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SOME OPEN PROBLEMS IN BANACH SPACE THEORY

by Joram LINDENSTRAUSS

My aim in this talk is to discuss briefly some directions of research in the isomorphic theory of Banach spaces. The directions which I plan to discuss are the following.

(a) Existence of operators,

(b) Local theory of Banach spaces,

(c) The approximation property,

(d) Existence of bases,

(e) Non linear problems,

(f) Non separable problems.

In each of these directions, I shall discuss only very few open questions so as to illustrate the specific direction. It should be emphasized that the choice of these questions is based on my personal interest and there are many other (often more fundamental) open questions in these directions which are currently being studied by various mathematicians. All the topics mentioned above are concerned mainly with general Banach spaces. Because of lack of time, I shall not enter into another very active domain of research in the isomorphic theory of Banach spaces, namely the structure of special Banach spaces. Let me just mention that among the special Banach spaces the most widely studied ones are the classical Banach spaces, i. e. spaces closely related to $L_p(\mu)$ and $C(K)$ spaces. In spite of the very significant progress made in recent years in the study of these spaces, there are still many challenging open problems concerning them. Recently, there began also a serious effort to study the structure of some non classical spaces (like spaces of analytic functions (especially the disc algebra), spaces of operators (especially the spaces C_p, $1 \leq p \leq \infty$, of operators on ℓ_2) and Orlicz spaces). The study of the structure of these spaces generates many interesting problems which are related to other areas of analysis like harmonic analysis, probability, complex analysis and operator theory.

As far as references are concerned, I shall give here mostly references to the latest results in the specific direction (these references contain in turn references to many earlier contributions).

Let us start with topic (a). The only general method for constructing operators between general Banach spaces X and Y is the Hahn-Banach theorem. This theorem ensures the existence of continuous linear functionals on X and thus of operators of rank 1 from X to Y . By taking sums, we get operators of finite rank, and by taking limits, we get compact operators from X to Y . At this stage, the general construction methods stop. We shall come back a little bit later to the question what kind of compact operators do we actually get. First, I want to discuss the question of existence of non compact operators. There are simple and well known examples of infinite dimensional Banach spaces X and Y such that every bounded linear operator from X to Y is compact (take e. g. $X = \ell_p$, $Y = \ell_r$ with $p > r$). In order to get a meaningful question, we have to restrict the pairs (X , Y) . An interesting situation occurs if X = Y . In this case, we always have a non compact operator namely the identity I . Are there also other non compact operators ? More precisely :

(Q.1) Does there exist an infinite dimensional Banach space X so that every bounded linear operator $T : X \to X$ is of the form $T = \lambda I + K$ with K compact ?

If such an X exists, it would have several other interesting properties.

(i) X is not isomorphic to its subspaces of finite-codimension (every $T : X \to X$ is either compact or a Fredholm operator of index 0).

(ii) X is indecomposable, i. e. every decomposition of X into a direct sum $X = Y \oplus Z$ is trivial in the sense that either $\dim Y < \infty$ or $\dim Z < \infty$.

(iii) Every bounded linear operator $T : X \to X$ has non trivial closed invariant subspaces.

There is no known example of a Banach space which has either one of these three properties (concerning (iii), we have to add the assumption that X is separable to make it non trivial. It should be pointed out also that P. ENFLO [7] constructed an example of an operator on some Banach space which fails to have non trivial invariant subspaces).

Coming back to the general available methods for constructing operators from X to Y , we point out that the use of the Hahn-Banach theorem insures in general only the existence of operators $T : X \to Y$ having the form

$$T(x) = \sum_{n=1}^{\infty} x_n^*(x)\, y_n \, , \quad x_n^* \in X^* \, , \quad y_n \in Y \text{ and } \sum_n \|x_n^*\|\, \|y_n\| < \infty \, .$$

Such operators were called by GROTHENDIECK nuclear operators. He posed the following question.

(Q.2) Does there exist a pair of infinite dimensional Banach spaces X and Y so that every compact operator from X to Y is nuclear ?

This question is still open, but there are by now several strong partial results which seem to suggest that the answer to (Q.2) is negative. The answer is e. g. negative if X has an unconditional basis [10], or if either X or Y is uniformly convex [5].

