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Complexes and ideals' resolutions

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COMPLEXES AND IDEALS' RESOLUTIONS

by David A. BUCHSBAUM

1. - The following theorem was proved by J.-P. SERRE :

THEOREM. - Let R be an unramified regular local ring, and let M , N be finitely generated R-modules such that $M \otimes_R N$ has finite length. Denote by $e(M, N)$ the finite sum $\sum_i (-1)^i \ell(\mathrm{Tor}_i^R(M, N))$, where $\ell(*)$ means the length of the module $*$. Then :

(a) if $\dim M + \dim N < \dim R$, then $e(M, N) = 0$;

(b) if $\dim M + \dim N = \dim R$, then $e(M, N) > 0$.

A conjecture of J.-P. SERRE is that the conclusion of the theorem remains valid even if R is a ramified regular local ring.

2. - To prove SERRE's conjecture, it is enough to consider the case when R is complete. It is known, however, that a complete ramified regular local ring R is of the form $R = S/(x)$ where S is regular and unramified. A. GROTHENDIECK made the following observation : let M be an R-module of finite type, where $R = S/(x)$. We say that an S-module $\tilde{M}$ is a lifting of M if

(a) $\tilde{M}/x\tilde{M} \simeq M$ and

(b) x is not a zero divisor for $\tilde{M}$.

If we can show that every R-module M has a lifting to S , then an easy argument (namely that $\mathrm{Tor}_i^R(M, N) \simeq \mathrm{Tor}_i^S(\tilde{M}, N)$) shows that J.-S. SERRE's conjecture is correct. A. GROTHENDIECK therefore suggested attacking the problem of whether modules over a (complete) ramified regular local ring R admit a lifting to S where $R = S/(x)$ and S is unramified.

3. - We consider the lifting problem in a slightly more general setting. We let $R = S/(x)$ where S is any local ring, and x is a regular element of S (i. e. x is not a zero divisor). Then clearly any finitely generated free R-module, F , admits a lifting, $\tilde{F}$, to S since $F \simeq R^n$ and we may take $\tilde{F} \cong S^n$. It is also clear that any morphism $\varphi : F \to G$ of free R-modules "lifts" to a morphism $\tilde{\varphi} : \tilde{F} \to \tilde{G}$ (i. e. $\tilde{\varphi} \otimes R = \varphi$).

4. - Let $\underset{\sim}{F}$: $\dots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \dots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$ be a free resolution of the R-module M , i. e. $H_i(\underset{\sim}{F}) = 0$ for $i > 0$, and $H_0(\underset{\sim}{F}) = M$. Let $\tilde{\underset{\sim}{F}}$ denote the sequence of free S-modules and maps

$$\cdots \longrightarrow \tilde{F}_n \xrightarrow{\tilde{\varphi}_n} \tilde{F}_{n-1} \longrightarrow \cdots \longrightarrow \tilde{F}_1 \xrightarrow{\tilde{\varphi}_1} \tilde{F}_0 \longrightarrow 0 \ ,$$

and suppose that the $\tilde{\varphi}_n$ are chosen so that $\tilde{\underset{\sim}{F}}$ is a complex. Then $H_0(\tilde{\underset{\sim}{F}}) = \tilde{M}$ is a lifting of M (and, in fact, $\tilde{\underset{\sim}{F}}$ is then a free S-resolution of $\tilde{M}$). For from the exact sequence of complexes :

$$0 \longrightarrow \tilde{\underset{\sim}{F}} \xrightarrow{x} \tilde{\underset{\sim}{F}} \longrightarrow \underset{\sim}{F} \longrightarrow 0$$

one obtains the exact sequence of homology :

$$\cdots \longrightarrow H_1(\tilde{\underset{\sim}{F}}) \longrightarrow H_1(\underset{\sim}{F}) \longrightarrow H_0(\tilde{\underset{\sim}{F}}) \xrightarrow{x} H_0(\tilde{\underset{\sim}{F}}) \longrightarrow H_0(\underset{\sim}{F}) \longrightarrow 0 \ .$$

Since $H_i(\underset{\sim}{F}) = 0$ for $i > 0$, the result follows.

