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ONE DIMENSIONAL ALGEBRA AND ITS GEOMETRIC ORIGINS

par David G. NORTHCOTT

It is well known that there is a close connection between the theory of algebraic numbers and that of fields of algebraic functions of a single variable ; on the other hand such an algebraic function field consists of the rational functions on an irreductible curve. It follows that the theories of algebraic numbers and algebraic curves have much in common.

Now there is a good deal known about curve singularities and at first sight it would appear that there is nothing really comparable in number theory. The reason for this is easy to explain. Normally when one considers an algebraic number field F one deals with the ring of all algebraic integers in F and this, by its definition, is integrally closed. The corresponding situation for a curve would be that in which the coordinate ring was integrally closed. As is well known, such a curve has no singularities.

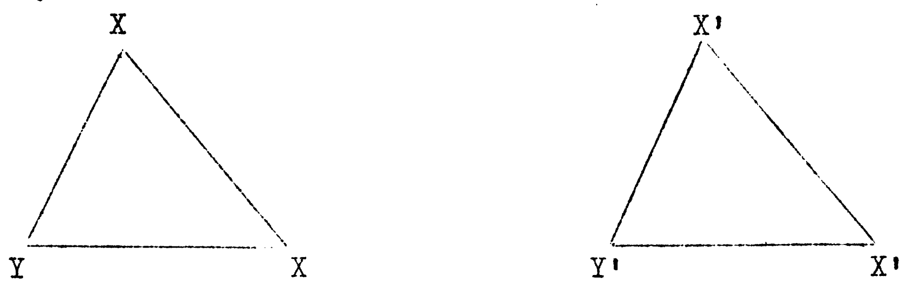
It follows that to find the theory we are seeking it is natural to consider rings of algebraic integers which are not integrally closed, in other words what are known as orders. There were studied by Dedekind, but in his correspondence he recorded his disappointment with the results he was able to obtain.

I do not know what Dedekind hoped to establish in this direction but it is clear that, in a suitably modified form, the theory of curve singularities carries over to Dedekind orders and forms an essential part of the local theory of such systems. Indeed one can say that what was previously done for curves alone applies in surprising detail to a very general kind of one-dimensional local algebra. What I shall try to do is to sketch this general theory which is applicable both to curves and orders as special cases.

The methods by which the abstract results are obtained must, because of the nature of the situation, be different from those used by geometers. However the pattern of results and the principal concepts are most readily appreciated in the geometric model and so I shall begin by giving a very rough sketch of the theory of curve singularities as propounded by M. NOETHER.

Let us consider a curve in the projective plane. Of course, dimension 2 has a special attraction for geometers which is not shared by algebraists, but this restriction will make the reasoning easier to follow without obscuring the important features. The geometric method of analysing singularities is to apply a transformation which has a fundamental point at the singularity. Any transformation satisfying certain general requirements will do, but of the available ones the so-called quadratic transformations are far the simplest. Let me therefore recall the definition and simpler properties of these transformations.

Suppose that π and π' are two planes. We shall use (x , y , z) as homogeneous coordinates in π while (x' , y' , z') will denote coordinates in π'. A standard quadratic



transformation of π into π' is obtained by mapping $P(x , y , z)$ into $P'(x' , y' , z')$ where

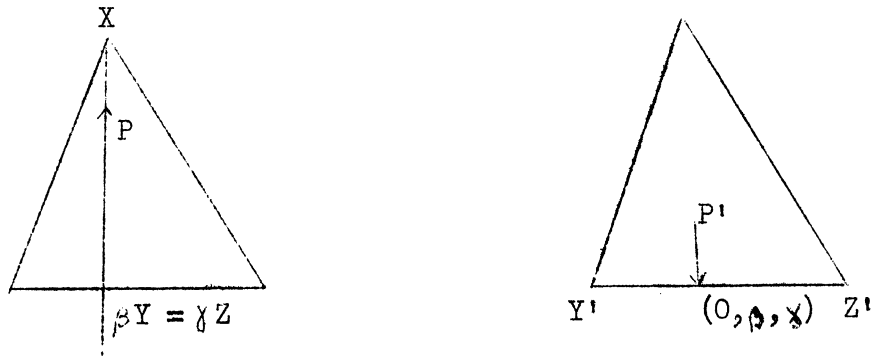
$$\lambda x' = yz , \quad \lambda y' = zx , \quad \lambda z' = xy ,$$