The question (Q.2) is actually already a question in item (b) of the list above namely the local theory of Banach spaces. This theory is devoted to the study of the structure of finite-dimensional subspaces of Banach spaces or in other words to the study of convex sets in the euclidean space R^n for n large (but finite). If n is large, there occur several phenomena which defy perhaps the common intuition (derived from the cases $n = 2, 3$). A central result in the local theory of Banach spaces is the theorem of Dvoretzky which asserts that for every $\varepsilon > 0$ and integer k there is an integer $n = n(\varepsilon, k)$ so that every Banach space X of dimension $\geqslant n$ (in particular, if $\dim X = \infty$) has a subspace Y with $\dim Y = k$ and

$$d(Y, \ell_2^k) \leqslant 1 + \varepsilon .$$

The existence of finite-dimensional almost hilbertian subspaces in an arbitrary Banach spaces enables one to reduce some questions on general Banach spaces to the much simpler case of Hilbert space. In general the almost hilbertian subspaces of concrete finite-dimensional Banach spaces are of a surprisingly high dimension (cf. [8], and this fact increases very much the effectiveness ot the use of almost hilbertian subspaces in the study of general Banach spaces. They would be even more useful if one could show that there are "nicely situated". This leads to the following question.

(Q.3) <u>Let X be an infinite-dimensional uniformly convex Banach space. Does there exist a constant C so that for every n there is a subspace $Y_n \subset X$ so that $d(Y_n, \ell_2^n) \leqslant C$ and there is a projection $P_n : X \to Y_n$ with $\|P_n\| \leqslant C$?</u>

TZAFRIRI proved that the answer is positive if X has an unconditional bases. Some stronger results can be found in JOHNSON and TZAFRIRI [9]. Let us point out that the assumption that X be uniformly convex is essential in (Q.3) since if e. g. $X = C(0, 1)$ or $X = L_1(0, 1)$ it is easily verified that X does not have nicely complemented copies of ℓ_2^n. These spaces have however nicely complemented copies of ℓ_∞^n (resp. ℓ_1^n). This observation leads to a version of (Q.3) which makes sense for arbitrary Banach spaces.

(Q.4) <u>Let X be an infinite-dimensional Banach space. Does there exist a constant C and a p (equal either to $1, 2$ or ∞) so that for every n there is a subspace $Y_n \subset X$ with $d(Y_n, \ell_p^n) \leqslant C$ and a projection P_n from X on to Y_n with $\|P_n\| \leqslant C$?</u>

The strongest known partial answers to (Q.4) are contained in the above mentioned paper of JOHNSON and TZAFRIRI. Let me point out that a positive answer to (Q.4) implies immediately that (Q.2) has a negative answer (this illustrates the claim made above that (Q.2) is a question in the "local theory").

In the study of the structure of non uniformly convex Banach spaces the spaces ℓ_1^n and ℓ_∞^n enter in a natural way (it is this fact which makes (Q.4) a reasonable problem). This is shown most clearly in the work of R. C. JAMES. A result which started a research direction in the local theory of Banach spaces which is parallel to the direction initiated by Dvoretzkij's theorem is the following theorem of James. Let X be a non reflexive Banach space. Then for every $\varepsilon > 0$ there is a subspace Y of X so that $\dim Y = 2$ and $d(Y, \ell_1^2) \leq 1 + \varepsilon$. For a long time, it was not known whether the same is true for ℓ_1^n for every finite n. JAMES settled this problem by proving that the answer is negative for $n = 3$ (and thus for all $n \geq 3$). There is however a stronger non reflexivity condition which implies the existence of almost isometric copies of ℓ_1^3. This is obtained in the following way : To say that X is not reflexive means that the canonical isometry $J_X : X \to X^{**}$ is not onto : There are two canonical isometries from X^{**} into $X^{(IV)}$ namely $(J_X)^{**}$ and $J_{X^{**}}$. Either one of these maps is onto if and only if X is reflexive. One can ask whether both maps together yield all of $X^{(IV)}$, i. e. whether

$$X^{(IV)} = J_{X^{**}}(X^{**}) + (J_X)^{**} X^{**} .$$

(this is the case if and only if X^{**}/X is reflexive). It was shown in [6] that if this is not the case then X contains almost isometric copies even of ℓ_1^n. Among the very many open problems in this direction let me mention the following.

(Q.5) Let X be a Banach space such that X^{**}/X is not reflexive. Is it true that for every ε and every integer n there is a subspace Y of X with

$$d(Y, \ell_1^n) \leq 1 + \varepsilon \ ?$$

In particular, what is the situation for $n = 5$?

