$ at $(x,\xi) \in T^*(X)\setminus 0$ means that $u = u_1 + u_2$ with $u_1 \in H_{(s)}$ and $(x,\xi) \notin WF(u_2)$. From this property $E_n^\pm$ we obtain the following regularity theorem.

<u>Theorem 2.1</u> : Let $P \in L_1^m(X)$ have a real and homogeneous principal symbol. If $u \in \mathcal{D}'(X)$ and $Pu = f$ then

(i) if $f \in H_{(s)}$ at (x,ξ), $(x,\xi) \notin N$, then $u \in H_{(s+m)}$ at (x,ξ).

(ii) if $f \in H_{(s)}$ in the open cone $\Gamma \subset T^*(X)\setminus 0$ and $u \in H_{(t)}$ at $(x,\xi) \in N \cap \Gamma$, $t \leq s+m-1$, then $u \in H_{(t)}$ on the whole interval of the bicharacteristic strip through (x,ξ) which is contained in Γ.

Proof : (i) follows from the usual elliptic theory.
For (ii) note that we may assume that $m = 1$. Cut off u to some $u' = Au$, $A \in L^0(X)$ such that $WF(u') \subset \Gamma_0$, Γ_0 a conic neighborhood in Γ of (x,ξ) where the transformation to D_n can be carried out, let v be the distribution in $\mathbb{R}^n$ corresponding to u'. Suppose that $\gamma(\tau)$ is the bicharacteristic strip with $\gamma(0) = (x,\xi)$ and let A be such that

$$\gamma(\tau) \notin WF(A) \text{ for } \tau \leq 0$$

$$\gamma(\tau) \notin WF(A - I) \text{ for } 0 < \tau < \varepsilon, \ \varepsilon \text{ small.}$$

The $P\mu' \in H_{(t)}$ for all $\tau < \varepsilon$. Transforming to v, applying E_n^+ and then transforming back this implies $u' \in H_{(t)}$ for all $\tau < \varepsilon$. In this way we see that $\mu \in H_{(t)}$ after (x,ξ) on the bicharacteristic strip through (x,ξ). For the other side apply E_n^-.

Note that we may allow in theorem 2.1 that H_p is tangent to the cone axis because on conic bicharacteristic strips (ii) is trivial.

Corollary 2.2 : Same assumptions as in Th. 2.1 but assume in addition that no complete bicharacteristic curve (= projection into X if a bicharacteristic strip) is contained in a compact subset K of X. Then

(i) $u \in \mathcal{E}'(K)$, $Pu \in H_{(S)} \Rightarrow u \in H_{(s+m-1)}$

(ii) the space $N(K) = \{u \in \mathcal{E}'(K) \ ; \ Pu = 0\}$ is a finite-dimensional subspace of $C_0^\infty(K)$ orthogonal to ${}^tP\,\mathcal{D}'(X)$.

(iii) if $g \in H_{(\sigma)}(X)$, resp. $g \in C^\infty(X)$ and g is orthogonal to N(K) then one can find $v \in H_{(\sigma+m-1)}(X)$, resp. $v \in C^\infty(X)$ such that ${}^tPv = g$ in a neighborhood of K.

Proof : (i) follows directly from Theorem 2.1 and (ii), (iii) follow from (i) by standard functional analysis. Note that the bicharacteristic strips for ${}^tP$ are the antipodals of the bicharacteristic strips for P, so we obtain a semi-global existence theorem for P under the same conditions.

The $E_n^{\pm}$ also can be used to construct parametrics for P. Note that $E_n^+ - E_n^- = i \cdot \delta(x'-y')$ which is a Fourier integral operator with phase function $< x'-y', \theta >$, $\theta \in \mathbb{R}^{n-1}$ and amplitude $(2\pi)^{-(n-1)} i$. It follows that

$$(2.3) \qquad E_n^+ - E_n^- \in I^{-1/2}(\mathbb{R}^n, \mathbb{R}^n ; C_n),$$

where

$$(2.4) \qquad C_n = \{(x, \xi, y, \eta) ; x' = y', \ \xi' = \eta', \ \xi_n = \eta_n = 0\}$$

is the <u>bicharacteristic relation</u> for D_n, that is $(x,\xi,y,\eta) \in C_n \Leftrightarrow (x,\xi)$ and (y,η) are on the same bicharacteristic strip for D_n. It follows that $WF'(E_n^{\pm})$ is the union of the diagonal Δ^* in $T^*(\mathbb{R}^n)\backslash 0 \times T^*(\mathbb{R}^n)\backslash 0$ and the part of C_n where $x_n \gtrless y_n$, and $\chi E_n^{\pm} \in I^{-1/2}(\mathbb{R}^n, \mathbb{R}^n; C_n)$ if $\chi \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ vanishes near the diagonal.

