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ON THE LONG TIME BEHAVIOR OF KDV TYPE EQUATIONS [after Martel-Merle]

by **Nikolay TZVETKOV**

1. INTRODUCTION

A central problem in the theory of dispersive PDE's is to understand the interplay between nonlinearity and dispersion. In the context of the water waves problem (see *e.g.* [1]) the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + \partial_x(u^2) = 0, \quad x \in \mathbb{R} \tag{1}$$

appears to be the simplest (asymptotic) model where both dispersive and nonlinear effects are taken into account. If we neglect the nonlinear interaction $\partial_x(u^2)$ we deal with the Airy equation

$$u_t + u_{xxx} = 0. \tag{2}$$

The solutions of (2) are known to "disperse" in the sense that every solution u of (2) issued from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ initial data $u(0,\cdot)$, has its L^2 mass conserved but

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} = 0.$$

If we neglect the dispersive term u_{xxx}, we deal with the Burgers equation which is known to develop singularities in finite time, even for smooth initial data. The KdV equation (1) displays a balance between dispersion and nonlinearity since the dynamics of (1) is well defined, globally in time, for a very large class of initial data and moreover the solutions of (1) enjoy a rich dynamics as $t \to \infty$. A very special role among the solutions of (1) is played by the so-called solitary wave solution

$$u_c(t,x) = Q_c(x-ct) = \frac{3c}{2}\,\mathrm{ch}^{-2}\Big(\frac{\sqrt{c}}{2}(x-ct)\Big), \quad c > 0. \tag{3}$$

The solution (3) *does not disperse* and represents the displacement of the profile Q_c with speed c from left to right as the time t increases. Using the inverse scattering method (see [16, 29, 58]), it turns out that for sufficiently large t, any solution of (1) issued from well localized smooth initial data decomposes as a sum of solitary waves of type (3) plus a radiation term moving in the opposite direction. A natural generalization of (1), with stronger nonlinear effects, is the equation

$$u_t + u_{xxx} + \partial_x(u^p) = 0, \tag{4}$$

where p is a positive integer. The case $p = 3$ (modified KdV) is a very special case since, as in the case of (1), it can be treated with the inverse scattering method. Unfortunately, the integrability machinery does not seem to apply anymore for the equation (4) when $p \neq 2, 3$. Therefore the qualitative study of (4) in these cases is much less understood.

The equation (4) is a Hamiltonian PDE and its solutions enjoy, at least formally, the conservation laws

$$\|u(t,\cdot)\|_{L^2} = \|u(0,\cdot)\|_{L^2} \tag{5}$$

and

$$\frac{1}{2}\|u_x(t,\cdot)\|_{L^2}^2 - \frac{1}{p+1}\int_{-\infty}^{\infty} u^{p+1}(t,x)dx = \frac{1}{2}\|u_x(0,\cdot)\|_{L^2}^2 - \frac{1}{p+1}\int_{-\infty}^{\infty} u^{p+1}(0,x)dx. \tag{6}$$

Using the Gagliardo-Nirenberg inequalities

$$\|u(t,\cdot)\|_{L^{p+1}(\mathbb{R})}^{p+1} \leqslant C\|u(t,\cdot)\|_{L^2(\mathbb{R})}^{(p+3)/2}\|u_x(t,\cdot)\|_{L^2(\mathbb{R})}^{(p-1)/2},$$

we deduce from (5) and (6) that, for $p < 5$, the H^1 norm of $u(t,\cdot)$ is bounded independently of t as $u(0,\cdot) \in H^1(\mathbb{R})$. Consequently, the H^1 local well-posedness result of Kenig-Ponce-Vega [27] implies the existence of well-defined global dynamics of (4), for $p < 5$, in the energy space $H^1(\mathbb{R})$.

If $p \geqslant 5$, the H^1 local well-posedness result of Kenig-Ponce-Vega still applies (see Theorem 2.1 below) but the conservation laws (5), (6) provide no longer an H^1 control and hence solutions developing singularities in finite time may appear. The existence of such solutions has been a long standing open problem. In the case $p = 5$, this problem has been solved by Martel-Merle in a series of recent papers. The goal of this exposé is to discuss the main ideas developed by Martel-Merle, together with a presentation of previously known closely related results. One can extract from the results of Martel-Merle the following statement.