5. - Problem, therefore, is to find liftings of the maps φ_n so that $\tilde{\underset{\sim}{F}}$ is a complex. In general, this is impossible, even if one assumes that the module M has finite projective dimension over R . To see this, consider an example due to C. PESKINE and L. SZPIRO. Let S be a regular local ring of dimension $d + 1 \geq 3$ and let $R = S/(x)$. Then R is Cohen-Macaulay of dimension $d \geq 2$. Suppose that

$$F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow k \longrightarrow 0$$

is exact, where k is the residue class field of R , and $F_1 , \ldots , F_d$ are free R-modules. Then we have an exact sequence :

$$0 \longrightarrow R \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{d-1}^* \longrightarrow F_d^* \longrightarrow L \longrightarrow 0$$

where X^* means $\mathrm{Hom}_R(X , R)$, and L has projective dimension d with

$$\mathrm{Ext}_R^i(L , R) = 0$$

for $i \neq 0 , d$. If, now, L admits a lifting, it is clear that R is regular. However, if we choose $x \in \mathfrak{M}^2$ where $\mathfrak{M}^2$ is the maximal ideal of S , this is impossible. Thus the module L , which is of finite projective dimension over R , cannot in general be lifted.

6. - In the above example, the module L is not cyclic. Yet, to prove J.-P. SERRE's conjecture, it is known that it suffices to consider cyclic modules M and N . Thus, if the lifting problem were solvable for cyclic modules, that would be enough. M. HOCHSTER recently gave an example to show that in general the lifting problem is not solvable even for cyclic modules. First he proves the following lemma :

LEMMA. - *Let* R *be a commutative, noetherian ring,* J *an ideal of* R *containing an element* x *which is regular in* R . *If* J *contains an ideal* I *such that*

(a) $J = (I , x)$ *and*

(b) x *is regular for* R/I ,

then the morphism $R/J \longrightarrow J/J^2$ *which sends* $\bar{1}$ *to* $\bar{x}$ *in* J/J^2 *splits. Thus this morphism is injective,* i. e. $J^2 : x = J$ (*clearly the condition that* J *contains* such an ideal I is simply that R/J, considered as an $R/(x)$-module admit a lifting to R).

Now let $\mathcal{O}$ be the 2-adic integers, let $R = \mathcal{O}[[X_{ij}]]$ where X_{ij} are indeterminates with $1 \leq i \leq 3$ and $1 \leq j \leq 2$, let $x = 2$, and let J the ideal generated by 2, $\{X_{ij}^2\}$, and the element $\sum_{i=1}^{3} X_{i1} X_{ir}$. Then M. HOCHSTER shows that $J^2 : 2 \neq J$ so that the cyclic module R/J, as a module over $R/(2)$, does not admit a lifting to R.

7. - Notice that in M. HOCHSTER example, the module to be lifted is a module over a regular local ring of positive characteristic and it is required to lift it to a module over a ring of characteristic zero. There are as yet no counter-examples to the lifting of cyclic modules when the rings in question have the same characteristic. This is, of course, the case that must be considered if one has Serre's conjecture in mind.

8. - Returning to lifting problem in general, we have seen that free modules are always liftable. The liftable lemma (n° 4) tell us that all module of projective dimension 1 are liftable. Since we ultimately are concerned with regular local rings and cyclic modules, the next question we ask is : What about cyclic modules of projective dimension 2 ? If $M = R/I$ is such a module, it has a resolution :

$$0 \longrightarrow R^n \xrightarrow{\varphi_2} R^{n+1} \xrightarrow{\varphi_1} R \longrightarrow \frac{R}{I} \longrightarrow 0 .$$

A theorem of L. BURCH says, essentially, that φ_1 is the composition of the following maps :

$$R^{n+1} \simeq \Lambda^n R^{(n+1)*} \xrightarrow{\Lambda^n \varphi_2} \Lambda^n R^{n*} \simeq R \xrightarrow{a} R$$

where R^{n+1} is identified with $\Lambda^n R^{(n+1)*}$ by means of an orientation of R^{n+1}, R is identified with $\Lambda^n R^{n*}$ by an orientation of R^n, and a is a non-zero divisor of R. Thus, the generators of I are the multiples of the $n \times n$ minors of the matrix of φ_2 by a fixed regular element of R. A lifting of the resolution R/I can thus be affected by first lifting the map φ_2 to $\tilde{\varphi}_2$, the element a to $\tilde{a}$ and defining $\tilde{\varphi}_1$ as the composite :

$$S^{n+1} \simeq \Lambda^n S^{(n+1)*} \xrightarrow{\Lambda^n \tilde{\varphi}_2^*} \Lambda^n S^{n*} \simeq S \xrightarrow{\tilde{a}} S .$$

This shows that all cyclic R-modules of projective dimension 2 may be lifted, but even more importantly it shows that there are strong connections between the minors of the matrices in a free resolution. The rest of these lectures will be devoted mainly to outline some of the results D. EISENBUD, and I have obtained about finite free resolutions.