Cutting off the $E_n^{\pm}$ in a suitable way, transforming the back to the manifold X, adding these "local parametrices" for P along the diagonal $\Delta_N \subset T^*(X)\backslash 0 \times T^*(X)$ of $N \times N$ and finally also adding the local parametrices obtained outside N by the classical elliptic theory, we obtain operators $F_{\nu}^{\pm}$ such that :

$$(2.5) \qquad P F_{\nu}^{\pm} = I + R_{\nu}^{\pm},$$

$$(2.6) \qquad WF'(F_{\nu}^{\pm}) \subset \Delta^* \cup C_{\nu,loc}^{\pm}, \quad F_{\nu}^+ - F_{\nu}^- \in I^{1/2-m}(X, X ; C_{loc})$$

$$(2.7) \qquad R_{\nu}^{\pm} \in I^{-1/2}(X, X ; C_{loc}), \quad WF'(R_{\nu}^{\pm}) \subset C_{\nu,loc}^{\pm} .$$

Her C_{loc} denotes the relation in $T^*(X)\backslash 0 \times T^*(X)\backslash 0$ defined by (x,ξ) and (y,η) are on the same bicharacteristic strip, sufficiently close to each other. It follows that this "local bicharacteristic relation" is automatically a homogeneous canonical relation. ν denotes an open and closed subset of N, $C_{\nu,loc}^+$ the part of C_{loc} where $(y,\eta) \in \nu$ and (x,ξ) is after (y,η) on the bicharacteristic strip on $(y,\eta) \in N\backslash\nu$ and (x,ξ) is before (y,η) on the bicharacteristic strip. $C_{\nu|loc}^-$ is the complement of $C_{\nu,loc}^+$ in C_{loc}

If we want to get genuine parametrices for P we must get rid of the term $R_\nu^{\pm}$. For this we need two lemmas.

Lemma 2.3 : Let C denote the bicharacteristic relation defined by : (x,ξ) and (y,η) are on the same bicharacteristic strip. Then the following conditions are equivalent

a) There are no periodic bicharacteristic strips and C is a closed C^∞ submanifold of $(T^*(X)\times T^*(X))\setminus 0$.

b) 1) No complete bicharacteristic curve is contained in a compact subset of X and

2) For every compact $K\subset X$ there exists a compact $K'\subset X$ such that every interval on a bicharacteristic strip with end points in K is contained in K'

c) There exists a conic manifold N_0, an open neighborhood N_1 of $N_0\times\{0\}$ in $N_0\times\mathbb{R}$ which is convex in the $\mathbb{R}$-direction, and a diffeomorphism $N\to N_0$, homogeneous of degree 1, which carries H_q into the operator $\frac{\partial}{\partial t}$. Here points in $N_0\times\mathbb{R}$ are denoted by (y_0,t), $q=ap$, $a\neq 0$ homogeneous of degree 1-m.

The manifold X will be called pseudo-convex with respect to the operator P if b) is satisfied.

Lemma 2.4 : Let C be a homogeneous canonical relation from $T^*(Y)\setminus 0$ to $T^*(X)\setminus 0$, such that p vanishes on the projection of C into $T^*(X)\setminus 0$. If $A\in I_\rho^\mu(X,Y;C)$ with principal symbol a, then $PA\in I_\rho^{m+\mu}(X,Y;C)$ with principal symbol equal to

$$i^{-1}\mathcal{L}_{H_{\tilde p}}a+\tilde c\,a\,. \tag{2.8}$$

Here $\tilde p$, $\tilde c$ are the pull-backs to $T^*(X)\setminus 0\times T^*(Y)\setminus 0$ of p, C by the projection on the first factor, $C\in S^{m-1}(T^*(X)\setminus 0)$ is the so-called subprincipal symbol of P and finally $\mathcal{L}$ denotes Lie-derivative.

