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THE VERIFICATION OF THE NIRENBERG-TREVES CONJECTURE

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Abstract. — In a series of recent papers, Nils Dencker proves that condition (ψ) implies the local solvability of principal type pseudodifferential operators (with loss of $\frac{3}{2}+\epsilon$ derivatives for all positive ϵ), verifying the last part of the Nirenberg-Treves conjecture, formulated in 1971. The origin of this question goes back to the Hans Lewy counterexample, published in 1957. In this text, we follow the pattern of Dencker's papers, and we provide a proof of local solvability with a loss of $\frac{3}{2}$ derivatives.

INTRODUCTION

The Hans Lewy counterexample. — In 1957, Hans Lewy stunned the mathematical world by showing that very simple and natural linear PDE could fail to have solutions. The Hans Lewy operator L_0, introduced in [30], is the following complex vector field in $\mathbb{R}^3$

$$L_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3}. \tag{0.1}$$

There exists $f \in C^\infty$ such that the equation $L_0 u = f$ has no distribution solution, even locally. This discovery came as a great shock for several reasons. First of all, L_0 has a very simple expression and is natural as the Cauchy-Riemann operator on the boundary of the pseudo-convex domain

$$\{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + 2\Im z_2 < 0\}.$$

Moreover L_0 is a non-vanishing vector field so that no pathological behaviour related to multiple characteristics is to be expected. In the fifties, it was certainly the conventional wisdom that any "reasonable" operator should be locally solvable, and obviously (0.1) was indeed very reasonable, so the conclusion was that the CW should be revisited(1). One of the questions posed by such a counterexample was to find some geometric explanation for this phenomenon. This was done in 1960 by L. Hörmander

(1)Gaston Bachelard did not use the words conventional wisdom, but wrote in [1] "La science, dans son besoin d'achèvement comme dans son principe, s'oppose absolument à l'opinion. S'il lui arrive, sur un point particulier, de légitimer l'opinion, c'est pour d'autres raisons que celles qui fondent l'opinion ; de sorte que l'opinion a, en droit, toujours tort. L'opinion pense mal ; elle ne pense pas : elle traduit des besoins en connaissances. En désignant les objets par leur utilité, elle s'interdit de les connaître."

in [15] who proved that if p is the symbol of a differential operator such that, at some point (x,ξ) in the cotangent bundle,

$$p(x,\xi)=0 \quad \text{and} \quad \{\Re p, \Im p\}(x,\xi)>0, \tag{0.2}$$

then the operator P with principal symbol p is not locally solvable at x; in fact, there exists $f\in C^\infty$ such that, for any neighborhood V of x the equation $Pu=f$ has no solution $u\in \mathcal{D}'(V)$. Of course, in the case of differential operators, the sign >0 in (0.2) can be replaced by $\neq 0$ since the Poisson bracket $\{\Re p, \Im p\}$ is then an homogeneous polynomial with odd degree in the variable ξ. Nevertheless, it appeared later (in [16]) that the same statement is true for pseudodifferential operators, so we keep it that way. Since the symbol of $-iL_0$ is $\xi_1-x_2\xi_3+i(\xi_2+x_1\xi_3)$, and the Poisson bracket $\{\xi_1-x_2\xi_3, \xi_2+x_1\xi_3\}=2\xi_3$, the assumption (0.2) is fulfilled for L_0 at any point x in the base and the nonsolvability property follows. This gives a necessary condition for local solvability of pseudodifferential equations: a locally solvable operator P with principal symbol p must satisfy

$$\{\Re p, \Im p\}(x,\xi)\le 0 \quad \text{at} \quad p(x,\xi)=0. \tag{0.3}$$

Naturally, condition (0.3) is far from being sufficient for solvability (see e.g. the nonsolvable M_3 below in (0.4)). After the papers [30], [15], the curiosity of the mathematical community was aroused in search of a geometric condition on the principal symbol, characterizing local solvability of principal type operators. It is important to note that for principal type operators with a real principal symbol, such as a non-vanishing real vector field, or the wave equation, local solvability was known after the 1955 paper of L. Hörmander [14]. In fact these results extend quite easily to the pseudodifferential real principal type case. As shown by the Hans Lewy counterexample and the necessary condition (0.3), the matters are quite different for complex-valued symbols.

Some examples. — It is certainly helpful to take a look now at some simple models. For $t,x\in\mathbb{R}$, with the usual notations

$$D_t=-i\partial_t, \quad (\widehat{|D_x|u})(\xi)=|\xi|\hat{u}(\xi),$$

where $\hat{u}$ is the x-Fourier transform of u, $l\in\mathbb{N}$, let us consider the operators defined by

$$M_l=D_t+it^lD_x, \quad N_l=D_t+it^l|D_x|. \tag{0.4}$$

It is indeed rather easy to prove that, for $k\in\mathbb{N}$, $M_{2k}, N_{2k}, N^*_{2k+1}$ are solvable whereas M_{2k+1}, N_{2k+1} are nonsolvable. In particular, the operators M_1, N_1 satisfy (0.2). On the other hand, the operator $N_1^*=D_t-it|D_x|$ is indeed solvable since its adjoint operator N_1 verifies the *a priori* estimate

$$T\|N_1u\|_{L^2(\mathbb{R}^2)}\ge \|u\|_{L^2(\mathbb{R}^2)},$$

for a smooth compactly supported u vanishing for $|t| \geq T/2$. No such estimate is satisfied by $N_1^* u$ since its x-Fourier transform is

$$-i\partial_t v - it|\xi|v = (-i)(\partial_t v + t|\xi|v),$$

where v is the x-Fourier transform of u. A solution of $N_1^* u = 0$ is thus given by the inverse Fourier transform of $e^{-t^2|\xi|/2}$, ruining solvability for the operator N_1. A complete study of solvability properties of the models M_l was done in [33] by L. Nirenberg and F. Treves, who also provided a sufficient condition of solvability for vector fields; the analytic-hypoellipticity properties of these operators were also studied in a paper by S. Mizohata [31]. The simplicity of the two-dimensional models (0.4) is somewhat misleading, since they can be reduced via the Fourier transform, to the study of an ODE. It is not the case of the following examples, which are genuinely three-dimensional. The operators

$$P_{klm} = D_{x_1} - ix_1^{2k}(D_{x_2} + x_1^{2l+1}x_2^{2m}|D_x|), \quad x \in \mathbb{R}^3, k, l, m \in \mathbb{N}, \tag{0.5}$$

are locally solvable since their adjoints are subelliptic (see chapter 27 in the fourth volume of [19]). On the other hand the operators

$$D_{x_1} + ia(x)(D_{x_2} + x_1^{2l+1}x_2^{2m}|D_x|), \quad a \in C^\infty(\mathbb{R}^3; (-\infty, 0]), l, m \in \mathbb{N}, \tag{0.6}$$

are also locally solvable, but the proof is not elementary.

The expression of the Nirenberg-Treves conjecture. — Let us look first at the operator

$$L = D_t + iq(t, x, D_x), \tag{0.7}$$

where q is a real-valued first-order symbol. The symbol of L is thus $\tau + iq(t, x, \xi)$. The bicharacteristic curves of the real part are oriented straight lines with direction $\partial/\partial t$; now we examine the variations of the imaginary part $q(t, x, \xi)$ along these lines. It amounts only to check the functions $t \mapsto q(t, x, \xi)$ for fixed (x, ξ). The good cases in (0.4) (when solvability holds) are $t^{2k}\xi, -t^{2k+1}|\xi|$: when t increases these functions do not change sign from $-$ to $+$. The bad cases are $t^{2k+1}|\xi|$: when t increases these functions do change sign from $-$ to $+$; in particular, the nonsolvable case (0.2), tackled in [15], corresponds to a change of sign of $\Im p$ from $-$ to $+$ at a simple zero. The general formulation of condition (ψ) for a principal type operator with principal symbol p is as follows: for all $z \in \mathbb{C}$, $\Im(zp)$ does not change sign from $-$ to $+$ along the oriented bicharacteristic curves of $\Re(zp)$. It is a remarkable and non-trivial fact (due to the articles [3] of J.-M. Bony and [6] of H. Brézis) that this condition is invariant by multiplication by an elliptic factor. The *Nirenberg-Treves conjecture*, proved in several cases in [33], [34], [35], such as for differential operators with analytic coefficients, states that, *for a principal type pseudodifferential equation, condition (ψ) is equivalent to local solvability.* Using the Malgrange-Weierstrass theorem on normal forms of complex-valued non-degenerate C^∞ functions and the Egorov theorem on quantization of homogeneous canonical transformations, there is no loss of generality considering only first order operators of type (0.7). The expression of condition (ψ) for L is then very simple since it reads

$$q(t,x,\xi) < 0 \quad \text{and} \quad s > t \Longrightarrow q(s,x,\xi) \le 0. \tag{0.8}$$

The necessity of condition (ψ) for local solvability. — In 1981, following an idea given by R.D. Moyer [32] for a result in two dimensions, L. Hörmander proved in [18] that condition (ψ) is necessary for local solvability: assuming that condition (ψ) is not satisfied for a principal type operator P, he was able to construct some approximate non-trivial solutions u for the adjoint equation $P^*u = 0$, which implies that P is not solvable. Although the construction is elementary for the model operators N_{2k+1} in (0.4) (as sketched above for N_1), the multidimensional proof is rather involved and based upon a geometrical optics method adapted to the complex case. The details can be found in the proof of theorem 26.4.7′ of [19].