9. - First a criterion for exactness of a finite free resolution. If $F \xrightarrow{\varphi} G$ is a map of free modules, we define the rank of φ to be the largest integer r

such that $\Lambda^r \varphi : \Lambda^r F \longrightarrow \Lambda^r G$ is not the zero map. If $r = \mathrm{rank}(\varphi)$, define $I(\varphi)$ to be the ideal generated by the minors of order r of a matrix of φ (i. e. $I(\varphi)$ is the image of $\Lambda^r G^* \otimes \Lambda^r F \longrightarrow R$).

Let $\underset{\sim}{F} : 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$ be a complex of free modules over a noetherian, commutative ring R. Then $\underset{\sim}{F}$ is exact if, and only if,

(a) $\mathrm{rank}(\varphi_{k+1}) + \mathrm{rank}(\varphi_k) = \mathrm{rank}(F_k)$ for $k = 1, \ldots, n$;

(b) $\mathrm{grade}\,(I(\varphi_k)) \geq k$ for $k = 1, \ldots, n$.

The proof of this result is particularly simple when R is a domain, for in this case, it is clear that localisation does not change the ranks of the maps involved.

The sufficiency of these conditions is essentially based on the lemme d'acyclicité of C. PESKINE - L. SZPIRO. The necessity is essentially a consequence of the fact that the finitistic global dimension of a local ring is equal to its codimension.

10. - An immediate corollary of the exactness criterion is the following.

COROLLARY. - The complex $\underset{\sim}{F}$ above is exact if, and only if, $\underset{\sim}{F} \otimes R_p$ is exact for all prime ideals p such that $\mathrm{codim}\, R_p < n$.

A particularly useful aspect of this corollary is that it often enables us to convert a problem of commutative algebra into a problem of linear algebra, i. e. the study of morphisms of free modules over local rings whose cokernels are free.

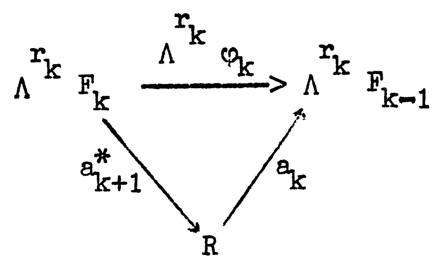
11. - From now on, $\underset{\sim}{F}$ will always denote the exact sequence of oriented free R-modules of finite type :

$$(\underset{\sim}{F}) \qquad 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 .$$

The orientation of each module F_k enable us to canonically identify $\Lambda^\lambda F_k$ and $\Lambda^\mu F_k^*$ if $\lambda + \mu = \mathrm{rank}(F_k)$. We shall also denote by r_k the rank of the map φ_k, and the notation $I(\varphi_k)$ will be the same as in n° 9. Our first result is that there are unique maps $a_k : R \longrightarrow \Lambda^{r_k} F_{k-1}$ for $k = 1, \ldots, n$ such that :

(i) $a_n = \Lambda^{r_n} \varphi_n : R \simeq \Lambda^{r_n} F_n \longrightarrow \Lambda^{r_n} F_{n-1}$ and

(ii) the diagram



is commutative.

(The map a^*_{k+1} is really the composite $\Lambda^{r_k} F_k \simeq \Lambda^{r_{k+1}} F_k^* \xrightarrow{a^*_{k+1}} R$.)

<u>Moreover, for each</u> $k = 1, \dots, n$, <u>we have</u> $\sqrt{I(a_k)} = \sqrt{I(\varphi_k)}$, where $\sqrt{I}$ means the radical of the ideal I.

12. - To see that the above result is not just a formal consequence of exactness, but requires the finiteness of the exact sequence $(\underset{\sim}{F})$, consider the ring $R = k[[X, Y]]/(X^3 - Y^2)$ where k is a field. Then the following sequence is exact :

$$\dots \longrightarrow R^2 \xrightarrow{\begin{pmatrix} y & -x \\ x^2 & -y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & -x \\ x^2 & -y \end{pmatrix}} R^2 \xrightarrow{\binom{x}{y}} R$$

where x and y denote the residue classes of X and Y respectively in R. Each matrix in the above sequence is of rank 1. If maps $a_1, a_2, \dots$ of the sort described in n° 11 existed, we could first see that $(x, y) \subset I(a_1)$. Since $I(a_1)$ is a principal ideal, this tell us that $I(a_1) = R$. So we may assume that $a_1 = 1$ and $a_2 = (x, y)$. However, using the map a_3, we see that this implies that $(y) \subset (x)$ which is absurd. Thus the maps a_k do not necessarily exist unless the exact sequence of free modules is finite.