<u>Theorem 2.5</u> : Let X be pseudo-convex for P. Then one can find $E_\nu^\pm$ such that

(2.9) $$PE_\nu^\pm \equiv I$$

(2.10) $$WF'(E_\nu^\pm) \subset \Delta^* \cup C_\nu^\pm$$

(2.11) $$E_\nu^+ - E_\nu^- \in I^{1/2-m}(X,X\,;\,C)\ .$$

The $E_\nu^\pm$ are automatically also left parametrices and any right or left parametrix satisfying (2.10) must be equal to $E_\nu^\pm$ mod C^∞. The parametrices $E_\nu^\pm$ are called the <u>distinguished parametrices</u>, because there exist many other parametrices for P. For these questions and many other details, see [1], 6.5, 6.6.

Finally we mention as another application of Lemma 2.4 the following converse of the regularity Theorem 2.1.

<u>Theorem 2.6</u> : Let $I \subset \mathbb{R}$ be an open interval and let $\gamma : I \to T^*(X)\setminus 0$ be an interval on a bicharacteristic strip which does not return to the same cone axis. Denote by Γ the closed conic hull of $\gamma(I)$ and by Γ' the limit points $= \cap\{$closed conic hull of $\gamma(I\setminus I_o)\,;\, I_o$ compact in $I\}$.

For any $s \in \mathbb{R}$ one can then find $u \in \mathcal{D}'(X)$ such that

(i) $u \in H_{(t)}(X)$ for all $t < s$

(ii) $WF(Pu) \subset \Gamma'$

(iii) $WF(u)\setminus\gamma' = WF_{(s)}(u)\setminus\Gamma' = \Gamma\setminus\Gamma'$.

Here $WF_{(s)}(u)$ is defined by $(x,\xi) \notin WF_{(s)}(u) \Leftrightarrow u \in H_{(s)}$ at (x,ξ) .

§ 3. A CAUCHY-TYPE PROBLEM

If X_0 is a submanifold of X of codimension k, then the restriction operator $\rho : C^\infty(X) \to C^\infty(X_0)$ is a Fourier integral operator of class $I^{k/n}(X_0, X ; R_0)$, where

(3.1) $$R_0 = \{(x_0, \xi_0, x, \xi) \; ; \; x_0 = x, \; \xi_0 = \zeta|T_{x_0}(X_0)\} .$$

To see this it suffices to consider the case $X = \mathbb{R}^n$, $X_0 = \mathbb{R}^{n-k}$ and then we have

(3.2) $$(\rho u)(x_0) = (2\pi)^{-n} \iint e^{i<x_0 - y, \eta>} u(y) \, dy d\eta .$$

From the calculus of wave front sets it follows that ρ can be continuously extended to the distributions u with

(3.3) $$(x_0, \xi) \notin WF(u) \text{ whenever } x_0 \in X, \; \xi|T_{x_0}(X_0) = 0 .$$

So in particular, if X_0 has codimension 1 and P has homogeneous real principal symbol $p(x,\xi)$, then ρ can be continuously extended to all distribution solutions u of $Pu = f \in C^\infty(X)$, if X_0 is <u>non-characteristic with respect to P</u>, that is if $p(x_0, \xi) = 0$, $\xi \neq 0$ implies $\xi|T_{x_0}(X_0) \neq 0$.

If $Q_j \in L^{m_j}(X)$, $j = 1, \ldots, u$, are given we want to construct operators $E_k : \mathcal{E}'(X_0) \to \mathcal{D}'(X)$, $k = 1, \ldots, l$ such that

(3.4) $$PE_k \equiv 0$$

(3.5) $$\rho \, Q_j \, E_k \equiv \delta_{jk} \text{ , where } I = \text{identity operator on } X_0 .$$

In this case we have for any choice of $f_j \in \mathcal{E}'(X_0)$, $j = 1, \ldots, \mu$ that $Pu \equiv 0$, $\rho Q_j u \equiv f_j$ for all $j = 1, \ldots, \mu$ if we take $u = \sum_k E_k f_k$. So the operators E_k solve a Cauchy-type problem modulo C^∞. We try

$$E_k \in I^{-m_k - 1/4}(X, X_o ; C_o) \tag{3.6}$$

for a suitable homogeneous canonical relation C_o from $T^*(X_o)\backslash 0$ to $T^*(X)\backslash 0$. (Also the order should be filled in later).

