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THE COVARIANT SYSTEMS OF TODD AND SEGRE

by David B. SCOTT

1. Summary.

We give an account of the relationship between the invariant (EGER-TODD) and covariant classes of an algebraic variety according to the differing treatments of TODD and SEGRE, and then discuss the simplified treatment of the invariants due to GROTHENDIECK.

Following this we emphasise the geometrical nature of the Grothendieck approach to the invariants by identifying what we call the "Grothendieck primal" of the tangent-direction bundle of an algebraic variety with the "invariant-lift" of INGLETON-SCOTT, and we then show how the Grothendieck approach can be extended also to the covariants.

2. The equivalence ring of a variety.

We shall consider the category of non-singular complex projective varieties with the usual morphisms (everywhere defined rational transformations). We shall take the old-fashioned view that an algebraic variety is merely the aggregate of zeros, in a complex projective space, of a set of forms. Of course, as we say in England, "maintenant nous avons changé tout cela", but it doesn't make much difference to our subject matter.

We are concerned with the graded ring $\mathcal{R}(V)$ of rational equivalence on a variety V (which we shall suppose to be of dimension $n + 1$). This ring, the so-called Chow-ring, was effectively pioneered by SEVERI, and developed more recently by SAMUEL ([4]) and in the Chevalley seminar of 1958 ([8]). If A^r is a subvariety of V of dimension r, we shall, by a customary abuse of notation, use the same symbol for the variety A (which may be virtual, i. e. a formal difference of respectable objects of the category) and its rational equivalence class (and note that the discussion of the equivalence class presupposes the acceptance of an ambient variety). Indexing equivalence classes by dimension is not however a clever idea : the functorial approach to, and the grading of, $\mathcal{R}(V)$ requires that we emphasise the co-dimension (cf. [8]). So we shall also use the symbol $a_\lambda = A^r$ where

$$r + \lambda = \dim V = n + 1 \quad ,$$

and we shall use small or large letters according as we wish to emphasise the co-dimension or the dimension.

The ring $\mathcal{R}(V)$ has properties similar to those of a cohomology ring. If we have a morphism f,

$$f : U \to V \quad ,$$

there is a contravariant ring-homomorphism f^* (preserving the grading by co-dimension),

$$f^* : \mathcal{R}(V) \to \mathcal{R}(U) \quad ,$$

precisely analogous to the cohomology situation.

There is also a covariant homomorphism f_* of the graded groups of $\mathcal{R}$ preserving <u>dimension</u>, but having no relation to the ring-structure

$$f_* : \mathcal{R}(U) \to \mathcal{R}(V) \quad .$$

This corresponds to the mapping of cohomology groups of U into those of V obtained by first dualising in U, then using the covariant mapping of homology groups, and then dualising back in V.

We shall be mostly concerned with the case where f is an inclusion mapping i. In this case, $i^*(a)$ is the equivalence class on U of the intersection with U of a suitable representative of the class a on V. Co-dimensions are preserved. And $i_*(b)$ is the class on V to which a suitable representative of the class b on U belongs if it is regarded as a subvariety of V. In this case, <u>dimensions</u> are preserved. If k is the co-dimension of U in V,

$$i^*(a_\lambda) = (i^* a)_\lambda \qquad \text{and} \qquad i_*(b_\lambda) = (i_* b)_{k+\lambda} \quad .$$

3. The canonical systems.

The canonical systems on a non-singular variety V were pioneered by SEVERI and B. SEGRE, and developed by EGER and TODD independently just before the war. A full history is provided by TODD ([10]).

For each dimension r $(0 \leqslant r \leqslant n+1)$, there is an Eger-Todd class $X^r(V)$ which is a rational equivalence class on V. We shall write $X^r(V) = x_\lambda(V)$, where $r + \lambda = \dim V = n+1$. In fact, there is a possible improvement of sign (of which TODD was not unaware, but he felt that he was bound by the then existing conventions for canonical primals), and we shall take as our canonical systems not the Eger-Todd classes but the classes $c_\lambda(V)$ where

$$c_\lambda(V) = (-1)^\lambda \, x_\lambda(V) \quad .$$

(It is immaterial whether we regard the c as the initial letter of "canonical" or "Chern".)