13. - The proof of the theorem of n° 11 rests on a combination of the exactness criterion for a free complex and some multilinear algebra. The multilinear algebra comes in as follows : if $\varphi : F \longrightarrow G$ is a map of free modules of rank r, then for each integer $i \geq 0$, we have the map $h : \Lambda^r F \otimes \Lambda^i G \longrightarrow \Lambda^{r+i} G$ given by $h(x \otimes y) = \Lambda^r \varphi(x) \wedge y$. Moreover, the map φ corresponds to an element c_φ in $G \otimes F^*$ under the isomorphism $\mathrm{Hom}(F, G) \simeq G \otimes F^*$. Thinking of $\Lambda^{r+i} G$ as $\Lambda^{r+i} G \otimes \Lambda^{\circ} F^*$, we may multiply an element $z \in \Lambda^{r+i} G$ by c_φ, obtaining an element in $\Lambda^{r+i+1} G \otimes F^*$. We thus get a morphism $g : \Lambda^{r+i} G \longrightarrow \Lambda^{r+i+1} G \otimes F^*$. The fundamental result is that $gh = 0$ and that if $I(\varphi) = R$, the sequence

$$\Lambda^r F \otimes \Lambda^i G \longrightarrow \Lambda^{r+i} G \longrightarrow \Lambda^{r+i+1} G \otimes F^*$$

is exact.

14. - We may use n° 11 to prove that <u>every regular local ring is factorial</u>. For it is known that to prove this, it is sufficient to prove that every ideal generated by two elements has projective dimension at most 1. If $I = (x, y)$, then R/I has a finite free resolution :

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_2 \xrightarrow{\varphi_2} R^2 \xrightarrow{\varphi_1} R$$

with $r_1 = 1$. Thus φ_1 factors as follows : $R^2 \xrightarrow{a_2^*} R \xrightarrow{a_1} R$ where a_1 is a homothety, and

$$I = a_1 \, I(a_2^*) = a_1 (x', y')$$

where

$$\mathrm{grade}\,(x', y') = \mathrm{grade}\; I(\varphi_2) \geq 2 .$$

Thus (x', y') is an R-sequence and has projective dimension 1. Since

$I \simeq I(a_2^*)$, the result follows.

15. – The result in n° 11 also gives us very quickly the result of n° 8. For if R/I has projective dimension 2 , it has a resolution :

$$0 \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R \quad \text{and} \quad \varphi_1 = a_1\, a_2^*$$

with a_2^* essentially equal to $\Lambda^{r_2} \varphi_2^*$. Since

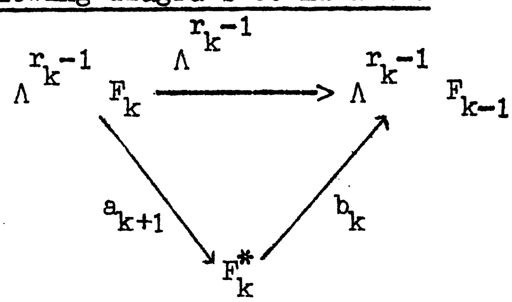
$$\text{grade } (I(a_1)) = \text{grade } (I(\varphi_1)) \geqslant 1 \ ,$$

and a_1 is a homothety, we see that a_1 is a regular element of R .

16. – A second structure theorem for a finite exact sequence (F) of free modules asserts the existence of maps

$$b_k : F_k^* \longrightarrow \Lambda^{r_k - 1} F_{k-1} \quad \text{for} \quad k = 2 , \ldots , n .$$

These maps make the following diagrams commutative :



where a'_{k+1} is the composite :

$$\Lambda^{r_k-1} F_k \simeq \Lambda^{r_k-1} F_k \otimes R \xrightarrow{1\otimes a_{k+1}} \Lambda^{r_k-1} F_k \otimes \Lambda^{r_{k+1}} F_k \longrightarrow \Lambda^{r_k+r_{k+1}-1} F_k \simeq F_k^* .$$

Again the proof of this result is based on a combination of the exactness criterion for free complexes and the multilinear algebra described in n° 13.

17. – If I is a 3-generator ideal such that R/I has projective dimension 3 (R is a local ring), then R/I has a free resolution

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R .$$

Using n° 16, we have

$$b_2 : F_2^* \longrightarrow R^3 \quad \text{and} \quad \Lambda^2 b_2^* : \Lambda^2 R^{3*} \longrightarrow \Lambda^2 F_2 .$$

Since $a_3^* : \Lambda^2 F_2 \longrightarrow R$, we have the composite

$$R^3 \simeq \Lambda^2 R^{3*} \xrightarrow{\Lambda^2 b_2^*} \Lambda^2 F_2 \xrightarrow{a_3^*} R$$

which can be shown to be $a_2^* : R^3 \longrightarrow R$. Thus the maps φ_1 and φ_2 can be completely described in terms of the highest order minors of φ_3 , the map b_2 and a non-zero divisor a_1 (since $\varphi_1 = a_1\, a_2^*$). This description makes it is possible to prove the liftability of R/I . However, if I is generated by more than three elements, or if it has projective dimension larger than 2 , our maps a_k and b_k

do not seem to provide as much information as we need, say, to prove liftability of R/I .