In view of (3.4) we demand that the projection of C_o on the first factor $T^*(X)\backslash 0$ is contained in the set N of zeros of p. Note that this implies that C_o is invariant under $H_{\tilde{p}}$, $\tilde{p}$ = pull-back of p to $T^*(X)\backslash 0 \times T^*(X_o)\backslash 0$.
In view of (3.5) we need that $R_o \times C_o$ intersects the diagonal in $T^*(X_o) \times T^*(X) \times T^*(X) \times T^*(X_o)$ transversally and that $R_o \circ C_o$ = diagonal in $T^*(X_o)\backslash 0 \times T^*(X_o)\backslash 0$. (See Theorem 1.1). The last property means that $(x, \xi, x_o, \xi_o) \in C_o$, $x \in X_o \Rightarrow x = x_o$ and $\xi|_{T_{x_o}(X_o)} = \xi_o$, and that every (x_o, ξ_o) occurs in this way.

This leads to the definition of C_o as the set of all (y, η, x_o, ξ_o) where (y, η) is the bicharacteristic strip through a point (x_o, ξ) with

$$p(x_o, \xi) = 0 \text{ and } \xi|_{T_{x_o}(X_o)} = \xi_o \, . \tag{3.7}$$

<u>Theorem 3.1</u> : Let X_o be a connected submanifold of X of codimension 1, $n \geq 3$. Suppose that X_o is non-characteristic for P and that every bicharacteristic curve intersects X_o at most once and transcersally. Assume finally that

(i) no bicharacteristic curve starting on X_o stays is a compact subset of X and

(ii) for every compact $K_o \subset X_o$, $K \subset X$ there is a compact $K' \subset X$ such that if γ is an interval on a bicharacteristic curve with one end point in K_o and the other in K, then $\gamma \subset K'$.

Then the number μ of solutions $\zeta = \xi_k(x_o, \xi_o)$ of (3.7) is finite and does not depend on $(x_o, \xi_o) \in T^*(X_o)\backslash 0$, the relation C_o defined above is a homogeneous canonical relation from $T^*(X_o)\backslash 0$ to $T^*(X)\backslash 0$,

$R_o \times C_o$ intersects the diagonal in $T^*(X_o) \times T^*(X) \times T^*(X) \times T^*(X_o)$ transversally and $R_o \circ C_o = $ identity in $T^*(X_o)\backslash 0$.

Finally, if $Q_j \in L^{m_j}(X)$, $j = 1,\dots,\mu$ has homogeneous principal symbol $q_j(x,\xi)$ such that

(3.8) the matrix $q_j(x_o,\xi_k(x_o,\xi_o))$, $j,k=1,\dots,\mu$ is non-singular for every $(x_o,\xi_o) \in T^*(X_o)\backslash 0$,

then we can find operators E_k satisfying (3.4), (3.5), (3.6).

<u>Proof</u> : If $p(x_o,\xi) = 0$, then the condition that the projection into X of the bicharacteristic strip through (x_o,ξ) intersects X_o transversally means that $d_\xi\, p_{(x_o,\xi)} \notin T_{x_o}(X_o)$, or $d_\xi\, p_{(x_o,\xi)}|_{T_{(x_o)}(X_o)^\perp} \neq 0$.

This means that the zeros ξ of $p(x_o,\xi)$, ξ restricted to the line $\xi|_{T_{x_o}'(X_o)} = \xi_o$, are simple.

So the zeros are isolated and their number μ is finite because otherwise we could find a sequence $\xi^{(j)}$ such that $p(x_o,\xi^{(j)}) = 0$, $\xi^{(j)}|_{T_{x_o}(X_o)} = \xi_o$, $|\xi^{(j)}| \to \infty$ and $\xi^{(j)}/|\xi^{(j)}| \to \xi$ for $j \to \infty$.