The proof of the conjecture for differential operators and in 2D. — For differential operators, condition (ψ) is equivalent to ruling out any change of sign of $\Im p$ along the bicharacteristics of $\Re p$ (the latter condition is called condition (P)); this fact is due to the identity $p(x,-\xi) = (-1)^m p(x,\xi)$, valid for an homogeneous polynomial of degree m in the variable ξ. Note that the expression of condition (P) for L in (0.7) is simply $q(t,x,\xi)q(s,x,\xi) \ge 0$. In 1973, R. Beals and C. Fefferman [2] took as a starting point the aforementioned results of L. Nirenberg and F. Treves on differential operators with analytic coefficients and, removing that analyticity assumption, were able to prove the sufficiency of condition (P) for local solvability, obtaining thus the sufficiency of condition (ψ) for local solvability of differential equations. The key ingredient was a drastically new vision of the pseudodifferential calculus, devised to obtain a factorization of the function q in (0.7) of the type

$$q(t,x,\xi) = a(t,x,\xi)b(x,\xi), \quad a \le 0 \text{ of order } 0 \text{ and } b \text{ of order } 1, \tag{0.9}$$

in regions of the phase space much smaller than cones or semi-classical "boxes" $\{(x,\xi), |x| \le 1, |\xi| \le h^{-1}\}$. Considering the family $\{q(t,x,\xi)\}_{t\in[-1,1]}$ of classical homogeneous symbols of order 1, they define, via a Calderón-Zygmund decomposition, a pseudodifferential calculus depending on the family $\{q(t,\cdot)\}$, in which all these symbols are first order but also such that, at some level t_0, some ellipticity property of $q(t_0,\cdot)$ or $\nabla_{x,\xi}q(t_0,\cdot)$ is satisfied. Although a factorization (0.9) can be obtained for differential operators with analytic regularity satisfying condition (ψ), such a factorization is not true in the C^∞ case, even microlocally in the standard sense[(2)]. This is why R. Beals and C. Fefferman had to resort to a much finer microlocalization scheme

[(2)]Consider the C^∞ function q defined on $\mathbb{R}^3$ by $q(t,x,\xi) = \begin{cases} (\xi - te^{-1/x})^2 & \text{if } x > 0, \\ \xi(\xi - e^{1/x}) & \text{if } x < 0. \end{cases}$ For every fixed (x,ξ), the function $t \mapsto q(t,x,\xi)$ does not change sign since $q(t,x,\xi)q(s,x,\xi) \ge 0$. Nevertheless than the classical one. In fact, the proof of R. Beals and C. Fefferman marked the day when microlocal analysis stopped being only homogeneous or semi-classical, thanks to methods of harmonic analysis such as Calderón-Zygmund decomposition made compatible with the Heisenberg uncertainty principle. In 1988, N. Lerner [23] proved the sufficiency of condition (ψ) for local solvability of pseudodifferential equations in two dimensions and as well for the classical oblique-derivative problem [24]. The method of proof of these results is also based upon a factorization analogous to (0.9) but where $b(x,\xi)$ is replaced by $\beta(t,x)|\xi|$ and β is a smooth function such that $t \mapsto \beta(t,x)$ does not change sign from $+$ to $-$ when t increases. Then a properly defined sign of $\beta(t,x)$ appears as a non-decreasing operator and the Nirenberg-Treves energy method can be adapted to this situation. The Beals-Fefferman result mentioned above proved the local existence of H^{s+m-1}_{loc} solutions u to the equation $Lu = f$ with a source f in H^s_{loc}, whenever L is an operator of order m satisfying condition (P); since the size of the neigbourhood where the equation is satisfied may depend on the index s, this is not enough to get C^∞ solutions whenever f is smooth. The existence of C^∞ solutions for C^∞ sources was proved by L. Hörmander in [17] for pseudodifferential equations satisfying condition (P). We refer the reader to the paper [21] for a more detailed historical overview of this problem. On the other hand, it is clear that our interest is focused on solvability in the C^∞ category. Let us nevertheless recall that the sufficiency of condition (ψ) in the analytic category (for microdifferential operators acting on microfunctions) was proved by J.-M. Trépreau [37] (see also [20], chapter VII).

Counting the loss of derivatives. — Let us consider a principal-type pseudodifferential operator L of order m. We shall say that L is locally solvable with a loss of μ derivatives whenever the equation $Lu = f$ has a local solution u in the Sobolev space $H^{s+m-\mu}_{\mathrm{loc}}$ for a source f in H^s_{loc}. Note that the loss is zero if and only if L is elliptic. Since for the simplest principal type equation $\partial/\partial x_1$, the loss of derivatives is 1, we shall consider that 1 is the "ordinary" loss of derivatives. When L satisfies condition (P) (e.g. if L is a differential operator satisfying condition (ψ)), or when L satisfies condition (ψ) in two dimensions, the estimates

$$C\|L^* u\|_{H^s} \geq \|u\|_{H^{s+m-1}}, \tag{0.10}$$

valid for smooth compactly supported u with a small enough support, imply local solvability with loss of 1 derivative, the ordinary loss referred to above. For many years, repeated claims were made that condition (ψ) for L implies (0.10), that is solvability with loss of 1 derivative. It turned out that these claims were wrong, as shown by N. Lerner in [25] by the following result (see also section 6 in the survey article [21] by L. Hörmander). *There exists a principal type first-order pseudodifferential operator L*

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one can show that it is not possible to find some C^∞ functions a, b such that a is nonnegative and b independent of t such that $q = ab$.

in three dimensions, satisfying condition (ψ), *a sequence* u_k *of* C_c^∞ *functions with* $\operatorname{supp} u_k \subset \{x \in \mathbb{R}^3, |x| \le 1/k\}$ *such that*

$$\|u_k\|_{L^2(\mathbb{R}^3)} = 1, \quad \lim_{k\to+\infty} \|L^* u_k\|_{L^2(\mathbb{R}^3)} = 0. \tag{0.11}$$

As a consequence, for this L, there exists $f \in L^2$ such that the equation $Lu = f$ has no local solution u in L^2. Thus condition (ψ) does not imply local solvability with loss of one derivative for pseudodifferential equations in three or more dimensions. The main result of Dencker discussed in this report is that, although local solvability with loss of one derivative does not follow from condition (ψ), that condition implies solvability with a larger loss. Namely the estimate

$$C\|L^* u\|_{H^s} \ge \|u\|_{H^{s+m-\mu}}, \tag{0.12}$$

holds (with $\mu = \epsilon + 3/2$ for all positive ϵ) for smooth compactly supported u with a small enough support, provided that L is a principal-type operator satisfying condition (ψ). Following the pattern of Dencker's paper, we show in fact ([22]) that it is possible to get rid of the ϵ and obtain $\mu = 3/2$ in (0.12). This proves that condition (ψ) implies local solvability with a loss of $3/2$ derivatives.

Preliminary comments. — The known counterexamples of [25], [21] do not rule out a loss of $1+\epsilon$ derivatives for any $\epsilon > 0$, so the loss $3/2$ may be not optimal under condition (ψ). One of the difficulties related to the handling of (0.12) when the loss μ is > 1 is the following: condition (ψ) is only concerned with the principal symbol of L^*, so that solvability and the estimate (0.12) should be preserved when the principal-type L^* is perturbed by a pseudodifferential operator of order $m-1$. However, the estimate (0.12) is too weak to absorb directly a perturbation of order $m-1$ and there is no way to avoid this situation under the sole condition (ψ) since (0.10) is not a consequence of (ψ) (it could be possible that the analyticity of the symbol and (ψ) imply (0.10)). The method of proof used by N. Dencker is based upon an energy method, rather classical in its principles, which was introduced by L. Nirenberg and F. Treves and developed by R. Beals and C. Fefferman. But although these authors were able to separate sharply the forward and backward regions of propagation for operators satisfying condition (P), N. Dencker defines these regions in the more general case of condition (ψ) and construct a multiplier smoother than a sign function. Although that smoothness forces a loss of derivatives larger than one, he can take advantage of it to handle some calculus of pseudodifferential operators. A version of one of his most striking arguments appears below as Lemma 2.10 and shows that the rigidity of condition (ψ) entails strong regularity properties for the set where the key change of sign occurs.

1. RESULTS AND NOTATIONS

1.1. Statement of the results

Let P be a properly supported principal-type pseudodifferential operator in a C^∞ manifold $\mathcal{M}$, with principal symbol p. The symbol p is assumed to be a C^∞ positively homogeneous function of degree m on $\dot{T}^*(\mathcal{M})$, the cotangent bundle minus the zero section. The principal type assumption that we shall use here is that

$$(x,\xi) \in \dot{T}^*(\mathcal{M}), \quad p(x,\xi) = 0 \Longrightarrow \partial_\xi p(x,\xi) \neq 0. \tag{1.1}$$

Also, the operator P will be assumed of polyhomogeneous type, which means that its total symbol is equivalent to $p + \sum_{j\geq 1} p_{m-j}$, where p_k is a smooth positively homogeneous function of degree k on $\dot{T}^*(\mathcal{M})$.

Definition 1.1 (Condition (ψ)). — *Let p be a C^∞ homogeneous function on $\dot{T}^*(\mathcal{M})$. The function p is said to satisfy condition (ψ) if, for $z = 1$ or i, $\Im(zp)$ does not change sign from $-$ to $+$ along an oriented bicharacteristic of $\Re(zp)$.*

For more properties of symbols satisfying this condition, we refer the reader to section 26.4 in [19].