We turn now to the approximation property. Let us recall that a Banach space has the approximation property (AP in short), if for every compact subset K of X and every $\varepsilon > 0$ there is an operator $T : X \to X$ which

$$\dim TX < \infty \quad \text{and} \quad \|Tx - x\| \leq \varepsilon$$

for every $x \in K$. ENFLO was the first who proved that there are Banach spaces which fail to have the approximation property. Since Enflo's work several other examples of spaces which fail to have the AP were constructed. We mention, in particular Davie's paper [4] which contains elegant examples of subspaces of ℓ_p, $2 < p < \infty$, which fail to have the AP and Szankowski's example [16] of a Banach lattice which fails to have the AP. It is however still far from being clear to what extent the AP holds or fails to hold "in general". For example, the following problem is open.

(Q.6) Let X be an infinite-dimensional Banach space which is not isomorphic to a Hilbert space. Does X have a subspace which fails to have the AP ?

All known examples of spaces which fail to have the AP are in some sense "artificial". It is of interest to exhibit spaces which appear naturally in analysis and fail to have the AP. This leads to the following.

(Q.7) Let $B(\ell_2)$ be the space of all bounded operators from ℓ_2 into itself with the operator norm. Does $B(\ell_2)$ have the AP ? Does the space of all bounded analytic functions $f(z)$ on $\{z\ ;\ |z| < 1\}$ with the sup norm have the AP ?

Note that both spaces mentioned in (Q.7) are non separable. There are several strong and simple criteria for checking the AP in separable spaces (the existence of a Schauder basis for example). By rising such a criteria or even more simply by arguing directly it is easy to prove that the common separable spaces which appear in analysis have the AP. There is a variant of the AP called the uniform approximation property (UAP in short) which leads to interesting and perhaps difficult problems in the setting of several common separable spaces. This property was introduced by PELCZYNSKI and ROSENTHAL and the point in it is that it does not only ask whether a suitable $T: X \to X$ with $\dim TX < \infty$ exists but asks for an estimate on $\dim TX < \infty$. A Banach space X is said to have the λ-UAP if there is a function $n \to f(n)$ on the integers so that for every n vectors $\{x_i\}_{i=1}^n$ in X there is a $T: X \to X$ with $\|T\| \leq \lambda$, $Tx_i = x_i$ for $1 \leq i \leq n$, and $\dim TX \leq f(n)$. A Banach space is said to have the UAP if it has the λ-UAP for some $\lambda < \infty$. PELCZYNSKI and ROSENTHAL proved that the spaces $L_p(\mu)$ and $C(K)$ have the UAP ; they use however a method which works only for these spaces. There are no known general criteria which can be used to verify the UAP in other concrete spaces. The existence of a basis in X and even the existence of an unconditional basis does not ensure that X has the UAP (this follows from [16]). The only spaces (besides those closely related to $C(K)$ and L_p spaces) which are known to have the UAP are reflexive Orliez spaces [12].

(Q.8) <u>Does the space</u> $C(\ell_2)$ (<u>the space of compact operators on</u> ℓ_2 <u>with the usual operator norm</u>) <u>have the UAP</u> ? <u>Does the disc algebra have the UAP</u> ?

Problems (Q.7) and (Q.8) are closely related. This follows from the observation [12] that X has the UAP if and only if X^{**} has the UAP (this is known to be false if UAP is replaced by AP). In this connection, I would like to mention also.

(Q.9) <u>Let</u> X <u>have the UAP; does</u> X^* <u>have the UAP</u> ? It is known that the answer is positive if we restrict ourselves to uniformly convex spaces [12].

We pass now to topic (d) – existence of bases. Of course the existence of separable spaces failing to have the AP implies in particular that there are separable spaces which fail to have a basis. A quite simple fact which was known already to Banach is that every infinite-dimensional Banach space has a subspace with a basis. This fact is one of the very few known results which ensure the existence of nice "infinite-dimensional objects" in a general Banach space (there are many more results which ensure the existence of concrete finite-dimensional objects in general Banach spaces, e. g. Dvoretzky's theorem). It is therefore of much interest to investigate whether the result on existence of a subspace with a basis can be improved. In particular, the following question arises.

(Q.10) <u>Let</u> X <u>be an infinite-dimensional Banach space. Does</u> X <u>have an infinite-dimensional subspace with an unconditional basis</u> ?