But this leads to $p(x_o,\xi) = 0$, $|\xi| = 1$, $\xi|_{T_{x_o}(X_o)} = 0$ in contradiction

with the assumption that X_o is non-characteristic. From the implicit function theorem it follows that μ is locally constant on $T^*(X_o)\backslash 0$, and because $T^*(X_o)\backslash 0$ is connected it follows that μ is constant. (If $n = 2$ we must add the condition that μ does not depend on the component of $T^*(X_o)\backslash 0$).

All the asertions concerning C_o now are readily verified. According to Lemma 2.4 the equation (3.4) leads to the equation

(3.9) $$i^{-1} \mathcal{L}_{H_{\tilde{p}}}\, e_k + \tilde{c} \,.\, e_k = 0$$

for the principal symbol e_k of E_k, whereas (3.3) leads in view of

Theorem 1.1 to the algebraic equations

$$\sum_{l=1}^{\mu} q_j(x_o,\xi_l(x_o,\xi_o)).r((x_o,\xi_o),(x_o,\xi_l(x_o,\xi_o))) \times \tag{3.10}$$
$$\times\ e_k((x_o,\xi_l(x_o,\xi_o)),(x_o,\xi_o)) = \delta_{jk}\ .$$

Here r denotes the principal symbol of ρ.
Because of (3.8) these equations can uniquely be solved and because $C_p \cap \pi^{-1}(X_o \times X_o)$ is transversal to $H_{\tilde{p}}$ we can treat the solutions $e_k((x_o,\xi_l(x_o,\xi_o)),(x_o,\xi_o))$ as initial values for e_k in the equation (3.9)

Taking $E_k^{(0)}$ with this principal symbol we obtain that $PE_k^{(0)} \in I^{m-m_k-1/4-1}(X,X_o,C_o)$, $\rho\, Q_j\, E_k^{(0)} - \delta_{jk}\, I \in L^{-1}(X_o)$.

With a recurrent procedure of the usual type we can find $E_k^{(l)}$, $l = 0,1,2,\dots$ such that

$$P(E_k^{(0)} + \dots + E_k^{(l)}) \in I^{m-m_k-1/4-l-1}(X,X_o,C_o)\ ,$$

$$\rho\, Q_j(E_k^{(0)} + \dots + E_k^{(l)}) - \delta_{jk}\, I \in L^{-l-1}(X_o)\ ,$$

so taking for E_k an asymptotic sum of $E_k^{(l)}$ the proof is ready.

This theorem can be considered as an interpretation of the results of Lax [11] and Ludwig [13] for strictly hyperbolic differential operators. For such operators the usual assumption is :

(3.11) $X \cong X_o \times \mathbb{R}$, for each t, $X_t = X_o \times (t)$ is non-characteristic with transversal bicharacteristique curves.

It is easily seen that (3.11) implies the conditions of Theorem 3.1. For the operators Q_j one can take D_n^{j-1} , D_n = differentiation in a normal direction of X_o. In this case (3.8) follows from the study of Vandermonde determinant. These are the usual Cauchy-data. For strictly hyperbolic differential operators we have $\mu = m$, but for general pseudo-differential operators we cannot expect any relation between m and μ.

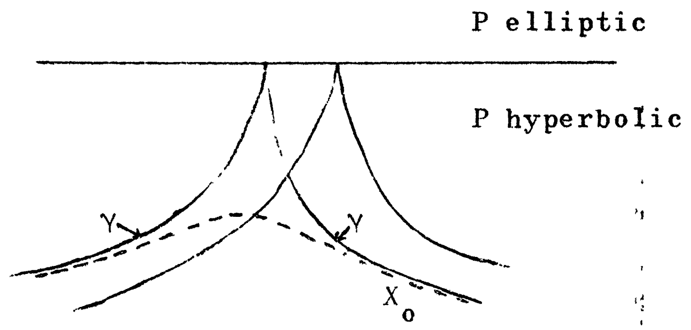
Note that in the case (3.11) the operators $\rho_{X_t} \circ E_k$ are Fourier integral operators : $\mathcal{E}'(X_o) \to \mathcal{E}'(X_t)$ defined by the relation $((x_o', \xi_o'), (x_o, \xi_o)) \in C \Leftrightarrow \exists\ \xi,\ \xi'$ such that (x_o', ξ') and (x_o, ξ) are on the same bicharacteristic strip and $\xi|_{T_{x_o}(X_o)} = \xi_o$, $\xi'|_{T_{x_o'}(X_t)} = \xi_o'$.