Theorem 1.2. — *Let P be as above, such that its principal symbol p satisfies condition (ψ). Let s be a real number. Then, for all $x \in \mathcal{M}$, there exists a neighborhood V such that for all $f \in H^s_{loc}$, there exists $u \in H^{s+m-\frac{3}{2}}_{loc}$ such that*

$$Pu = f \text{ in } V.$$

Remark 1.3. — Theorem 1.2 will be proved by a multiplier method, involving the computation of

$$\langle Pu, Mu\rangle$$

with a suitably chosen operator M. It is interesting to notice that, the greater is the loss of derivatives, the more regular should be the multiplier in the energy method. As a matter of fact, the Nirenberg-Treves multiplier of [35] is not even a pseudodifferential operator in the $S^0_{1/2,1/2}$ class, since it could be as singular as the operator $\operatorname{sign} D_{x_1}$; this does not create any difficulty, since the loss of derivatives is only 1. On the other hand, in [9], [28], where estimates with loss of 2 derivatives are handled, the regularity of the multiplier is much better than $S^0_{1/2,1/2}$, since we need to consider it as an operator of order 0 in an asymptotic class defined by an admissible metric on the phase space.

1.2. Some notations

First of all, we recall the definition of the Weyl quantization a^w of a function $a \in \mathcal{S}(\mathbb{R}^{2n})$: for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\xi} a(\frac{x+y}{2}, \xi) u(y) dy d\xi.$$

Our definition of the Fourier transform $\hat{u}$ of $u \in \mathcal{S}(\mathbb{R}^n)$ is $\hat{u}(\xi) = \int e^{-2i\pi x\xi} u(x) dx$ and the usual quantization $a(x, D_x)$ of $a \in \mathcal{S}(\mathbb{R}^{2n})$ is $(a(x, D_x)u)(x) = \int e^{2i\pi x\xi} a(x, \xi)\hat{u}(\xi) d\xi$. The phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is a symplectic vector space with the standard symplectic form

$$\big[(x, \xi), (y, \eta)\big] = \langle \xi, y\rangle - \langle \eta, x\rangle. \tag{1.2}$$

Definition 1.4. — *Let g be a metric on $\mathbb{R}^{2n}$, i.e. a mapping $X \mapsto g_X$ from $\mathbb{R}^{2n}$ to the cone of positive definite quadratic forms on $\mathbb{R}^{2n}$. Let M be a positive function defined on $\mathbb{R}^{2n}$.*
(i) *The metric g is said to be slowly varying whenever $\exists C > 0, \exists r > 0, \forall X, Y, T \in \mathbb{R}^{2n}$,*

$$g_X(Y - X) \le r^2 \Longrightarrow C^{-1} g_Y(T) \le g_X(T) \le C g_Y(T).$$

(ii) *The symplectic dual metric g^σ is defined as $g^\sigma_X(T) = \sup_{g_X(U)=1} [T, U]^2$. The parameter of g is defined as $\lambda_g(X) = \inf_{T\ne 0}\big(g^\sigma_X(T)/g_X(T)\big)^{1/2}$ and we shall say that g satisfies the uncertainty principle if $\inf_X \lambda_g(X) \ge 1$.*
(iii) *The metric g is said to be temperate when $\exists C > 0, \exists N \ge 0, \forall X, Y, T \in \mathbb{R}^{2n}$,*

$$g^\sigma_X(T) \le C g^\sigma_Y(T)\big(1 + g^\sigma_X(X - Y)\big)^N.$$

When the three properties above are satisfied, we shall say that g is admissible. The constants appearing in (i) *and* (iii) *will be called the structure constants of the metric g.*
(iv) *The function M is said to be g-slowly varying if $\exists C > 0, \exists r > 0, \forall X, Y \in \mathbb{R}^{2n}$,*

$$g_X(Y - X) \le r^2 \Longrightarrow C^{-1} \le \frac{M(X)}{M(Y)} \le C.$$

(v) *The function M is said to be g-temperate if $\exists C > 0, \exists N \ge 0, \ \forall X, Y \in \mathbb{R}^{2n}$,*

$$\frac{M(X)}{M(Y)} \le C\big(1 + g^\sigma_X(X - Y)\big)^N.$$

When M satisfies (iv) *and* (v), *we shall say that M is a g-weight.*

Definition 1.5. — *Let g be a metric on $\mathbb{R}^{2n}$ and M be a positive function defined on $\mathbb{R}^{2n}$. The set $S(M, g)$ is defined as the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that, for all $l \in \mathbb{N}$, $\sup_X \|a^{(l)}(X)\|_{g_X} M(X)^{-1} < \infty$, where $a^{(l)}$ is the l-th derivative. It means that $\forall l \in \mathbb{N}, \exists C_l, \forall X \in \mathbb{R}^{2n}, \forall T_1, \dots, T_l \in \mathbb{R}^{2n}$,*

$$|a^{(l)}(X)(T_1, \dots, T_l)| \le C_l M(X) \prod_{1\le j\le l} g_X(T_j)^{1/2}.$$

We discuss now some basic facts about partitions of unity. We refer the reader to the chapter 18 in [19] for the basic properties of admissible metrics as well as for the following lemma.

Lemma 1.6. — *Let g be an admissible metric on $\mathbb{R}^{2n}$. There exist a sequence $(X_k)_{k\in\mathbb{N}}$ of points in the phase space $\mathbb{R}^{2n}$ and positive numbers r_0, N_0, such that the following properties are satisfied. We define U_k, U_k^*, U_k^{**} as the $g_k = g_{X_k}$ balls with center X_k and radius $r_0, 2r_0, 4r_0$. There exist two families of non-negative smooth functions on $\mathbb{R}^{2n}$, $(\chi_k)_{k\in\mathcal{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ such that*

$$\sum_k \chi_k(X) = 1,\ \ supp\,\chi_k \subset U_k,\ \psi_k \equiv 1 \quad on\ U_k^*,\ \ supp\,\psi_k \subset U_k^{**}.$$

Moreover, $\chi_k, \psi_k \in S(1, g_k)$ with semi-norms bounded independently of k. The overlap of the balls U_k^{**} is bounded, i.e.

$$\bigcap_{k\in\mathcal{N}} U_k^{**} \neq \varnothing \implies \#\mathcal{N} \leq N_0.$$

Also we have $g_X \sim g_k$ all over U_k^{**} (i.e. the ratios $g_X(T)/g_k(T)$ are bounded above and below by a fixed constant, provided that $X \in U_k^{**}$).

The next lemma is proved in [4] (see also Lemma 6.3 in [27]).

Lemma 1.7. — *Let g be an admissible metric on $\mathbb{R}^{2n}$ and $\sum_k \chi_k(x,\xi) = 1$ be a partition of unity related to g as in the previous lemma. There exists a positive constant C such that for all $u \in L^2(\mathbb{R}^n)$*

$$C^{-1}\|u\|^2_{L^2(\mathbb{R}^n)} \leq \sum_k \|\chi_k^w u\|^2_{L^2(\mathbb{R}^n)} \leq C\|u\|^2_{L^2(\mathbb{R}^n)},$$

where a^w stands for the Weyl quantization of the symbol a.

The following lemma is proved in [5].

Lemma 1.8. — *Let g be an admissible metric on $\mathbb{R}^{2n}$, m be a weight for g, U_k and g_k as in Lemma 1.6. Let (a_k) be a sequence of bounded symbols in $S\big(m(X_k), g_k\big)$ such that, for all non-negative integers l, N*

$$\sup_{k\in\mathbb{N}, T\in\mathbb{R}^{2n}} |m(X_k)^{-1} a_k^{(l)}(X) T^l \big(1 + g_k^\sigma(X - U_k)\big)^N g_k(T)^{-l/2}| < +\infty.$$

Then the symbol $a = \sum_k a_k$ makes sense and belongs to $S(m, g)$. The important point here is that no support condition is required for the a_k, but instead some decay estimates with respect to g^σ. The sequence (a_k) will be called a confined sequence in $S(m, g)$.

2. THE GEOMETRY OF CONDITION (ψ)

In this section and also in section 3, we shall consider that the phase space is equipped with a *symplectic quadratic form* Γ (Γ is a positive definite quadratic form such that $\Gamma = \Gamma^\sigma$, see the definition 1.4(ii) above). It is possible to find some linear symplectic coordinates (x, ξ) in $\mathbb{R}^{2n}$ such that $\Gamma(x,\xi) = |(x,\xi)|^2 = \sum_{1\le j\le n} x_j^2 + \xi_j^2$. The running point of our Euclidean symplectic $\mathbb{R}^{2n}$ will be usually denoted by X or by an upper-case letter such as Y, Z. The open Γ-ball with center X and radius r will be denoted by $B(X, r)$.

2.1. The basic structure

Let $q(t, X, \Lambda)$ be a smooth real-valued function defined on $\Xi = \mathbb{R} \times \mathbb{R}^{2n} \times [1, +\infty)$, vanishing for $|t| \ge 1$ and satisfying(3)

$$\forall k \in \mathbb{N}, \sup_{\Xi} \|\partial_X^k q\|_\Gamma \Lambda^{-1+\frac{k}{2}} = \gamma_k < +\infty, \text{ i.e. } q(t,\cdot) \in S(\Lambda, \Lambda^{-1}\Gamma), \tag{2.1}$$

$$s > t \quad \text{and } q(t, X, \Lambda) > 0 \Longrightarrow q(s, X, \Lambda) \ge 0. \tag{2.2}$$

Notation. In this section and in the next section, the Euclidean norm $\Gamma(X)^{1/2}$ is fixed and the norms of the vectors and of the multilinear forms are taken with respect to that norm. We shall write everywhere $|\cdot|$ instead of $\|\cdot\|_\Gamma$. Furthermore, we shall say that C is a "fixed" constant if it depends only on a finite number of γ_k above and on the dimension n.