We also consider, in the polynomial ring $\mathcal{R}(V)[t]$, the polynomial

$$C(V , t) = c_0 + c_1 t + \dots + c_{n+1} t^{n+1} ,$$

where c_0 is the unit v_0 of $\mathcal{R}(V)$. As no element of $\mathcal{R}$ has grade exceeding $n + 1$, we also have a formal inverse polynomial

$$D(V , t) = (C(V , t))^{-1} = d_0 + d_1 t + \dots + d_{n+1} t^{n+1}$$

introduced by TODD for reasons of manipulative convenience. TODD used the symbol $Y^r(V)$ instead of our $D^r(V)$. We shall call the d_λ the "inverse canonical classes".

4. The Todd covariant systems.

Having introduced the invariants c_λ , d_μ of a single variety, TODD set out [9] to consider the covariants of two varieties U^{m+1} and V^{n+1} where $U \subset V$ and i is the inclusion mapping of U into V (and $k = n - m$ is the co-dimension of U on V).

The mapping $i : U \to V$ enables us to derive from the polynomials $C(V , t)$ and $D(V , t)$ in $\mathcal{R}(V)[t]$ the (mutually inverse) polynomials $i^* C$ and $i^* D$ in $\mathcal{R}(U)[t]$ where

$$i^* C(V , t) = i^* c_0(V) + i^* c_1(V)t + \dots + i^* c_{n+1}(V)t^{n+1}$$

and

$$i^* D(V , t) = i^* d_0(V) + i^* d_1(V)t + \dots + i^* d_{n+1}(V)t^{n+1}$$

(which are in fact only of degree $m + 1$).

TODD picked out two particular covariant systems $\hat{c}_\lambda(U , V)$ and $\hat{d}_\mu(U , V)$ defined (as classes of U) by

$$\hat{C}(U , V , t) = \hat{C}(t) = \sum_0^{m+1} \hat{c}_\lambda t^\lambda = C(U , t) . i^* D(V , t) ,$$

and

$$\hat{D}(t) = \sum_0^{m+1} \hat{d}_\mu t^\mu = D(U , t) . i^* C(V , t) .$$

These covariants were picked out as having the property of "section invariance".

This means that if we cut U and V by a variety W of the ambient space of V which is in "general position" with respect to U and V, and if j is the inclusion,

$$j : U.W \to U$$

(assuming that the intersections $U.W$ and $V.W$ are non-singular), then

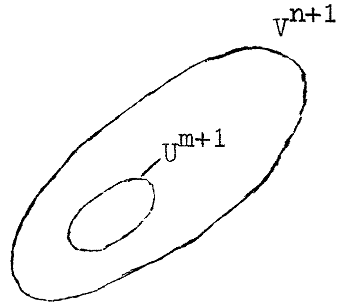
$$\hat{c}_\lambda(U.W , V.W) = j^* \hat{c}_\lambda(U , V) \quad ,$$

$$\hat{d}_\mu(U.W , V.W) = j^* \hat{d}_\mu(U , V) \quad .$$

TODD, in [9], expressed the view that the $\hat{d}_\mu$ were more interesting : in fact, $\hat{D}(U , V , t)$ is formally similar to the (then unknown) Chern polynomial of the normal bundle of U in V.

5. The Segre covariants of immersion.

The Segre covariants were introduced in two papers ([6], [7]). On our variety V, we take a number of primals A_i^n or $a_1^{(i)}$ according as we index by dimension or co-dimension. For simplicity, we shall write A_i instead of A_i^n, and a_i instead of $a_1^{(i)}$, but we must remember that, in a_i, i is an index and not a co-dimension.



If we put, through U, k general primals $A_1 , \ldots , A_k$ of V, we get a residual intersection W^{m+1} or w_k. (There is however a technical difficulty that even though U be non-singular, the A's need not be if k is small enough.)