In the case of the wave operator, C is the normal bundle of the relation $|x_o - x_o'| = ct$, and the relation

(3.12) $$WF(\rho_{X_t} \circ E_k f_j) \subset WF(C) \circ WF(f_j)$$

can be considered as a refined form of the (weak) Huygens principle.

The example of the <u>Tricomi operator</u> $P = X_2 \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ shows that the conditions of Theorem 3.1 are much weaker than (3.11). The bicharacteristic curves are sketched below. The dotted line is a possible submanifold X_o satisfying the hypotheses of Theorem 3.1, but it is obvious that (3.11) cannot be satisfied. Every bicharacteristic curve apart from γ intersects X_o once and transversally. The solutions of $Pu \in C^\infty$ with sing supp $\mu = \gamma$, which exist according to Theorem 2.7, are precisely those non-smooth for which the Cauchy data on X_o are smooth.



§ 4. Operators with complex principal symbols

As already remarked in the introduction, Egorov [3] and Nirenberg-Trèves [15] used Fourier integral operators to simplify their study of subelliptic and local solvability for operators with essentially non-real principal symbols. See also Hörmander [10], Prop. 3.3.5.

In this section I want to describe two cases which are in some sense opposite to each other, and where the operator can be reduced to a very special one.

Case I :

(4.1) $\{p,\bar{p}\} = 0$ when $p=0$

(4.2) $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$ are linearly independent and the cone direction is not contaiend in their span when $p=0$.

Condition (4.1) is the necessary local solvability condition for both P and ${}^tP$ of Hörmander [1], ch. 6 (see Hörmander [8] for the case of pseudo-differential operators). Because of (4.2) also $d\,\mathrm{Re}\,p$ and $d\,\mathrm{Im}\,p$ are linearly independent at $p=0$, so $p=0$ defines a conic C^∞ submanifold N of $T^*(X)\setminus 0$ of codimension 2. Because of (4.1), $H_{\mathrm{Re}\,p}$ and $H_{\mathrm{Im}\,p}$ are tangent to N and span an integrable tangent system, so they define a 2-dimensional foliation of N. In analogy with the real case the leafs of this foliation are called the bicharacteristic strips for the operator P.

Again we reduce first to the case $m=1$ by multiplying with an elliptic factor of degree $1-m$. We now can try to find a homogeneous canonical transformation χ transforming $\mathrm{Re}\,p$ to ε_{n-1} and $\mathrm{Im}\,p$ to ε_n. It turns out that this is possible if and only if we have instead of (4.1) :

(4.3) $\{p,\bar{p}\} = 0$ on a conic neighborhood of (x_0,ξ_0) .

The necessity of (4.3) is obvious : we have $\{\xi_{n-1}, \xi_n\} = 0$ and because Poisson brackets are preserved by canonical transformations, we must have $\{\mathrm{Re}\, p, \mathrm{Im}\, p\} = 0$.
The following lemma shows that (4.3) can be obtained if we multiply with a suitable elliptic factor :

Lemma 4.1 : Suppose that $m = 1$ and that (4.1), (4.2) are valid in a conic neighborhood of (x_o, ξ_o). Then we can find a homogeneous C^∞ function a of degree 0, such that $a \neq 0$ and $\{q, \bar{q}\} = 0$ on a canonic neighborhood of (x_o, ξ_o) if $q = ap$.

Using this we can ultimately transform P by Fourier integral operators to the Cauchy-Riemann operator $D_{n-1} + i\, D_n$, locally in $T^*(X)\backslash 0$. This leads to analogues of all the theorems wich we have in the real case. (But with important differences ! See [1], ch. 7).

Case II :

(4.4) $\{p, \bar{p}\} \neq$ when $p = 0$.

Here the situation is geometrically completely different. It follows from (4.4) that $H_{\mathrm{Re}\, p}$ and $H_{\mathrm{Im}\, p}$ are linearly independent, so again $p = 0$ is a submanifold N of condimension 2. But this time the plane spanned by $H_{\mathrm{Re}\, p}$ and $H_{\mathrm{Im}\, p}$ is transversal to the tangent space of $p = 0$. Because $p = 0$ is conic this also implies that the cone direction is not contained in the span of $H_{\mathrm{Re}\, p}$ and $H_{\mathrm{Im}\, p}$ at $p = 0$.