it being understood that λ is just a factor of proportionality. More simply we may observe that the transformation and its inverse are given by

$$(x , y , z) \rightarrow (x^{-1} , y^{-1} , z^{-1}) \quad \text{and} \quad (x' , y' , z') \rightarrow (x'^{-1} , y'^{-1} , z'^{-1}) .$$

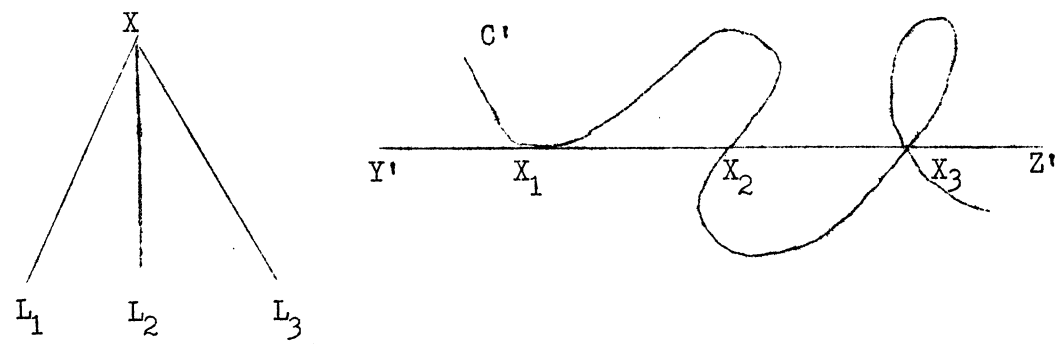
This makes it clear that the mapping is birational for we have a one-to-one correspondence between π and π' if we first remove the sides of the triangles of reference. But consider the exceptional elements. Each point of YZ is mapped into X' and similarly, for the inverse transformation, the whole of $Y'Z'$ goes into X . Reversing out point of view, we may say that the quadratic transformation inflates the <u>point</u> X into the <u>line</u> $Y'Z'$. It is this property which is so valuable in the study of singularities.

We shall now examine what happens at X more closely by letting P tend to X along a line $\beta Y = \gamma Z$. Accordingly put $P \equiv (1 , \varepsilon \gamma , \varepsilon \beta)$ then $P' \equiv (\varepsilon^2 \beta\gamma , \varepsilon\beta , \varepsilon\gamma) \equiv (\varepsilon\beta\gamma , \beta , \gamma)$ consequently (letting $\varepsilon \to 0$) as $P \to X$ along $\beta Y = \gamma Z$ its image P' will tend to $(0 , \beta , \gamma)$.



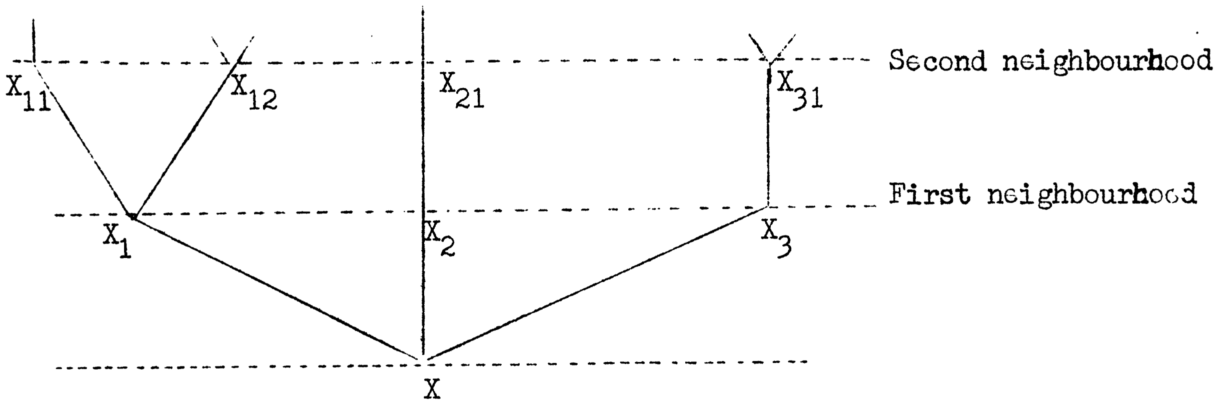
Thus there is a one-to-one correspondence between the directions through X and the points of the line Y'Z' into which it is inflated. It is customary to call Y'Z' the first neighbourhood of X . In this terminology, the directions through X correspond to the points in its first neighbourhood.