We shall always omit the dependence of q with respect to the large parameter Λ and write $q(t, X)$ instead of $q(t, X, \Lambda)$. The operator $Q(t) = q(t)^w$ will stand for the operator with Weyl symbol $q(t, X)$. We introduce now for $t \in \mathbb{R}$, following [13],

$$\mathbb{X}_+(t) = \cup_{s\le t}\{X \in \mathbb{R}^{2n}, q(s, X) > 0\},\ \mathbb{X}_-(t) = \cup_{s\ge t}\{X \in \mathbb{R}^{2n}, q(s, X) < 0\}, \tag{2.3}$$

$$\mathbb{X}_0(t) = \mathbb{X}_-(t)^c \cap \mathbb{X}_+(t)^c. \tag{2.4}$$

Thanks to (2.2), $\mathbb{X}_+(t), \mathbb{X}_-(t)$ are disjoint open subsets of $\mathbb{R}^{2n}$; moreover $\mathbb{X}_0(t), \mathbb{X}_0(t) \cup \mathbb{X}_\pm(t)$ are closed since their complements are open. The three sets $\mathbb{X}_0(t), \mathbb{X}_\pm(t)$ are two by two disjoint with union $\mathbb{R}^{2n}$ (note also that $\overline{\mathbb{X}_\pm(t)} \subset \mathbb{X}_0(t) \cup \mathbb{X}_\pm(t)$ since $\mathbb{X}_0(t) \cup \mathbb{X}_\pm(t)$ are closed). When t increases, $\mathbb{X}_+(t)$ increases and $\mathbb{X}_-(t)$ decreases. The following three lemmas are easy and can be found as Lemmas 2.1.1-2-3 in [22].

LEMMA 2.1. — *Let (E, d) be a metric space, let $A \subset E$ and $\kappa > 0$ be given. We define $\Psi_{A,\kappa}(x) = \kappa$ if $A = \varnothing$ and if $A \ne \varnothing$, we define*

$$\Psi_{A,\kappa}(x) = \min\bigl(d(x, A), \kappa\bigr).$$

(3)The attentive reader should not be scared by the fact that (2.2) is different from (0.8): we shall deal from now on with the adjoint of an operator satisfying condition (ψ) and we are willing to prove an *a priori* estimate as in Theorem 4.1 below.

The function $\Psi_{A,\kappa}$ is valued in $[0,\kappa]$, Lipschitz continuous with a Lipschitz constant ≤ 1. Moreover, the following implication holds: $A_1 \subset A_2 \subset E \Longrightarrow \Psi_{A_1,\kappa} \ge \Psi_{A_2,\kappa}$.

LEMMA 2.2. — *For each $X \in \mathbb{R}^{2n}$, the function $t \mapsto \Psi_{\mathbb{X}_+(t),\kappa}(X)$ is decreasing and for each $t \in \mathbb{R}$, the function $X \mapsto \Psi_{\mathbb{X}_+(t),\kappa}(X)$ is supported in $\mathbb{X}_+(t)^c = \mathbb{X}_-(t) \cup \mathbb{X}_0(t)$. For each $X \in \mathbb{R}^{2n}$, the function $t \mapsto \Psi_{\mathbb{X}_-(t),\kappa}(X)$ is increasing and for each $t \in \mathbb{R}$, the function $X \mapsto \Psi_{\mathbb{X}_-(t),\kappa}(X)$ is supported in $\mathbb{X}_-(t)^c = \mathbb{X}_+(t) \cup \mathbb{X}_0(t)$. As a consequence the function $X \mapsto \Psi_{\mathbb{X}_+(t),\kappa}(X)\Psi_{\mathbb{X}_-(t),\kappa}(X)$ is supported in $\mathbb{X}_0(t)$.*

LEMMA 2.3. — *For $\kappa > 0, t \in \mathbb{R}, X \in \mathbb{R}^{2n}$, we define*(4)

$$\sigma(t,X,\kappa) = \Psi_{\mathbb{X}_-(t),\kappa}(X) - \Psi_{\mathbb{X}_+(t),\kappa}(X). \tag{2.5}$$

The function $t \mapsto \sigma(t,X,\kappa)$ is increasing and valued in $[-\kappa,\kappa]$, the function $X \mapsto \sigma(t,X,\kappa)$ is Lipschitz continuous with Lipschitz constant less than 2; we have

$$\sigma(t,X,\kappa) = \begin{cases} \min(|X-\mathbb{X}_-(t)|,\kappa) & \text{if } X \in \mathbb{X}_+(t), \\ -\min(|X-\mathbb{X}_+(t)|,\kappa) & \text{if } X \in \mathbb{X}_-(t). \end{cases}$$

We have $\{X \in \mathbb{R}^{2n}, \sigma(t,X,\kappa) = 0\} \subset \mathbb{X}_0(t) \subset \{X \in \mathbb{R}^{2n}, q(t,X) = 0\}$, and

$$\{X \in \mathbb{R}^{2n}, \pm q(t,X) > 0\} \subset \mathbb{X}_\pm(t) \subset \{X \in \mathbb{R}^{2n}, \pm\sigma(t,X,\kappa) > 0\} \subset \{X \in \mathbb{R}^{2n}, \pm\sigma(t,X,\kappa) \ge 0\} \subset \{X \in \mathbb{R}^{2n}, \pm q(t,X) \ge 0\}. \tag{2.6}$$

DEFINITION 2.4. — *Let $q(t,X)$ be as above. We define*

$$\delta_0(t,X) = \sigma(t,X,\Lambda^{1/2}) \tag{2.7}$$

and we notice that from the previous lemmas, $t \mapsto \delta_0(t,X)$ is increasing, valued in $[-\Lambda^{1/2},\Lambda^{1/2}]$, satisfying

$$|\delta_0(t,X) - \delta_0(t,Y)| \le 2|X-Y|, \tag{2.8}$$

and such that

$$\{X \in \mathbb{R}^{2n}, \delta_0(t,X) = 0\} \subset \{X \in \mathbb{R}^{2n}, q(t,X) = 0\}, \tag{2.9}$$

$$\{X \in \mathbb{R}^{2n}, \pm q(t,X) > 0\} \subset \{X, \pm\delta_0(t,X) > 0\} \subset \{X, \pm q(t,X) \ge 0\}. \tag{2.10}$$

The following lemma is elementary and is a good introduction to the Calderón-Zygmund methods. This is lemma 2.1.5 in [22].

LEMMA 2.5. — *Let f be a symbol in $S(\Lambda^m, \Lambda^{-1}\Gamma)$ where m is a positive real number. We define*

$$\lambda(X) = 1 + \max_{\substack{0 \le j < 2m \\ j \in \mathbb{N}}} \big(\|f^{(j)}(X)\|_\Gamma^{\frac{2}{2m-j}}\big). \tag{2.11}$$

(4)If the distances of X to both $\mathbb{X}_\pm(t)$ are less than κ, we have $\sigma(t,X,\kappa) = |X-\mathbb{X}_-(t)| - |X-\mathbb{X}_+(t)|$.

Then $f \in S(\lambda^m, \lambda^{-1}\Gamma)$ and the mapping from $S(\Lambda^m, \Lambda^{-1}\Gamma)$ to $S(\lambda^m, \lambda^{-1}\Gamma)$ is continuous. Moreover, with $\gamma = \max_{\substack{0\le j<2m\\ j\in\mathbb{N}}} \gamma_j^{\frac{2}{2m-j}}$, where the γ_j are the semi-norms of f, we have for all $X \in \mathbb{R}^{2n}$,

$$1 \le \lambda(X) \le 1 + \gamma\Lambda. \tag{2.12}$$

The metric $\lambda^{-1}\Gamma$ is admissible (Def. 1.4), with structure constants depending only on γ. It will be called the m-proper metric of f. The function λ above is a weight for the metric $\lambda^{-1}\Gamma$ and will be called the m-proper weight of f.

The following two lemmas are more involved and appear as lemmas 2.1.6-7 in [22].

LEMMA 2.6. — *Let $q(t, X)$ and $\delta_0(t, X)$ be as above. We define, with $\langle s\rangle = (1 + s^2)^{1/2}$,*

$$\mu(t, X) = \langle\delta_0(t, X)\rangle^2 + |\Lambda^{1/2} q'_X(t, X)| + |\Lambda^{1/2} q''_{XX}(t, X)|^2. \tag{2.13}$$

The metric $\mu^{-1}(t, \cdot)\Gamma$ is slowly varying with structure constants depending only on a finite number of semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$. Moreover, there exists $C > 0$, depending only on a finite number of semi-norms of q, such that

$$\mu(t, X) \le C\Lambda, \quad \frac{\mu(t, X)}{\mu(t, Y)} \le C(1 + |X - Y|^2), \tag{2.14}$$

and we have

$$\Lambda^{1/2} q(t, X) \in S(\mu(t, X)^{3/2}, \mu^{-1}(t, \cdot)\Gamma), \tag{2.15}$$

so that the semi-norms depend only the semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$.