A recent result of MAUREY and ROSENTHAL [13] suggests that the answer to (Q.10) may be negative. They showed that a natural approach to constructing unconditional basic sequences does not work in general (they exhibited a sequence of elements $\{x_n\}_{n=1}^{\infty}$ in a Banach space so that $\|x_n\| = 1$ for every n and $w - \lim x_n = 0$ but no subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ forms an unconditional basic sequence).

A question closely related to (Q.10) is the following.

(Q.11) _Let_ X _be an infinite-dimensional Banach space. Does_ X _contain an infinite-dimensional subspace_ Y _so that_ Y _is either reflexive or isomorphic to one of the spaces_ C_0 _and_ ℓ_1 ?

An old result of JAMES shows that a positive answer to (Q.10) implies a positive answer to (Q.11). It is however not unlikely that (Q.10) has a negative answer while (Q.11) has a positive answer. There are several results which are related to (Q.11). The deepest one is due to ROSENTHAL [15] who characterized those Banach spaces which have a subspace isomorphic to ℓ_1 .

A different question on bases is concerned with their uniqueness, i. e. in what sense is a coordinate system generated by a suitable basis in a Banach space a canonical coordinate system. Let me recall that a basis $\{x_n\}_{n=1}^{\infty}$ is said to be equivalent to a basis $\{y_n\}_{n=1}^{\infty}$ if a series $\sum_n \lambda_n x_n$ with $\{\lambda_n\}_{n=1}^{\infty}$ scalars converges if and only if $\sum_n x_n y_n$ converges. Unconditional bases if they exist are rarely unique. In fact, the only Banach spaces which have up to equivalence a unique normalized unconditional basis are C_0 , ℓ_1 and ℓ_2 (a basis $\{x_n\}_{n=1}^{\infty}$ is said to be normalized if $\|x_n\| = 1$ for every n). If a Banach space X has two non equivalent normalized unconditional bases then it is easily seen to have a normalized unconditional basis $\{x_n\}_{n=1}^{\infty}$ which is not symmetric (an unconditional basis $\{x_n\}_{n=1}^{\infty}$ is said to be symmetric if $\{x_n\}_{n=1}^{\infty}$ is equivalent to $\{x_{\pi(n)}\}_{n=1}^{\infty}$ for every permutation π of the integers). It is then easy to see that the family $\{x_{\pi(n)}\}_{n=1}^{\infty}$ where π ranges over all permutations of the integers contains uncountably many mutually non equivalent normalized unconditional bases on X . If we restrict ourselves to symmetric bases then the pheaomenon of uniqueness is much more common. There are many examples of spaces which have up to equivalence a unique symmetric basis and thus in them the symmetric basis provides a canonical coordinate system (this is the case e. g. for ℓ_p , $1 \leq p < \infty$, Lorentz sequence spaces, many Orliez sequence spaces). In some situations (even for some Orlicz sequence spaces), a space may have more than one symmetric basis. In all known examples of spaces having more than one symmetric basis there are uncountably many mutually non equivalent symmmetric Bases. Unlike the situation with non uniqueness of unconditional bases (were we could simply use permutations to generate new equivalence classes), there seems to be no general procedure to generate new symmetric bases if we know that there exist two non equivalent ones. This gives rise to the following question.

(Q.12) _Let_ X _be a Banach space having at least two non equivalent symmetric bases._

Must X have infinitely many mutually non equivalent symmetric bases ?

(For further background to (Q.12), cf. [11]).

We pass now to some non linear problems. The structure of a Banach space and its convex subsets as metric spaces has been a subject of much research (of course, we take as the metric the natural one induced by the norm, i. e. $d(x , y) = \|x - y\|$). There are now many strong and beautiful theorems in this direction (cf. the book [2]). Actually, the study of Banach spaces as metric spaces created a new branch of topology called infinite-dimensional topology. That this subject is a branch of topology rather than Banach space theory stems from the fact that the linear topological properties of a Banach space have almost no topological meaning. For example, I recall the result of KADEC which states that any two separable infinite-dimensional Banach spaces are homeomorphic. The situation changes completely if we take into account not only the topology induced by $\|x - y\|$, but the uniform structure induced by it. The constructions used in infinite dimensional topology are almost never uniformly continuous. It is not true that every two infinite-dimensional separable Banach spaces are uniformly homeomorphic. In fact, the uniform structure of a Banach space gives much information on its linear structure, and it is not known whether it determines it completely.