After these preliminaries let C be given plane curve suppose that we wish to study a particular point on it. This can be any point, but it should be thought of as a very complicated singularity. The procedure is then to construct a triangle XYZ of reference so that X is the singularity in question and neither XY nor XZ is a branch tangent. Denote the distinct branch tangents at X by $L_1 , L_2 , \ldots , L_s$

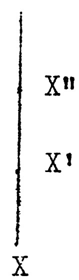


then their s directions through X will determine s points $X_1 , X_2 , \ldots , X_s$ in the first neighbourhood of X . Further, since the curve C passes through X in the directions of $L_1 , L_2 , \ldots , L_s$ the transform C' of C (when one applies the quadratic transformation) will pass through

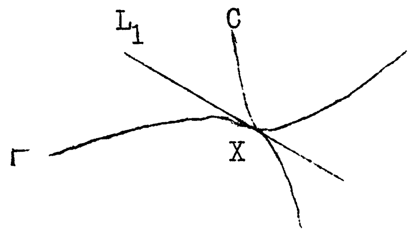
X_1 , X_2 , ... , X_s . For this reason these latter are called <u>the points of C which lie in the first neighbourhood of</u> X . Now there will be points X_{i1} , X_{i2} , ... , X_{it} in the first neighbourhood of X_i which lie on C' these being defined, in the same manner by means of a further quadratic transformation. The X_{ij} are known as the points of C in the second neighbourhood of X . Additional applications of quadratic transformations lead us successively to the points of C which belong to the third, fourth and higher neighbourhoods of X . I think it helps to imagine the system of points which make up the neighbourhoods of X , arranged in a diagram thus :

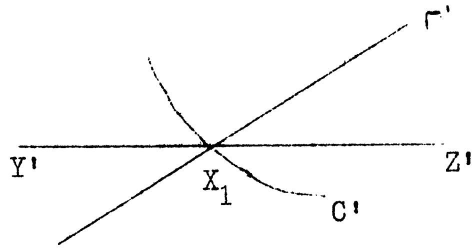


Let us call the complete diagram the <u>tree of neighbourhoods</u>, and a sequence X , X' , X" , ... , in which each of X' , X" , ... is in the first neighbourhood of the point which precedes it, a branch sequence. It is known that there are finitely many branch sequences and that they correspond, in a natural way, with the different analytic branches of C at X . For example, if X is a simple point or a cusp or, quite generally, if C is analytically irreductible at X , the whole tree of neighbourhoods reduces to a single branch



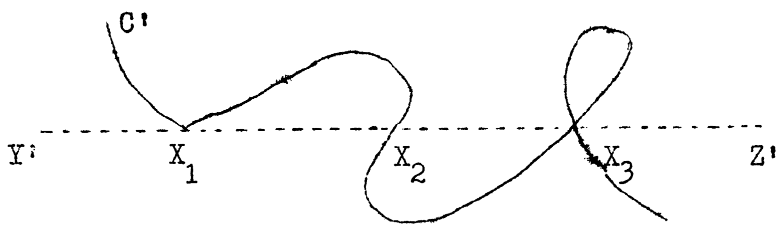
So far wa have considered only a single curve. We must now discuss, very briefly, the case of two curves C and Γ , both of which pass through X . If C and Γ have a common branch tangent L_1 (say) then their transforms





C' and Γ' both pass through X_1. This is expressed by saying that C and Γ have the point X_1 of the first neighbourhood in common. In like manner we explain what it means for C and Γ to have common points in the higher neighbourhoods of X ; and then it is found that the total number of points, infinitely near to X, which lie on both C and Γ is always finite.