LEMMA 2.7. — *Let $q(t, X), \delta_0(t, X), \mu(t, X)$ be as above. We define,*

$$\nu(t, X) = \langle\delta_0(t, X)\rangle^2 + |\Lambda^{1/2} q'_X(t, X)\mu(t, X)^{-1/2}|^2. \tag{2.16}$$

The metric $\nu^{-1}(t, \cdot)\Gamma$ is slowly varying with structure constants depending only on a finite number of semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$. There exists $C > 0$, depending only on a finite number of semi-norms of q, such that

$$\nu(t, X) \le 2\mu(t, X) \le C\Lambda, \quad \frac{\nu(t, X)}{\nu(t, Y)} \le C(1 + |X - Y|^2), \tag{2.17}$$

and we have

$$\Lambda^{1/2} q(t, X) \in S(\mu(t, X)^{1/2}\nu(t, X), \nu(t, \cdot)^{-1}\Gamma), \tag{2.18}$$

so that the semi-norms of this symbol depend only on the semi-norms of q in $S(\Lambda, \Lambda^{-1}\Gamma)$. Moreover the function $\mu(t, X)$ is a weight for the metric $\nu(t, \cdot)^{-1}\Gamma$.

We wish now to discuss the normal forms attached to the metric $\nu^{-1}(t, \cdot)\Gamma$ for the symbol $q(t, \cdot)$. In the sequel of this section, we consider that t is fixed.

Definition 2.8. — *Let $0 < r_1 \le 1/2$ be given. With ν defined in* (2.16)*, we shall say that*
(i) Y *is a nonnegative* (*resp. nonpositive*) *point at level* t *if*

$$\delta_0(t,Y) \ge r_1\nu(t,Y)^{1/2}, \quad (\textit{resp.} \quad \delta_0(t,Y) \le -r_1\nu(t,Y)^{1/2}).$$

(ii) Y *is a gradient point at level* t *if*

$$|\Lambda^{1/2}q'_Y(t,Y)\mu(t,Y)^{-1/2}|^2 \ge \nu(t,Y)/4 \quad \textit{and} \quad \delta_0(t,Y)^2 < r_1^2\nu(t,Y).$$

(iii) Y *is a negligible point in the remaining cases*

$$|\Lambda^{1/2}q'_Y(t,Y)\mu(t,Y)^{-1/2}|^2 < \nu(t,Y)/4 \quad \textit{and} \quad \delta_0(t,Y)^2 < r_1^2\nu(t,Y).$$

Note that this implies $\nu(t,Y) \le 1 + r_1^2\nu(t,Y) + \nu(t,Y)/4 \le 1 + \nu(t,Y)/2$ *and thus* $\nu(t,Y) \le 2$.

Note that if Y is a nonnegative point, from (2.8) we get, for $T \in \mathbb{R}^{2n}$, $|T| \le 1$, $0 \le r \le r_1/4$

$$\delta_0\big(t, Y + r\nu^{1/2}(t,Y)T\big) \ge \delta_0(t,Y) - 2r\nu^{1/2}(t,Y) \ge \frac{r_1}{2}\nu^{1/2}(t,Y)$$

and from (2.10), this implies that $q(t,X) \ge 0$ on the ball $B(Y, r\nu^{1/2}(t,Y))$. Similarly if Y is a nonpositive point, $q(t,X) \le 0$ on the ball $B(Y, r\nu^{1/2}(t,Y))$. Moreover if Y is a gradient point, we have $|\delta_0(t,Y)| < r_1\nu(t,Y)^{1/2}$ so that, if $Y \in \mathbb{X}_+(t)$, we have $\min(|Y - \mathbb{X}_-(t)|, \Lambda^{1/2}) < r_1\nu(t,Y)^{1/2}$ and if r_1 is small enough, since $\nu \lesssim \Lambda$, we get that $|Y - \mathbb{X}_-(t)| < r_1\nu(t,Y)^{1/2}$ which implies that there exists $Z_1 \in \mathbb{X}_-(t)$ such that $|Y - Z_1| < r_1\nu(t,Y)^{1/2}$. On the segment $[Y, Z_1]$, the Lipschitz continuous function is such that $\delta_0(t,Y) > 0$ ($Y \in \mathbb{X}_+(t)$, cf. Lemma 2.3) and $\delta_0(t,Z_1) < 0$ ($Z_1 \in \mathbb{X}_-(t)$); as a result, there exists a point Z (on that segment) such that $\delta_0(t,Z) = 0$ and thus $q(t,Z) = 0$. Naturally the discussion for a gradient point Y in $\mathbb{X}_-(t)$, is analogous. If the gradient point Y belongs to $\mathbb{X}_0(t)$, we get right away $q(t,Y) = 0$, also from the lemma 2.3. The function

$$f(T) = \Lambda^{1/2}q\Big(t, Y + r_1\nu^{1/2}(t,Y)T\Big)\mu(t,Y)^{-1/2}\nu(t,Y)^{-1} \tag{2.19}$$

satisfies for r_1 small enough with respect to the semi-norms of q and c_0, C_0, C_1, C_2 fixed positive constants, $|T| \le 1$, from (2.18),

$$|f(T)| \le |S - T|C_0r_1 \le C_1r_1^2, \quad |f'(T)| \ge r_1c_0, \quad |f''(T)| \le C_2r_1^2.$$

The standard analysis (see the appendix A.7 in [22]) of the Beals-Fefferman metric [2] shows that, on $B(Y, r_1\nu^{1/2}(t,Y))$

$$q(t,X) = \Lambda^{-1/2}\mu^{1/2}(t,Y)\nu^{1/2}(t,Y)e(t,X)\beta(t,X), \tag{2.20}$$

$$1 \le e \in S(1, \nu(t,Y)^{-1}\Gamma),\ \beta \in S(\nu(t,Y)^{1/2}, \nu(t,Y)^{-1}\Gamma), \tag{2.21}$$

$$\beta(t,X) = \nu(t,Y)^{1/2}(X_1 + \alpha(t,X')), \alpha \in S(\nu(t,Y)^{1/2}, \nu(t,Y)^{-1}\Gamma). \tag{2.22}$$

Lemma 2.9. — *Let $q(t,X)$ be a smooth function satisfying* (2.1-2) *and let $t \in [-1,1]$ be given. The metric g_t on $\mathbb{R}^{2n}$ is defined as $\nu(t,X)^{-1}\Gamma$ where ν is defined in* (2.16)*. There exists $r_0 > 0$, depending only on a finite number of semi-norms of q in* (2.1) *such that, for any $r \in]0, r_0]$, there exist a sequence of points (X_k) in $\mathbb{R}^{2n}$, and sequences of functions $(\chi_k), (\psi_k)$ satisfying the properties in the lemma 1.4 such that there exists a partition of $\mathbb{N}$,*

$$\mathbb{N} = E_+ \cup E_- \cup E_0 \cup E_{00}$$

so that, according to the definition 2.8, $k \in E_+$ means that X_k is a nonnegative point, ($k \in E_-$:X_k nonpositive point; $k \in E_0$:X_k gradient point, $k \in E_{00}$:X_k negligible point).

Proof. — This lemma is an immediate consequence of the definition 2.8, of lemma 1.4 and of lemma 2.7, asserting that the metric g_t is admissible.

2.2. Some lemmas on C^3 functions

We give in this section a key result on the second derivative f''_{XX} of a real-valued smooth function $f(t,X)$ such that $\tau - if(t,x,\xi)$ satisfies condition (ψ). The following claim gives a good qualitative version of what is needed for our estimates. Although we shall not use that (very simple) result, proving the following claim may serve as a good warm-up exercise for the more difficult sequel.

Claim. — *Let f_1, f_2 be two real-valued twice differentiable functions defined on an open set Ω of $\mathbb{R}^N$ and such that $f_1^{-1}(\mathbb{R}_+^*) \subset f_2^{-1}(\mathbb{R}_+)$ (i.e. $f_1(x) > 0 \Longrightarrow f_2(x) \geq 0$). If for some $\omega \in \Omega$, the conditions $f_1(\omega) = f_2(\omega) = 0, \quad df_1(\omega) \neq 0, df_2(\omega) = 0$ are satisfied, we have $f_2''(\omega) \geq 0$ (as a quadratic form).*

This claim has the following consequence: take three functions f_1, f_2, f_3, twice differentiable on Ω, such that, for $1 \leq j \leq k \leq 3$, $f_j(x) > 0 \Longrightarrow f_k(x) \geq 0$. Assume that, at some point ω we have $f_1(\omega) = f_2(\omega) = f_3(\omega) = 0, \quad df_1(\omega) \neq 0, df_3(\omega) \neq 0$, $df_2(\omega) = 0$. Then one has $f_2''(\omega) = 0$: indeed, the previous claim gives $f_2''(\omega) \geq 0$ and it can be applied to the couple $(-f_3, -f_2)$ to get $-f_2''(\omega) \geq 0$.

Notations. The open Euclidean ball of $\mathbb{R}^N$ with center 0 and radius r will be denoted by B_r. For a k-multilinear symmetric form A on $\mathbb{R}^N$, we shall note $\|A\| = \max_{|T|=1} |AT^k|$ which is easily seen to be equivalent to the norm $\max_{|T_1|=\dots=|T_k|=1} |A(T_1,\dots,T_k)|$ since the symmetrized $T_1 \otimes \dots \otimes T_k$ can be written a sum of k^{th} powers.