Thus in $\mathcal{R}(V)$, we have the relation

$$u_k + w_k = a_1 \, a_2 \ldots a_k \quad ,$$

which in principle gives the class of W^{m+1}. However we shall write

$$u_k = a_1 \, a_2 \ldots a_k - w_k \quad .$$

If now we have U' of dimension $m + 1 - r$, or co-dimension $k + r$, we could similarly put $k + r$ general primals through U' getting a similar relation

$$\text{(A)} \qquad u'_{k+r} = a_1 \, a_2 \, \dots \, a_{k+r} - w_{k+r} \quad .$$

Now let us, instead of putting the $k + r$ general primals through a variety of co-dimension $k + r$, put them through U itself. We still get a residual intersection w_{k+r} on the right-hand-side of (A), i. e. $a_1 \, a_2 \, \dots \, a_{k+r} - w_{k+r}$ is still (apart from the technical difficulty referred to above) a good element of $\mathcal{R}(V)$. SEGRE's first result in [6] is that this class is independent of the arbitrary choice of the a_i , so that there is a well-defined class u'_{k+r} in $\mathcal{R}(V)$.

The natural interpretation of this class is that it is the class on V of U regarded as having only <u>accidentally</u> the dimension $m + 1$, but having the virtual dimension $m + 1 - r$. Indeed we have simply treated U as if it had the virtual dimension instead of its actual dimension.

It is further shown (of course SEGRE's language was different) that there is a class u_r in $\mathcal{R}(U)$ such that

$$u'_{k+r} = i_* \, u_r \quad .$$

We thus have a formal polynomial

$$U_V(t) = u_0 + u_1 \, t + \dots + u_{m+1} \, t^{m+1}$$

in $\mathcal{R}(U)[t]$ (where $u_0 = u$ is the unit of $\mathcal{R}(U)$). This polynomial has a formal inverse

$$(U_V(t))^{-1} = \widetilde{U}_V(t) = \widetilde{u}_0 + \widetilde{u}_1 \, t + \dots + \widetilde{u}_{m+1} \, t^{m+1} \quad .$$

The two systems $U_V = (u_0 \, , \, u_1 \, , \, \dots \, , \, u_{m+1})$ and $\widetilde{U}_V = (\widetilde{u}_0 \, , \, \widetilde{u}_1 \, , \, \dots \, , \, \widetilde{u}_{m+1})$ of elements of $\mathcal{R}(U)$ are called the Segre covariant sequence of immersion and its formal inverse.

6. The relation between the Todd and Segre systems.

So far there is no obvious connection between the Todd and Segre covariants. But, whereas TODD had started with invariants and derived covariants from them, SEGRE started with covariants and ([7]) deduced the invariants from them. His trick for doing this is one of the oldest in the algebraic geometry repertoire, which is now widely believed to have been invented by topologists.

We consider the diagonal mapping Δ ,

$$\Delta : \; V \rightarrow V \times V = W \quad ,$$

and we derive the canonical classes on V , or more precisely on $\Delta(V)$ (incidentally the c's , not the x's), as the sequence $\widetilde{\Delta(V)}_W$. In fact, it is remarked by TODD ([10]), who attributes the essential step to VESENTINI ([11], [12]), that

$$\hat{C}(U , V , t) = U_V(t) ,$$

and

$$\hat{D}(U , V , t) = \widetilde{U}_V(t) .$$

Thus Segre's approach to the canonical systems justifies Todd's guess about the comparative importance of his two covariant sequences.

7. Canonical systems and the Grothendieck primal.

Taking as his starting point a property of the cohomology classes dual to the c_i (i. e. the Chern classes of the tangent bundle) given by CHERN ([1]), taking this over to rational equivalence, and then using it for a definition of the canonical classes, GROTHENDIECK ([2]) gave an approach to the canonical systems much simpler than the original ones of TODD and EGER.