After this remark we return to the consideration of a single given curve C. The initial quadratic transformation produces not only a curve C' but also a line Y'Z', namely the first neighbourhood of X, and these meet in the points $X_1, X_2, \dots, X_s$. One now says that



$X_1, X_2, \dots, X_s$, and all points infinitely near to them, which lie on both C' and Y'Z' are proximate to X on C. It is precisely this notion of proximity which dominates the more advanced theory of curve singularities. Time is too short to illustrate this in any detail and so I shall only mention one single property. This can be stated as follows :

The multiplicity of X (on C) is equal to the sum of the multiplicities (on C) of all the points proximate to X.

Although what I have said can give but a poor impression of the achievements of the geometric method, we must pass now to the general abstract theory which it suggests. To effect this transition, let us review what has already been said from a more algebraic point of view. The rational functions on C, which are finite and determinate at X, form a ring Λ, the so-called local ring of C at X, and we know, in an intuitive way, that the structure of this ring reflects the nature of the singularity at X. The quadratic transformation replaces C by C' and X by the points $X_1, X_2, \dots, X_s$. Let Λ_i be the local ring of C' at X_i then, since the transformation is birational, Λ and Λ_i have the same quotient field. Clearly the clue to the problem is to find how to derive $\Lambda_1, \Lambda_2, \dots, \Lambda_s$ by a process which only uses the ring structure of Λ. If we can do this then we may expect that the process will also be applicable to situations not found in geometry and in this way we shall arrive at a more general theory. To be brief, I shall begin immediately to describe the abstract theory which emerges when one follows out this idea.

From here on, Λ will denote a completely arbitrary one-dimensional local ring, with maximal ideal $\mathfrak{m}$ and residue field $\Lambda/\mathfrak{m} = K$. If now $\alpha \in \mathfrak{m}^s$ then we say that α <u>is superficial of degree</u> s if $\alpha\mathfrak{m}^\nu = \mathfrak{m}^{\nu+s}$ for all large values of ν. The superficial elements of degree zero are the units and there always exist superficial elements of degree s provided that s is sufficiently large. Observe that if α is superficial of degree s and β is superficial of degree t then $\alpha\beta$ is superficial of degree $s + t$; hence the set of superficial elements is closed under multiplication. Consider next the set of formal fractions a/c, where $a \in \mathfrak{m}^s$ and c is superficial of degree s. Here s is a freely variable degree of superficiality. If a_1/c_1 and a_2/c_2 are two such fractions, we put $a_1/c_1 \sim a_2/c_2$ if $cc_2 a_1 = cc_1 a_2$ for some superficial element c. This relation $\sim$ is an equivalence relation. Denote by $[\frac{a}{c}]$ the equivalence class to which $\frac{a}{c}$ belongs, then these classes can be made into a ring $\mathfrak{R}$ by setting

$$\left[\frac{a_1}{c_1}\right] + \left[\frac{a_2}{c_2}\right] = \left[\frac{c_2\, a_1 + c_1\, a_2}{c_1\, c_2}\right] , \quad \left[\frac{a_1}{c_1}\right]\left[\frac{a_2}{c_2}\right] = \left[\frac{a_1\, a_2}{c_1\, c_2}\right] .$$

$\mathfrak{R}$ is called the <u>first neighbourhood ring of</u> Λ. Observe that the mapping $a \to [\frac{a}{1}]$ is a ring-homomorphism $\Lambda \to \mathfrak{R}$. This is a fundamental mapping and it turns out that $\Lambda \to \mathfrak{R}$ is an isomorphism if and only if Λ is regular or, if you prefer it, a valuation ring. (This corresponds to the geometric situation when X is a simple point of C).