The next statement is a precise quantitative version of the previous claim and is lemma 2.2.2 in [22].

Lemma 2.10. — *Let $R_0 > 0$ and f_1, f_2 be real-valued functions defined in $\bar{B}_{R_0}$. We assume that f_1 is C^2, f_2 is C^3 and for $x \in \bar{B}_{R_0}$,*

$$f_1(x) > 0 \Longrightarrow f_2(x) \geq 0. \tag{2.23}$$

We define the non-negative numbers ρ_1, ρ_2, by

$$\rho_1 = \max\big(|f_1(0)|^{\frac{1}{2}}, |f_1'(0)|\big), \quad \rho_2 = \max\big(|f_2(0)|^{\frac{1}{3}}, |f_2'(0)|^{\frac{1}{2}}, |f_2''(0)|\big), \tag{2.24}$$

and we assume that, with a positive C_0,

$$0 < \rho_1, \quad \rho_2 \le C_0\rho_1 \le R_0. \tag{2.25}$$

We define the non-negative numbers C_1, C_2, C_3, by

$$C_1 = 1 + C_0\|f_1''\|_{L^\infty(\bar{B}_{R_0})}, \quad C_2 = 4 + \frac{1}{3}\|f_2'''\|_{L^\infty(\bar{B}_{R_0})}, \quad C_3 = C_2 + 4\pi C_1. \tag{2.26}$$

Assume that for some $\kappa_2 \in [0,1]$, with $\kappa_2 C_1 \le 1/4$,

$$\rho_1 = |f_1'(0)| > 0, \tag{2.27}$$

$$\max\big(|f_2(0)|^{1/3}, |f_2'(0)|^{1/2}\big) \le \kappa_2|f_2''(0)|, \tag{2.28}$$

$$B(0, \kappa_2^2\rho_2) \cap \{x \in \bar{B}_{R_0}, f_1(x) \ge 0\} \ne \varnothing. \tag{2.29}$$

Then we have

$$|f_2''(0)_-| \le C_3\kappa_2\rho_2, \tag{2.30}$$

where $f_2''(0)_-$ stands for the negative part of the quadratic form $f_2''(0)$. Note that, whenever (2.29) *is violated, we get $B(0, \kappa_2^2\rho_2) \subset \{x \in \bar{B}_{R_0}, f_1(x) < 0\}$ (note that $\kappa_2^2\rho_2 \le \rho_2 \le R_0$) and thus*

$$\mathit{distance}\big(0, \{x \in \bar{B}_{R_0}, f_1(x) \ge 0\}\big) \ge \kappa_2^2\rho_2. \tag{2.31}$$

2.3. Inequalities for symbols

The next statement (theorem 2.3.1 in [22]) is a (not-so-easy) consequence of the previous lemmas. A slightly weaker version of this theorem appeared for the first time in Dencker's preprint [7] and is certainly one of the main novelties brought forward by this author.

Theorem 2.11. — *Let q be a symbol satisfying* (2.1-2) *and δ_0, μ, ν as defined above in* (2.7), (2.13) *and* (2.16). *For the real numbers t', t, t'', and $X \in \mathbb{R}^{2n}$, we define*

$$N(t', t'', X) = \frac{\langle\delta_0(t', X)\rangle}{\nu(t', X)^{1/2}} + \frac{\langle\delta_0(t'', X)\rangle}{\nu(t'', X)^{1/2}}, \tag{2.32}$$

$$R(t, X) = \Lambda^{-1/2}\mu(t, X)^{1/2}\nu(t, X)^{-1/2}\langle\delta_0(t, X)\rangle. \tag{2.33}$$

Then there exists a constant $C_0 \ge 1$, depending only on a finite number of semi-norms of q in (2.1), *such that, for $t' \le t \le t''$, we have*

$$C_0^{-1}R(t, X) \le N(t', t'', X) + \frac{\delta_0(t'', X) - \delta_0(t, X)}{\nu(t'', X)^{1/2}} + \frac{\delta_0(t, X) - \delta_0(t', X)}{\nu(t', X)^{1/2}}. \tag{2.34}$$

2.4. Quasi-convexity

A differentiable function ψ of one variable is said to be quasi-convex on $\mathbb{R}$ if $\dot\psi(t)$ does not change sign from $+$ to $-$ for increasing t (see [20]). In particular, a differentiable convex function is such that $\dot\psi(t)$ is increasing and is thus quasi-convex.

Definition 2.12. — *Let $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be an increasing function, $C_1 > 0$ and let $\rho_1 : \mathbb{R} \to \mathbb{R}_+$. We shall say that ρ_1 is quasi-convex with respect to (C_1, σ_1) if for $t_1, t_2, t_3 \in \mathbb{R}$,*

$$t_1 \le t_2 \le t_3 \Longrightarrow \rho_1(t_2) \le C_1 \max\bigl(\rho_1(t_1), \rho_1(t_3)\bigr) + \sigma_1(t_3) - \sigma_1(t_1).$$

When σ_1 is a constant function and $C_1 = 1$, this is the definition of quasi-convexity. Let $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\omega : \mathbb{R} \to \mathbb{R}_+$. We define

$$\rho_1(t) = \inf_{t' \le t \le t''} \Bigl(\omega(t') + \omega(t'') + \sigma_1(t'') - \sigma_1(t')\Bigr). \tag{2.35}$$

Then the function ρ_1 is quasi-convex with respect to $(2, \sigma_1)$.

The following lemma (lemma 2.4.3 in [22]) is due to L. Hörmander [13].

Lemma 2.13. — *Let $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\omega : \mathbb{R} \to \mathbb{R}_+$. Let $T > 0$ be given. We consider the function ρ_1 as given in Definition 2.12 and we define*

$$\Theta_T(t) = \sup_{-T \le s \le t} \left\{ \sigma_1(s) - \sigma_1(t) + \frac{1}{2T} \int_s^t \rho_1(r) dr - \rho_1(s) \right\}. \tag{2.36}$$

Then we have

$$2T \partial_t (\Theta_T + \sigma_1) \ge \rho_1, \quad \text{and for } |t| \le T, \quad |\Theta_T(t)| \le \rho_1(t). \tag{2.37}$$

Definition 2.14. — *For $T > 0, X \in \mathbb{R}^{2n}, |t| \le T$, we define*

$$\omega(t,X) = \frac{\langle \delta_0(t,X) \rangle}{\nu(t,X)^{1/2}}, \ \sigma_1(t,X) = \delta_0(t,X), \ \eta(t,X) = \int_{-T}^t \delta_0(s,X) \Lambda^{-1/2} ds + 2T, \tag{2.38}$$

where δ_0, ν are defined in (2.7), (2.16). For $T > 0$, $(t,X) \in \mathbb{R} \times \mathbb{R}^{2n}$, we define $\Theta(t,X)$ by the formula (2.36),

$$\Theta(t,X) = \sup_{-T \le s \le t} \left\{ \sigma_1(s,X) - \sigma_1(t,X) + \frac{1}{2T} \int_s^t \rho_1(r,X) dr - \rho_1(s,X) \right\}, \tag{2.39}$$

where ρ_1 is defined by (2.35). We define also

$$m(t,X) = \delta_0(t,X) + \Theta(t,X) + T^{-1} \delta_0(t,X) \eta(t,X). \tag{2.40}$$

The next statement is theorem 2.4.5 in [22]. The reader may be interested in checking that it is indeed the term η, defined above in (2.38), which allows us to cut the loss of derivatives from 2 to 3/2.

Theorem 2.15. — *With the notations above for Θ, ρ_1, m, with R and C_0 defined in Theorem 2.11, we have for $T > 0$, $|t| \leq T$, $X \in \mathbb{R}^{2n}, \Lambda \geq 1$,*

$$|\Theta(t,X)| \leq \rho_1(t,X) \leq 2\frac{\langle\delta_0(t,X)\rangle}{\nu(t,X)^{1/2}}, \quad |\sigma_1(t,X)| = |\delta_0(t,X)|, \tag{2.41}$$

$$C_0^{-1}R(t,X) \leq \rho_1(t,X) \leq 2T\frac{\partial}{\partial t}\Big(\Theta(t,X) + \sigma_1(t,X)\Big), \tag{2.42}$$

$$0 \leq \eta(t,X) \leq 4T, \quad \frac{d}{dt}\big(\delta_0\eta\big) \geq \delta_0^2\Lambda^{-1/2}, \quad |\eta'_X(t,X)| \leq 4T\Lambda^{-1/2}, \tag{2.43}$$

$$T\frac{d}{dt}m \geq \frac{1}{2}\rho_1 + \delta_0^2\Lambda^{-1/2} \geq \frac{1}{2C_0}R + \delta_0^2\Lambda^{-1/2} \geq \frac{1}{2^{3/2}C_0}\langle\delta_0\rangle^2\Lambda^{-1/2}. \tag{2.44}$$

3. ENERGY ESTIMATES

3.1. Preliminaries

Definition 3.1. — *Let $T > 0$ be given. With m defined in* (2.40)*, we define for $|t| \leq T$,*

$$M(t) = m(t,X)^{\mathrm{Wick}}, \tag{3.1}$$

where the Wick quantization is given by the definition 5.1 in the appendix.