18. – Another approach to the study of finite resolutions of cyclic modules is provided by the multiplicative structure of such a resolution. In fact, if

$$0 \to F_n \to \dots \to F_1 \xrightarrow{\varphi_1} R$$

is an exact sequence of free modules, a commutative multiplication (in the graded sense) can always be defined which is homotopy associative. If $n = 3$, this multiplication is a fortiori associative, and it is an open question whether a multiplication which is both commutative and associative (and, of course, for which the boundary map is a derivation) can be put on the minimal resolution of a cyclic module of finite homological dimension over a local ring R .

19. – If R is a local ring and I is a perfect ideal of grade g in R , we say I is a Gorenstein ideal if $\mathrm{Ext}^g(R/I , R)$ is cyclic. If

$$0 \to F_g \to F_{g-1} \to \dots \to F_1 \xrightarrow{\varphi_1} R$$

is the minimal resolution of R/I when I is Gorenstein, then $F_g \simeq R$ (when R is regular, the ideal I in R is Gorenstein if, and only if, the ring R/I is Gorenstein). A multiplication on this resolution gives us, in particular, maps $F_k \otimes F_{g-k} \to F_g = R$ and hence maps $s_k : F_k \to F^*_{g-k}$. It can be proved that the maps s_k are isomorphisms, so that F_k may be identified with F^*_{g-k} . In particular, if I is a Gorenstein ideal of grade 3 , R/I has a resolution of the form

$$0 \to R \xrightarrow{\varphi_3} F^* \xrightarrow{\varphi_2} F \xrightarrow{\varphi_1} R .$$

The map φ_2 corresponds to an element of $F \otimes F$ and can be shown to be in the kernel of the canonical map $F \otimes F \to S_2 F$ where $S_2 F$ is the second symmetric product of F . Thus φ_2 corresponds to an element $\alpha \in \Lambda^2 F$. Since φ_2 is a square alternating matrix of rank $n - 1$ where $n = \mathrm{rank}\ (F)$, we must have $n - 1$ evence, hence n odd. If $\alpha^{((n-1)/2)}$ denotes the $(n - 1)/2$ divided power of α in ΛF , then $\alpha^{((n-1)/2} \in \Lambda^{n-1} F \simeq F^*$, so that $\alpha^{((n-1)/2)}$ may be considered as a map from F to R . The image of this map is an ideal denoted by $\mathrm{Pf}_{n-1}(\varphi_2)$, i. e. the ideal generated by the Pfaffians of φ_2 of order $n - 1$. It can be proved that φ_1 is the homomorphism $\alpha^{((n-1)/2)^2}$ and that $\varphi_3 = \varphi_1^*$. Thus Gorenstein ideals of grade 3 (and their resolutions) are completely describable and using this description, it is clear that R/I is liftable if I is Gorenstein of grade 3 .

20. – If R is a local ring, an ideal I of R is an almost complete intersection if I is perfect of grade g and minimally generated by $g + 1$ elements. Using the theory of liaison as developed by M. ARTIN and M. NAGATA and by C. PESKINE and L. SZPIRO, one proves that almost complete intersections and Gorenstein ideals

are linked. Making the multiplicative structure of the minimal resolution of a Gorenstein ideal of grade 3 explicit, and using liaison, one can then describe the minimal resolution of an almost complete intersection of grade 3 . This explicit description enables one to lift R/I if I is an almost complete intersection of grade 3 .

21. - The above techniques and results lead to the following natural areas of investigation :

(a) Study the ideals and relations on lower order minors of a matrix ;

(b) Study the number of "liaison classes" of ideals of given codimension in, say, a regular local rings (For codimension 2 , there is only one class, namely the complete intersections) ;

(c) Find a "parameter space" for resolutions with prescribed Betti numbers. M. HOCHSTER has done this for projective dimension 2 (unpublished). In all the specific cases studied so far, the parameter space is affine, and the resolution is liftable.

Is there a connection between the liftability of the resolution and the smoothness of the parameter space ? (An interesting special example to look at in this connection would be M. HOCHSTER's counter-example to lifting. This occurs in homological dimension 6).

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