Now $\mathfrak{R}$ is found to be a semi-local ring, which implies that it has only a finite number of maximal ideals. Denote by $\Lambda_1^{(1)}, \ldots, \Lambda_1^{(s)}$ the rings of fractions of $\mathfrak{R}$ with respect to its maximal ideals. These rings, which are all one-dimensional local rings, will be said to be in the first neighbourhood of Λ (They correspond precisely to the local rings of the curve C' at the points $X_1, X_2, \ldots, X_s$ in the first neighbourhood of X). Let Λ_1 be one of the local rings in the first neighbourhood of Λ, then we can repeat the construction with Λ_1 and so obtain rings in the first neighbourhood of Λ_1 which will be said to be in the second neighbourhood of Λ. Further repetitions lead, of course, to rings in the third, fourth and higher neighbourhoods of Λ giving a system of one-dimensional local rings which we call the tree of neighbourhoods of Λ. This tree is made up of branch sequences $\Lambda_0 = \Lambda, \Lambda_1, \Lambda_2, \ldots$, where Λ_{r+1} is in the first neighbourhood of Λ_r and, just as in the case of curves, it is not difficult to show that the number of different branch sequences is finite.

Let Λ_1 be in the first neighbourhood of Λ then Λ_1 is a ring of fractions of $\mathfrak{R}$. We have therefore a canonical mapping $\mathfrak{R} \to \Lambda_1$ which we can combine with the original mapping $\Lambda \to \mathfrak{R}$ to obtain a homomorphism $\Lambda \to \Lambda_1$. Denote by $\Lambda_1\mathfrak{m}$ the ideal generated by the image of $\mathfrak{m}$ in Λ_1 then, and this is a fundamental result, $\Lambda_1\mathfrak{m}$ is a principal ideal, that is to say it can be generated by a single element. It is this fact which gives rise to the theory of proximity relations.

As an illustration, let $\Lambda_0 = \Lambda, \Lambda_1, \Lambda_2, \ldots$ be a branch sequence and let $\mathfrak{m}_r$ be the maximal ideal of Λ_r. Then, as already observed, $\Lambda_{r+1}\mathfrak{m}_r$ is a principal ideal, say

$$\Lambda_{r+1}\,\mathfrak{m}_r = \Lambda_{r+1}\,\tau_{r+1} \qquad (\tau_{r+1} \in \Lambda_{r+1}) .$$

One now says that Λ_p <u>is proximate to</u> Λ_0 if

$$\frac{\tau_1}{\tau_2\ \tau_3 \cdots \tau_p} \in \mathfrak{m}_p$$

where, it is to be unterstood, the left hand side is to be computed in Λ_p. (This means that we first embed τ_r $(1 \leq r \leq p-1)$ in Λ_p by means of the mappings $\Lambda_r \to \Lambda_{r+1} \to \cdots \to \Lambda_p$). This definition opens the way to generalisations of results well known in the theory of curves. For instance, one can show that the multiplicity of Λ is equal to the sum, properly counted, of the multiplicities of the local rings proximate to it. I should say that the multiplicity of a local ring is an abstract concept, due to Professor Samuel, which corresponds to the multiplicity of a point on an algebraic variety. The definition is a little complicated so I shall not go into details.

