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Séminaire Dubreil. Algèbre et théorie des nombres, tome 25, nº 2 (1971-1972), exp. nº J14, p. J1-J2
<http://www.numdam.org/item?id=SD_1971-1972__25_2_A14_0>

Séminaire DUBREIL (Algèbre)
25e année, 1971/72, n° J14, 2 p.
Journées d'Algèbre [1972. Paris]
Demi-groupes. Demi-groupes ordonnés
(M.-L. DUBREIL-JACOTIN)

1er juillet 1972

EXTENSIONS OF ORDERED SEMIGROUPS

by Alfred H. CLIFFORD

In this talk, I present the main ideas of the doctoral dissertation (Tulane University, 1970) of my student, A. J. HULIN. An announcement of his results appeared in the Semigroup Forum, t. 2, 1971, p. 336-342.

By an ordered semigroup, we mean a semigroup S provided with a total order relation $<$ satisfying the monotone condition : $a \leqslant b$ implies $ac \leqslant bc$ and $ca \leqslant cb$ for all c .

An (ideal) extension of a semigroup S by a semigroup T with 0 is a semigroup Σ containing S as an ideal such that $\Sigma/S \cong T$. Let S and T be disjoint semigroups, T having a zero 0 , and let $T^* = T \setminus 0$. Let Φ be a partial homomorphism of T^* into S , i. e. a mapping of T^* into S such that if $tt' \neq 0$ then $(tt')\Phi = (t\Phi)(t'\Phi)$. Let $\Sigma = S \cup T^*$. Define a product $\circ$ on Σ as follows (where $t , t' \in T^*$; $s , s' \in S$) :

$$t \circ t' = \begin{cases} tt' & \text{if } tt' \neq 0 , \\ (t\Phi)(t'\Phi) & \text{if } tt' = 0 ; \end{cases}$$

$$t \circ s = (t\Phi)s \ ; \qquad s \circ t = s(t\Phi) \ ; \qquad s \circ s' = ss' \ .$$

Then Σ is an extension of S by T . Only extensions of this kind are considered.

By a null decomposition of T^* , we mean a pair of complementary subsets (X , Y) of T^* such that

(1) if $X \neq \emptyset \neq Y$, then $XY = YX = \{0\}$, and

(2) $X^2 \subseteq X \cup \{0\}$ and $Y^2 \subseteq Y \cup \{0\}$.

Let S and T be ordered semigroups, T having a zero 0 , and let $\Sigma(\circ)$ be an extension of S . By an extending order on Σ , we mean a monotone (total) order on Σ which extends the given orders on S and T^* . If $(<)$ is an extending order on Σ , and we define

$$X = \{t \in T^* : \ t(<)t\Phi\}$$

$$Y = \{t \in T^* : \ t(>)t\Phi\} ,$$

then the pair (X , Y) is a null decomposition of T^* .

On the other hand, not every null decomposition of T^* arises in this way from an extending order on Σ . In fact, Σ need not admit an extending order ; this is the case if S is an unbounded right zero semigroup. The main objective of this paper is to give conditions on a null decomposition of T^* in order that at least one extending order on Σ exist, and to describe all such in terms of the null decomposition.

For example, a Φ-admissible extending order, $(\leqslant)$ on Σ (meaning on such that $(t\ (<)\ s)$ implies $(t\Phi \leqslant s)$, and $(t\ (>)\ s)$ implies $(t\Phi \geqslant s)$) is determined by its null decomposition $(X\ ,\ Y)$ as follows :

$$(t\ (<)\ s) \Longleftrightarrow (t\Phi < s \ \text{ or } \ t\Phi = s \ \text{ and } \ t \in X)\ ;$$

$$(t\ (>)\ s) \Longleftrightarrow (t\Phi > s \ \text{ or } \ t\Phi = s \ \text{ and } \ t \in Y)\ .$$

If S is weakly reductive ($(ax = bx$ and $xa = xb$ for all x in $S)$ imply $(a = b)$), then every extending order on Σ is Φ-admissible.

When T has the property that every annihilator of T belongs to T^2 (e. g., when 0 is the only annihilator), then the only null decomposition of T^* are the pairs $(T^*\ ,\ \emptyset)$ and $(T^-\ ,\ T^*)$, where

$$T^- = \{t \in T^* :\ t < 0\}\ ,$$

$$T^+ = \{t \in T^* :\ t > 0\}\ .$$

It follows that Σ can admit at most four Φ-admissible extending orders in this case.

(Texte reçu le 19 mars 1973)

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