We consider the tangent bundle $T(V)$ of V, which is not, in the sense of this lecture, an algebraic variety. But the derived projective bundle or tangent direction bundle $PT(V)$ (which we shall henceforth abbreviate to V^T) is an algebraic variety and there is a morphism ρ (the natural bundle projection),

$$\rho : V^T \to V .$$

Note that V^T has dimension $2n + 1$ (the fibre being of dimension n).

Now $\mathcal{R}(V^T)$ has, given $\mathcal{R}(V)$, a very simple structure. The mapping

$$\rho^* : \mathcal{R}(V) \to \mathcal{R}(V^T)$$

is an isomorphism into, and in addition $\mathcal{R}(V^T)$ is generated by the adjunction to $\rho^*(\mathcal{R}(V))$ of a single class ξ , of co-dimension 1 , representing what I shall call the Grothendieck primal Ξ^{2n} of V^T . The class ξ is in fact the negative of the divisor class defined on V^T by the line-bundle over V^T associated with $T(V)$. The elements $(1 , \xi , \xi^2 , \dots , \xi^n)$ of $\mathcal{R}(V^T)$ are independent over $\rho^*(\mathcal{R}(V))$, and ξ satisfies the minimal equation

$$\xi^{n+1} + (\rho^* c_1)\xi^n + (\rho^* c_2)\xi^{n-1} + \dots + (\rho^* c_n)\xi + (\rho^* c_{n+1}) = 0 .$$

Thus, as ρ^* is an isomorphism, this defines the classes $c_1 , \dots , c_{n+1}$ in $\mathcal{R}(V)$ (c_0 being the unit).

8. Geometrical identification of the Grothendieck primal.

The purpose of this lecture is to add two points to the foregoing theory. In this section, we give a geometrical identification of the Grothendieck primal of V^T

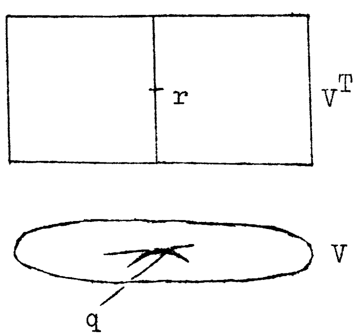
which is very similar to the classical geometrical approach to the canonical primals. In the next section, we shall use V^T to identify the covariant systems in a way similar to Grothendieck's identification of the invariants.

The identification of ξ is nominally due to INGLETON and myself ([3]), but INGLETON contributed the technical expertise. I shall first explain the case $n = 1$ (i. e. V is a surface), and then it is easy to see how the idea carries over to general n.

My original problem was to find a base (for homology, <u>not</u> the stricter rational equivalence) on V^T, given a base on V. At the time I was inexcusably, but perhaps fortunately, ignorant of the general theory. It started with a simple observation. If instead of V^T we had the locally isomorphic (in the complex topology) $V^2 \times P^1$, all we have to do to obtain a base on $V \times P$ is to take, for each base element Γ of V, the elements $\Gamma \times P$ and $\Gamma \times p$ (p being a point of P) on $V \times P$.

The first is the inverse image of Γ under the natural fibre projection. The second is the intersection of this inverse image with the global section $V \times p$.

In the case of V^T, the inverse image under the fibre projection ρ is straightforward. It is the non-existence (in general) of the global section which causes trouble. But nevertheless, the algebraic situation enables us to produce if not a global section, at least a "near-section".



Take a pencil $|A|$ of curves on the surface V. Through a generic point q of V passes one curve of $|A|$ which has a definite tangent, giving rise to a unique point r in the fibre over q. Trouble arises only at the base points of $|A|$ and at the points where a member of the pencil has a singularity, and at these places things get wrapped right round the fibre.

The upshot of all this is to give a near-section $V_{|A|}$, say of which r is the generic point. The obvious question now is how does $V_{|A|}$ change if we alter the pencil $|A|$? The answer is that if $|B|$ is another pencil, then on V^T (remembering that $\rho^* A$ is the class of $\rho^{-1} A$)

$$V_{|A|} - 2\rho^* A = V_{|B|} - 2\rho^* B \quad (\text{in } \mathcal{R}(V^T)) \ .$$

So the class of $V_{|A|} - 2\rho^* A$ is naturally called the "invariant lift" of V to V^T.