In my earlier remarks, I recalled that, for curves, the different branch sequences correspond to the different analytic branches as defined by means of power-series. By way of conclusion, therefore, I shall indicate how this fact becomes incorporated in the new theory. The powers of the maximal ideal of Λ define a metric topology on the ring. We can therefore form the completion Λ^* of Λ and this, too, is a one-dimensional local ring. Let

$$\mathfrak{p}^{*(1)}, \mathfrak{p}^{*(2)}, \ldots, \mathfrak{p}^{*(t)}$$

be the non-maximal prime ideals of Λ^* then I assert that there is a natural one-to-one correspondence between the branch sequences $\Lambda, \Lambda_1, \Lambda_2, \ldots$ and these prime ideals. The correspondence works in this way. The first neighbourhood ring of Λ^* is found to be the completion of the first neighbourhood ring of Λ. From this it follows that to each branch sequence $\Lambda, \Lambda_1, \Lambda_2, \ldots$ there corresponds a branch sequence $\Lambda^*, \Lambda_1^*, \Lambda_2^*, \ldots$ and conversely. In other terms, the trees of neighbourhoods of Λ and Λ^* have the same structure. Suppose now that a particular branch sequence $\Lambda, \Lambda_1, \Lambda_2, \ldots$ is given. Let $\Lambda^*, \Lambda_1^*, \Lambda_2^*, \ldots$ be the corresponding sequence then, when n is large enough, the zero ideal of Λ_n^* turns out to be a primary ideal. Let this primary ideal belong to the prime ideal $\mathfrak{p}_n^*$ then the inverse image of $\mathfrak{p}_n^*$, for the combined mapping $\Lambda^* \rightarrow \Lambda_1^* \rightarrow \ldots \rightarrow \Lambda_n^*$ is a prime ideal $\mathfrak{p}^*$ of Λ^* which does not depend on n. If now we associate $\mathfrak{p}^*$ with the sequence $\Lambda, \Lambda_1, \Lambda_2, \ldots$ this gives the required one-to-one correspondence between branch sequences and prime ideals.

As before let $\mathfrak{p}^{*(1)}, \ldots, \mathfrak{p}^{*(t)}$ be the non-maximal prime ideals of Λ^* and $\mathfrak{p}^*$ any particular one of them. Put $\bar{\Lambda} = \Lambda^*/\mathfrak{p}^*$ so that $\bar{\Lambda}$ is a complete one-dimensional local ring without zero-divisors. These rings, of which $\bar{\Lambda}$ is typical, are called the analytic components of Λ and, if μ is the length of the $\mathfrak{p}^*$-primary component of the zero-ideal of Λ^*, then we say that μ is the multiplicity of $\bar{\Lambda}$ as analytic component of Λ. Suppose now that $\Lambda, \Lambda_1, \Lambda_2, \ldots$ is a branch sequence, then to it corresponds a prime ideal $\mathfrak{p}^*$ and hence an analytic component $\bar{\Lambda}$; further the correspondence between branch sequences and analytic components is one-to-one.

What now is the interpretation of the multiplicity μ of the component $\bar{\Lambda}$? It is simply the terminal value of the multiplicities $m(\Lambda)$, $m(\Lambda_1)$, $m(\Lambda_2)$, ... of the rings in the branch sequence.

Finally $\bar{\Lambda}$ has only one analytic component, namely itself, and so its tree of neighbourhoods consists of a single branch

$$\bar{\Lambda}\ ,\ \bar{\Lambda}_1\ ,\ \bar{\Lambda}_2\ ,\ \ldots\ .$$

Let us compare this with the original branch

$$\Lambda\ ,\ \Lambda_1\ ,\ \Lambda_2\ ,\ \ldots$$

The connections between the two are very intimate and, in particular, the proximity relations are the same so that Λ_n is proximate to Λ_n if and only if $\bar{\Lambda}_n$ is proximate to $\bar{\Lambda}_n$. But $\bar{\Lambda}$, $\bar{\Lambda}_1$, $\bar{\Lambda}_2$, ... has the great advantage that it always terminates, which means that

$$\bar{\Lambda}_n = \bar{\Lambda}_{n+1} = \bar{\Lambda}_{n+2} = \ldots$$

for a suitable integer n . Now the moment at which the sequence comes to an end is just the moment when we reach a ring $\bar{\Lambda}_n$ of multiplicity one ; or, to use geometric language, the termination of the sequence marks the final resolution of the associated singularity.

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