Now we have the following analogue of Lemma 4.1.

Lemma 4.2 : Suppose that (4.4) is valid and that $m = 1$. Then there is a conic neighborhood U of N and a homogeneous C^∞ function a of degree 0, such that $a \neq 0$ and $\{q, \{q, \bar{q}\}\} = 0$ on U, if $q = ap$.

An important difference with Lemma 4.1 is that here the construction is global on a neighborhood of N (in Lemma 4.1 the construction can only made global if certain geometric conditions on the bicharacteristic foliation of N are satisfied). The reason is that the vector field H_p occuring in the proof is transversal to N.

Now write $p = p_1 - i\, p_2$ with p_1, p_2 real. Then the condition $\{p, \{p, \bar{p}\}\} = 0$ implies that

(4.5) $\{p_1, r_2\} = 1 \quad \{p_1, s_2\} = 0 \quad \{r_2, s_2\} = 0$,

if we write

(4.6) $$p_2 = r_2 s_2,\ s_2 = \{p_1, p_2\},\ \text{thus } r_2 = p_2/\{p_1, p_2\} .$$

By a local homogeneous canonical transformation we can make $p_1 = \xi_1$, $s_2 = \xi_2$, $r_2 = x_1$. So we can transform P locally in the cotangent bundle to an operator with principal symbol equal to

(4.7) $$\xi_1 - i\, x_1 \xi_2,\ \text{near } \xi_1 = x_1 = 0,\ \xi_2 \neq 0 .$$

(Compare this procedure with Sato [16]).
Such operators are extensively studied and Sjöstrand [17], [18], extending work of Egorov and Kondratev [5] on the oblique derivative problem, has given explicit parametrices

$$\mathcal{E} = \begin{pmatrix} E & E^+ \\ E^- & 0 \end{pmatrix}$$

for matrices of operators

$$\mathcal{E} = \begin{pmatrix} P & R^- \\ R^+ & 0 \end{pmatrix}$$

His results are valid for general operators P satisfying (4.4) and the additional assumption that the projection of N into X has constant rank.

Writing $N^{\pm} = \{(x,\xi) \in N ; \frac{1}{2}\{p,\bar{p}\} \gtrless 0\}$ and $\Sigma^{\pm}$ for the projection $N^{\pm}$ in X, then the operators R^{+}, resp. R^{-} are variants of the restriction operator :

$C^{\infty}(X) \to C^{\infty}(\Sigma^{+})$ resp. the adjoint of the restriction operator $C^{\infty}(X) \to C^{\infty}(\Sigma^{-})$.

Sjöstrand's constructions together with Lemma 4.2 lead in the general case (that is without the rank condition) to global operators E, F^{+}, F^{-} such that

$$EP + F^{+} \equiv I, \quad PE + F^{-} \equiv I, \quad PF^{+} \equiv 0, \quad F^{-}P \equiv 0 \tag{4.8}$$

$$WF'(E) \subset \Delta_{T^{*}(X)\backslash 0 \times T^{*}(X)\backslash 0} , \tag{4.9}$$

$$WF'(F^{\pm}) = N^{\pm} \times N^{\pm} ,$$

however, we are still working on this to get the best possible $H_{(s)}$ continuity properties for E, F^{+}, F^{-}. We also think that F^{+}, F^{-} should be Fourier integral operators, eventually of a somewhat more general type than those described in section 1.

Note that (4.8) implies the sequence

$$\mathcal{D}'(X)/C^{\infty}(X) \xrightarrow{F^{+}} \mathcal{D}'(X)/C^{\infty}(X) \xrightarrow{P} \mathcal{D}'(X)/C^{\infty}(X) \xrightarrow{F^{-}} \mathcal{D}'(X)/C^{\infty}(X)$$

is exact.

It is hoped that in the future results of similar form can also be obtained for more general overdetermined systems. In the category of hyperfunctions Sato and his pupils seem to have far-reaching results in this direction (see [16]).

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