A similar trick, using a pencil of primals, works if $n > 1$. Then q does not lift to a point in the overlying fibre, but to a primal in the fibre, and the locus of a generic point of this gives the lift $V_{|A|}$. Again the class of

$$\overline{V} = V_{|A|} - 2\rho^* A$$

is invariant (i. e. independent of the lifting pencil $|A|$). It is this invariant lift $\overline{V}$ which is the "Grothendieck primal" and thus provides all the canonical systems of V.

9. A new geometrical approach to the covariants.

We can use V^T to give an approach to the covariants as well as the invariants. The tangent direction bundle U^T of U is, if U is contained in V, naturally embedded in V^T. The class u^T_{2k} (of co-dimension $2k$) of U^T in $\mathcal{R}(V^T)$ will be called the "natural lift" of U to V^T. (Natural lifts are tricky, they don't give even an additive homomorphism of equivalence classes of $\mathcal{R}(V)$ into $\mathcal{R}(V^T)$.)

Now the class u^T_{2k} can be expressed in terms of the powers of ξ and the elements of $\rho^*(\mathcal{R}(V))$. In fact there is a relation

$$u^T_{2k} = (\rho^* \alpha_k)\xi^k + (\rho^* \alpha_{k+1})\xi^{k-1} + \dots + (\rho^* \alpha_{2k}) \quad .$$

(The fact that ξ^k is the highest power of ξ is easily established and so is the relation $\alpha_k = u_k$.)

What I have recently managed to show ([5]) is essentially that

$$\alpha_{k+\mu} = i_* \widetilde{u}_\mu = i_* \hat{d}_\mu \quad ,$$

where $\hat{d}_\mu = \widetilde{u}_\mu$ are the Todd-Segre covariants previously discussed.

In fact, my results are weaker than the Todd-Segre theory in two important ways, but these defects are doubtless inessential. In the first place, we have got as the coefficients of the powers of ξ in the expression for u^T_{2k} not the Todd-Segre covariant classes of $\mathcal{R}(U)$, but only their i_* images in $\mathcal{R}(V)$. But this is easily dealt with : instead of considering the class of U^T in the equivalence ring of V^T, we can confine ourselves to that part of V^T (i. e. $\rho^{-1} U$) lying over U, and consider the equivalence class of U^T in $\mathcal{R}(\rho^{-1} U)$.

The second difficulty is also technical. My results were obtained in terms of rational homology instead of the stronger relation of rational equivalence. This too can doubtless be repaired, but my technique for finding u^T_{2k} (which may well be capable of simplification, though neither INGLETON nor I have yet seen how) depends essentially on the "universal contact formula" of INGLETON-SCOTT ([3]). Once this formula has been reworked in terms of the Grothendieck approach instead of Chern's, there should be no further difficulty. After this has been done, we can deal geometrically with the covariants as effectively as the Grothendieck approach (with the Ingleton identification of the Grothendieck primal) enables us to treat the invariants.

There remain one or two minor points to be made. If U is a complete intersection of primals, the calculation of u^T_{2k} is very easy and thus gives simple formulae for the covariants (already found by SEGRE ([7])). Also our approach gives only k (not as one might expect $m + 1$) covariant classes. This enables us to show that $\hat{d}_\lambda$ vanishes if its co-dimension in U exceeds the co-dimension of U on V, and this is a non-trivial restriction on the covariants if the dimension of U exceeds half the dimension of V (again this was previously found by SEGRE).

Finally, it is not altogether surprising that the covariants arise in this way. It is difficult to see what there is to be known about the immersion of U in V which is not to be deduced from the way U^T lies in V^T, or even in $\rho^{-1} U$. But it is perhaps a little surprising that we don't lose more of this information by considering only the equivalence class of U^T.

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