(Q.13) Let X and Y be two uniformly homeomorphic Banach spaces. Is X isomorphic to Y ?

The strongest known partial result to (Q.13) is a result of RIBE [14] which says that if X is uniformly homeomorphic to Y then the local structure of X and Y as Banach spaces are the same (i. e. there is a constant $\lambda < \infty$ so that for every subspace B of X with $\dim B < \infty$ there is a $B' \subset Y$ with $d(B , B') < \lambda$). An interesting special case of (Q.13) is obtained if $X = C_0$ and $Y = C(0 , 1)$ (these spaces have the same local structure so RIBE's result does not say anything in this case). AHARONI [1] proved that there is mapping $T : Y \to X$ so that T and T^{-1} both satisfy a Lipschitz condition (it is well known that there is no linear T which satisfies this). Moreover there is a Lipschitz continuous projection from X onto (it non linear subset) TY . These results seem to suggest that C_0 may be uniformly homeomorphic to $C(0 , 1)$.

As a matter of fact, there is very little known on the uniform structure of Banach spaces. Let me mention just one more (among the very many natural) question on this subject.

(Q.14) Is the Hilbert space ℓ_2 uniformly homeomorphic to a bounded subset of itself ?

AHARONI [1] observed that the answer is positive if we replace ℓ_2 by C_0 .

We come to our last topic-non separable questions. Non separable questions have attracted much less attention than separable ones among the research in Banach space theory. The lack of knowledge on non separable spaces is illustrated by the fact that the following simple looking problem is still apparently open.

(Q.15) Does every infinitely dimensional Banach space X have an infinite dimensional separable quotient space ?

In many special cases (e. g. if X is reflexive), the answer to (Q.15) is positive and trivially so but no general construction of a separable quotient space is known.

The most extensively studied class of non separable Banach space for which a kind of structure theory is available is the class of WCG spaces (spaces which have a weakly compact set which generates the whole space). This class includes all reflexive spaces (where the unit ball is w-compact) as well as other spaces (e. g. $L_1(\mu)$ where μ is a finite measure on an arbitrary measure space). Among the many problems concerning WCG spaces which are still open I mention the following:

(Q.16) Let X be a Banach space. Assume that X is a Lindelöf space in its w topology ; is X isomorphic to a subspace of a WCG space ?

TALAGRAND [17] showed recently that conversely every WCG space (and thus every subspace of a WCG space) is Lindelöf in its ω topology.

In the non separable case, the question of classifying up to isomprphism the spaces $C(K)$ seems to pose very difficult questions. In the separable case (i. e. K compact metric), a complete classification up to isomorphism was carried out by BESSAGA and PELCZYNSKI and MILUTIN. The main step was the result of MILUTIN which showed that if K is compact metric uncountable then $C(K)$ is isomorphic to $C(\Delta)$ where Δ is the Cantor set. In the non-separable case (i. e. K compact Hausdorff non metrizable) there is some information on special classes of K (e. g. if K is the space of ordinals in the order topology, or a topological group or a Stone-Čech compactification of a metric space or an Eberlein compact). The general classification problem for non separable $C(K)$ spaces seems to be hopelessly complicated. However there are some concrete questions which may have a reasonable answer. The key point in the proof of MILUTIN's result was that every separable $C(K)$ space is isomorphic to such a space with K totally disconnected. Is this true in general ?

(Q.17) Let K be a compact Hausdorff space. Does there exist a totally disconnected compact Hausdorff space K_0 so that $C(K) \approx C(K_0)$?

In the setting of non separable spaces, there are many open questions about the existence of "nice norms". We mention here one question of this type.

(Q.18) Characterize those Banach spaces which have an equivalent strictly convex norm.

A norm is stictly convex if $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$ imply that $x = y$. (Q.18) is of course a vaguely stated problem. It is easily verified that every separable Banach space has an equivalent strictly convex norm. The same is true for a general WCG space (and also for duals of WCG spaces). On the other band, it was shown by DAY that there exist Banach spaces which do not have an equivalent strictly convex norm. Some conjectures concerning a possible answer to (Q.18) were shown to be false in [3]. This paper shows that even for C(K) spaces it seems to be a delicate and presumably difficult question to decide under which condition there exists an equivalent strictly convex norm.

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