3.2. Stationary estimates for the model cases

Let $T > 0$ be given and $Q(t) = q(t)^w$ given by (2.1-2). We define $M(t)$ according to (3.1). We consider

$$\Re\big(Q(t)M(t)\big) = \frac{1}{2}Q(t)M(t) + \frac{1}{2}M(t)Q(t) = P(t). \tag{3.2}$$

We have, omitting now the variable t fixed here,

$$P = \Re\Big[q^w\big(\delta_0(1 + T^{-1}\eta)\big)^{\mathrm{Wick}} + q^w\Theta^{\mathrm{Wick}}\Big]. \tag{3.3}$$

Following the section 3.2 in [22], we discuss now the various model cases that could occur for the symbol $q(t,X)$ when t is fixed.

3.2.1. The gradient points. — Let us assume first that $q = \Lambda^{-1/2}\mu^{1/2}\nu^{1/2}\beta e_0$ with $\beta \in S(\nu^{1/2}, \nu^{-1}\Gamma), 1 \leq e_0 \in S(1, \nu^{-1}\Gamma)$ and $\delta_0 = \beta$. Moreover, we assume $0 \leq T^{-1}\eta \leq 4$, $T^{-1}|\eta'| \leq 4\Lambda^{-1/2}$, $\quad |\Theta| \leq C\langle\delta_0\rangle\nu^{-1/2}$. Here Λ, μ, ν are assumed to be positive constants such that $\Lambda \geq \mu \geq \nu \geq 1$. After a rather simple but delicate discussion involving various properties of the Wick quantization, we get

$$\Re(QM) + S(\Lambda^{-1/2}\mu^{1/2}\nu^{-1/2}, \Gamma)^w \geq 0. \tag{3.4}$$

3.2.2. The nonnegative points. — Let us assume now that $q \geq 0$, $q \in S(\Lambda^{-1/2}\mu^{1/2}\nu, \nu^{-1}\Gamma)$, $\gamma_0 \nu^{1/2} \leq \delta_0 \leq \gamma_0^{-1}\nu^{1/2}$ with a positive fixed constant γ_0. Moreover, we assume $0 \leq T^{-1}\eta \leq 4, T^{-1}|\eta'| \leq 4\Lambda^{-1/2}$, $|\Theta(X)| \leq C$, Θ real-valued. Here Λ, μ, ν are assumed to be positive constants such that $\Lambda \geq \mu \geq \nu \geq 1$. We start over our discussion from the identity (3.3):

$$P = \Re\Big[q^w \Big(\delta_0(1 + T^{-1}\eta) + \Theta\Big)^{\text{Wick}}\Big]. \tag{3.5}$$

Some arguments of symbolic calculus and the Fefferman-Phong inequality ([12]) yield

$$\Re(QM) + S(\Lambda^{-1/2}\mu^{1/2}, \Gamma)^w \geq 0. \tag{3.6}$$

The discussion is analogous for the nonpositive points and the negligible points.

3.3. Stationary estimates

Following the section 3.3 in [22], we get the following result as a consequence of section 3.2.

LEMMA 3.2. — *Let p be the Weyl symbol of P defined in* (3.3) *and* $\widetilde{\Theta} = \Theta * 2^n \exp -2\pi\Gamma$*, where Θ is defined in* (2.39) (*and satisfies* (2.41)). *Then we have*

$$p(t, X) \equiv p_0(t, X) = q(t, X)\Big(\delta_0(1 + T^{-1}\eta) * 2^n \exp -2\pi\Gamma\Big) + q(t, X)\widetilde{\Theta}(t, X), \tag{3.7}$$

modulo $S(\Lambda^{-1/2}\mu^{1/2}\nu^{-1/2}\langle\delta_0\rangle, \Gamma)$.

Now, we shall use a partition of unity $1 = \sum_k \chi_k^2$ related to the metric $\nu(t, X)^{-1}\Gamma$ and a sequence (ψ_k) as in the lemma 1.6. We have, omitting the variable t, with p_0 defined in the previous lemma,

$$p_0(X) = \sum_k \chi_k(X)^2 q(X) \int \delta_0(Y)\big(1 + T^{-1}\eta(Y)\big)2^n \exp -2\pi\Gamma(X - Y)dY$$
$$+ \sum_k \chi_k(X)^2 q(X) \int \Theta(Y) 2^n \exp -2\pi\Gamma(X - Y)dY.$$

We obtain, assuming $\delta_0 = \delta_{0k}, \Theta = \Theta_k, q = q_k$ on U_k,

$$p_0 = \sum_k \chi_k^2 q_k\big(\delta_{0k}(1 + T^{-1}\eta) * 2^n \exp -2\pi\Gamma\big) + \sum_k \chi_k^2 q_k\big(\Theta_k * 2^n \exp -2\pi\Gamma\big) + S(\Lambda^{-1/2}\mu^{1/2}\nu^{-\infty}, \Gamma). \tag{3.8}$$

It is then rather straightforward to get the following lemma (cf. lemma 3.3.3 in [22]).

LEMMA 3.3. — *With* $\widetilde{\Theta}_k = \Theta_k * 2^n \exp -2\pi\Gamma$, $d_k = \delta_{0k}(1 + T^{-1}\eta) * 2^n \exp -2\pi\Gamma$ *and q_k, χ_k defined above, we have*

$$\sum_k \chi_k \sharp q_k d_k \sharp \chi_k + \sum_k \chi_k \sharp q_k \widetilde{\Theta}_k \sharp \chi_k = p_0 + S(\Lambda^{-1/2}\mu^{1/2}\nu^{-1/2}\langle\delta_0\rangle, \Gamma). \tag{3.9}$$

From this, we can obtain the following result (cf. proposition 3.3.4 in [22]).

Proposition 3.4. — *Let* $T>0$ *be given and* $Q(t)=q(t)^w$ *given by* (2.1-2). *We define* $M(t)$ *according to* (3.1). *Then, with a partition of unity* $1=\sum_k \chi_k^2$ *related to the metric* $\nu(t,X)^{-1}\Gamma$ *we have*

$$\Re(Q(t)M(t)) = \sum_k \chi_k^w \Re\big(q_k^w d_k^w + q_k^w \widetilde{\Theta}_k^w\big)\chi_k^w + S(\Lambda^{-1/2}\mu^{1/2}\langle\delta_0\rangle\nu^{-1/2},\Gamma)^w$$

$$\text{and}\quad \Re(Q(t)M(t)) + S(\Lambda^{-1/2}\mu^{1/2}\langle\delta_0\rangle\nu^{-1/2},\Gamma)^w \geq 0.$$

3.4. The multiplier method

Theorem 3.5. — *Let* $T>0$ *be given and* $Q(t)=q(t)^w$ *given by* (2.1-2). *We define* $M(t)$ *according to* (3.1). *There exist* $T_0>0$ *and* $c_0>0$ *depending only on a finite number of* γ_k *in* (2.1) *such that, for* $0<T\leq T_0$, *with* $D(t,X)=\langle\delta_0(t,X)\rangle$ (*D is Lipschitz continuous with Lipschitz constant* 2, *as* δ_0 *in* (2.8) *and thus a* Γ*-weight*),

$$\frac{d}{dt}M(t) + 2\Re\big(Q(t)M(t)\big) \geq T^{-1}\big(D^2\big)^{\mathrm{Wick}}\Lambda^{-1/2}c_0. \tag{3.10}$$

Moreover we have with m *defined in* (2.40), $\widetilde{m}(t,\cdot)=m(t,\cdot)*2^n\exp-2\pi\Gamma$,

$$M(t) = m(t,X)^{\mathrm{Wick}} = \widetilde{m}(t,X)^w,\ \text{with}\ \widetilde{m}\in S_1(D,D^{-2}\Gamma)+S(1,\Gamma), \tag{3.11}$$

where the set of symbols $S_1(D,D^{-2}\Gamma)$ *is defined in section 5.2 of our appendix. We have also*

$$m(t,X)=a(t,X)+b(t,X),\ |a/D|+|a'_X|+|b|\ \text{bounded},\ \dot m\geq 0, \tag{3.12}$$

$$a=\delta_0(1+T^{-1}\eta),\quad b=\widetilde{\Theta}.$$

This theorem is a direct consequence of the previous lemmas and propositions and is Theorem 3.4.1 in [22]. We shall not give its complete proof here, but we wish to make a few points about the loss of derivatives in a semi-classical framework.

Remark 3.6. — Let us check that this theorem gives an estimate with loss of 3/2 derivatives for

$$L = D_t + iQ(t). \tag{3.13}$$

We compute for $u\in C_c^1(\mathbb{R},L^2(\mathbb{R}^n)), \operatorname{supp} u\subset[-T_0,T_0]$, the quantity $\langle Lu, iMu\rangle$ and we use (3.10):

$$2\Re\langle Lu,iMu\rangle = \langle\dot{M}u,u\rangle + 2\Re\langle Qu,Mu\rangle \geq c_0T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\mathrm{Wick}}u,u\rangle.$$

We get, for all positive α,

$$c_0T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\mathrm{Wick}}u,u\rangle \leq \alpha^{-1}\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + \alpha\|Mu\|^2_{L^2(\mathbb{R}^{n+1})}$$

and from the lemma A.1.4 in [22], with a positive fixed constant C_1, we obtain

$$c_0T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\mathrm{Wick}}u,u\rangle \leq \alpha^{-1}\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + \alpha C_1\langle\big(1+\delta_0^2(t,\cdot)\big)^{\mathrm{Wick}}u,u\rangle.$$

Choosing now $\alpha = \frac{c_0}{2C_1 T\Lambda^{1/2}}$, we obtain

$$\frac{1}{2}c_0 T^{-1}\Lambda^{-1/2}\langle\big(1+\delta_0^2(t,\cdot)\big)^{\mathrm{Wick}}u,u\rangle \le \frac{2C_1T}{c_0}\Lambda^{1/2}\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} \tag{3.14}$$

and thus with a fixed positive constant c_1, $\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} \ge c_1^2 T^{-2}\Lambda^{-1}\|u\|^2_{L^2(\mathbb{R}^{n+1})}$, yielding

$$\|Lu\|_{L^2(\mathbb{R}^{n+1})} \ge c_1 T^{-1}\Lambda^{-1/2}\|u\|_{L^2(\mathbb{R}^{n+1})}, \tag{3.15}$$

which is indeed an estimate with loss of 3/2 derivatives with respect to the elliptic estimate $\|Lu\| \gtrsim \Lambda\|u\|$. We can notice also, that in the region where $\langle\delta_0\rangle \sim \Lambda^{1/2}$, the estimate (3.14) looses just one derivative and is an $L^2 - L^2$ estimate.

4. FROM SEMI-CLASSICAL TO LOCAL ESTIMATES

The theorem 3.5 and the discussion in the remark 3.6 indicate that we are in a good position to prove an estimate with loss of 3/2 derivatives for the transposed of an operator satisfying condition (ψ), as defined in definition 1.1. However since the loss of derivatives is strictly larger than 1, we cannot patch together these weak estimates, simply because the commutators with a partition of unity would be of order 0, giving L^2-norms as remainders whereas we dominate only a $H^{-1/2}$ norm. Another seemingly paradoxical feature of this problem is that the result should be invariant by "nice" perturbations of order 0, and since these perturbations have a size larger than what is controlled by the inequality, this triggers another difficulty. Of course the word "nice" in the previous sentence is important and means that the perturbations should be limited to have symbols in a very standard class of pseudodifferential operators. That situation of having to cope with a large loss of derivatives is quite common for multiple characteristics operators, and we shall use the inequalities (3.10) which can actually be patched together; they are in principle quite close to the construction of a parametrix. The details of the remaining arguments can be found in section 4 of [22]. Although the semi-classical result of remark 3.6 appears as the main step for the proof, it turns out that some significant difficulties remain to deal with the weak estimate and to sort out pseudodifferential operators in n dimensions depending on a real parameter from homogeneous pseudodifferential operators in $n+1$ dimensions. The following result is Theorem 4.1.9 in [22].

THEOREM 4.1. — *Let $f(t,x,\xi)$ be a smooth real-valued function defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, satisfiying* (2.2) *and*

$$\sup_{\substack{t\in\mathbb{R}\\(x,\xi)\in\mathbb{R}^{2n}}} |(\partial_x^\alpha\partial_\xi^\beta f)(t,x,\xi)|(1+|\xi|)^{-1+|\beta|} = C_{\alpha\beta} < \infty. \tag{4.1}$$

Let $f_0(t,x,\xi)$ be a smooth complex-valued function defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, such that $\langle\xi\rangle f_0(t,x,\xi)$ satisfies (4.1). Then there exists $T_0>0, c_0>0$ depending on a finite number of seminorms of f, f_0, such that, for all $T\le T_0$ and all $u\in C_c^\infty\big((-T,T);\mathcal{S}(\mathbb{R}^n)\big)$

$$\|D_t u + if(t,x,\xi)^w u + f_0(t,x,\xi)^w u\|_{L^2(\mathbb{R}^{n+1})} \ge c_0 T^{-1}\left(\int \|u(t)\|^2_{H^{-1/2}(\mathbb{R}^n)}dt\right)^{1/2}.$$

The remaining part of the proof is concerned with the delicate construction of an homogeneous microlocalization for a pseudodifferential operator in $n+1$ dimensions with some tensor-product structure related to the normal form of the principal symbol. The lengthy details are given in sections 4.2-3-4 of [22].

5. APPENDIX

5.1. Wick quantization

We recall here some facts on the so-called Wick quantization, as used in [26]-[27]-[28].

Definition 5.1. — *Let $Y=(y,\eta)$ be a point in $\mathbb{R}^{2n}$. The operator Σ_Y is defined as $\big[2^n e^{-2\pi|\cdot-Y|^2}\big]^w$. This is a rank-one orthogonal projection: $\Sigma_Y u = (Wu)(Y)\tau_Y\varphi$ with $(Wu)(Y)=\langle u,\tau_Y\varphi\rangle_{L^2(\mathbb{R}^n)}$, where $\varphi(x)=2^{n/4}e^{-\pi|x|^2}$ and $(\tau_{y,\eta}\varphi)(x)=\varphi(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}$. Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as*

$$a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{5.1}$$

The following proposition is classical and easy (see e.g. section 5 in [27]).

Proposition 5.2
(i) *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}} = W^* a^\mu W$ and $1^{\mathrm{Wick}} = Id_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_H = WW^*$ is the orthogonal projection on a closed proper subspace H of $L^2(\mathbb{R}^{2n})$. Moreover, we have*

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})}, \tag{5.2}$$

$$a(X)\ge 0 \text{ for all } X \text{ implies } a^{\mathrm{Wick}}\ge 0. \tag{5.3}$$

(ii) *Let m be a real number and $p \in S(\Lambda^m, \Lambda^{-1}\Gamma)$. Then $p^{\mathrm{Wick}} = p^w + r(p)^w$, with $r(p) \in S(\Lambda^{m-1}, \Lambda^{-1}\Gamma)$ so that the mapping $p \mapsto r(p)$ is continuous. More precisely, one has*

$$r(p)(X) = \int_0^1\int_{\mathbb{R}^{2n}} (1-\theta)p''(X+\theta Y)Y^2 e^{-2\pi\Gamma(Y)}2^n dY d\theta.$$

Note that $r(p) = 0$ if p is affine.

(iii) *For $a \in L^\infty(\mathbb{R}^{2n})$, the Weyl symbol of a^{Wick} is*

$$a * 2^n \exp -2\pi\Gamma \quad \text{which belongs to } S(1,\Gamma) \text{ with } k^{th}\text{-seminorm } c(k)\|a\|_{L^\infty}. \tag{5.4}$$

(iv) *Let $\mathbb{R} \ni t \mapsto a(t,X) \in \mathbb{R}$ such that, for $t \le s$, $a(t,X) \le a(s,X)$. Then, for $u \in C_c^1\big(\mathbb{R}_t, L^2(\mathbb{R}^n)\big)$, assuming $a(t,\cdot) \in L^\infty(\mathbb{R}^{2n})$,*

$$\int_{\mathbb{R}} \Re\langle D_t u(t), ia(t)^{\mathrm{Wick}} u(t)\rangle_{L^2(\mathbb{R}^n)} dt \ge 0. \tag{5.5}$$

(v) *With the operator Σ_Y given in Definition 5.1, we have the estimate*

$$\|\Sigma_Y\Sigma_Z\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}\Gamma(Y-Z)}. \tag{5.6}$$

Note that (5.5) is simply a way of writing that $\frac{d}{dt}\big(a(t)^{\mathrm{Wick}}\big) \ge 0$, which is a consequence of (5.3) and of the non-decreasing assumption made on $t \mapsto a(t,X)$.

5.2. Some classes of symbols

Let g be an admissible metric on $\mathbb{R}^{2n}$ and m be a g-weight (see Definition 1.4). Then, at each point $X \in \mathbb{R}^{2n}$, we can define a metric $g_X^\sharp$ by taking the geometric mean of g_X, g_X^σ so that in particular

$$g_X \le g_X^\sharp = (g_X^\sharp)^\sigma \le g_X^\sigma. \tag{5.7}$$

We define

$$h(X) = \sup_{g_X^\sharp(T)=1} g_X(T) \tag{5.8}$$

and we note that whenever $g^\sigma = \lambda^2 g$ we get from the definition 1.4 that $g^\sharp = \lambda_g g$ and $\lambda_g = 1/h$.

Definition 5.3. — *Let l be a nonnegative integer. We define the set $S_l(m,g)$ as the set of smooth functions a defined on $\mathbb{R}^{2n}$ such that a satisfies the estimates of $S(m,g)$ for derivatives of order $\le l$, and the estimates of $S(m,g^\sharp)$ for derivatives of order $\ge l+1$, which means*

$$|a^{(k)}(X)T^k| \le C_k m(X) \times \begin{cases} g_X(T)^{k/2} & \text{if } k \le l, \\ g_X^\sharp(T)^{k/2} h(X)^{\frac{l+1}{2}} & \text{if } k \ge l+1, \\ & \text{with } h(X) = \sup_{g_X^\sharp(T)=1} g_X(T). \end{cases}$$

Note that since $h \leq 1$ *and* $g \leq hg^\sharp$, *we get* $S(m,g) \subset S_l(m,g)$. *If* $g = \lambda(X)^{-1}\Gamma_X$, *where* $\lambda(X)$ *is positive* (*scalar*) *and* $\Gamma_X = \Gamma_X^\sigma$, *then* $g_X^\sharp = \Gamma_X$ *and* a *belongs to* $S_l(m, \lambda^{-1}\Gamma)$ *means*

$$|a^{(k)}(X)|_{\Gamma_X} \leq C_k m(X) \times \begin{cases} \lambda(X)^{-k/2} & \text{if } k \leq l, \\ \lambda(X)^{-l/2} & \text{if } k \geq l+1. \end{cases}$$

Moreover, if $g \equiv g^\sharp$, *then for all* l, $S(m,g) = S_l(m,g)$.

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