Theorem 1.1 (Martel-Merle [35, 44, 36, 37]). — *Let $p = 5$. There exists $u_0 \in H^1(\mathbb{R})$ such that the local solution of (4) with initial data u_0 blows up in finite time. More precisely there exists $T > 0$ such that $\lim_{t\to T}\|u(t,\cdot)\|_{H^1(\mathbb{R})} = \infty$.*

We refer to section 8 below for a more precise statement. Let us make a comment on the choice of the initial data u_0. Equation (4) still has solutions of type (3). Namely, the solitary waves of (4) have the form $u_c(t,x) = Q_c(x - ct)$, $c > 0$ with $Q_c(x) = c^{1/(p-1)}Q(\sqrt{c}x)$ and

$$Q(x) = \Big[\frac{p+1}{2\,\mathrm{ch}^2\left(\frac{p-1}{2}x\right)}\Big]^{1/(p-1)}.$$

The crux of the Martel-Merle analysis is the deep understanding of the flow of (4) *close to a solitary wave.* It turns out that the solutions developing singularities in finite time constructed by Martel-Merle are issued from initial data close to $Q(x)$ and are essentially of the form $Q_{c(t)}(x + x(t))$ with $c(t) \to \infty$ as $t \to \infty$.

The study of solutions of PDE's developing singularities in finite time is an active research field. Let us briefly recall a few of the existing results and compare them with the analysis in the context of (4). In the case of semi-linear wave equations, due to the "finite propagation speed", the blow-up dynamics can be approximated by an ODE developing singularities in finite time (see [2] and the references therein). In the case of quasi-linear wave equations, a Burgers type behavior is behind the blow up dynamics (see [15] and the references therein). The equation (4) does not enjoy similar finite propagation speed properties and the qualitative study of (4) offers new features. Probably the closest models to (4) are the nonlinear Schrödinger equations (NLS). In the case of NLS, we have a functional (viriel functional) giving a simple obstruction for the existence of global dynamics (see [59] and the references therein). A similar functional is not known to exist in the context of (4). Due to a conformal invariance(1) of some Nonlinear Schrödinger equations, one can construct explicit blow-up solutions (see [41, 43, 60]). Similar invariance is not known in the context of (4).

The rest of this text is organized as follows. In the next section we recall some basic facts on the Cauchy problem for (4). Next, we recall results on the stability of the solitary waves for (4). Starting from section 4, we concentrate on the case $p = 5$. In section 4, we present a characterization of the solitary waves among the solutions with data close to the profile Q. Then, in sections 5 and 6, we present two applications of that characterization result. Section 5 is devoted to an asymptotic stability result while in section 6 we present a result showing the existence of solutions blowing up in finite or infinite time. The last two sections are devoted to the existence of solutions blowing up in finite time. In section 7, we present a result on the blow-up profile which is essential to prove the blow-up in finite time. Section 8 is devoted to the argument providing finite-time blow-up solutions. Finally, in section 9 we present some remarks and open problems.

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(1)The viriel functional is a consequence of that invariance too.

2. THE CAUCHY PROBLEM

In this section, we collect some preliminary results on the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \partial_x(u^p) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{7}$$

where $p \geqslant 1$ is an integer. The following theorem[(2)], which can be extracted from the work of Kenig-Ponce-Vega [27], is the starting point for the study of (7) in H^1.

THEOREM 2.1 ([27]). — *For every $u_0 \in H^1(\mathbb{R})$, there exist $T \in]0,+\infty]$, bounded from below by a positive constant which only depends on $\|u_0\|_{H^1}$, and a functional space X_T continuously embedded in $C([0,T]\,;\,H^1(\mathbb{R}))$ such that the Cauchy problem (7) has a unique maximal solution $u \in X_T$. Moreover, if $T < +\infty$ then $\lim_{t\to T} \|u(t,\cdot)\|_{H^1} = \infty$.*

Of course, a similar statement holds for negative times t. One can also prove the local well-posedness of (7) in H^s for suitable $s < 1$. This fact plays an important role in the Martel-Merle work. For example, it is used to prove that the flow enjoys a continuity property with respect to the weak H^1 topology.

Let us give some indications on the proof of Theorem 2.1 in the case $p = 5$. The proof of the other cases follows similar lines. In the case $p = 5$, one can prove that (7) is well-posed for data in H^s, $s > 0$. The proof is based on applying the contraction mapping principle to the integral formulation (Duhamel principle) of (7)

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)\partial_x(u^p(\tau))d\tau. \tag{8}$$

In (8), $S(t) = \exp(-t\partial_x^3)$ is the generator of the free evolution. This is the operator of convolution with respect to x with $(3t)^{-1/3}\,\mathrm{Ai}(x(3t)^{-1/3})$, where Ai is the Airy function. Let us recall that the Airy function is exponentially decaying on the right and it decays as $|x|^{-1/4}$ on the left (see *e.g.* [24]). Using the smoothing properties of $S(t)$ one can prove (see [27, Corollary 2.11]) that for $u_0 \in H^1$, the right-hand side of (8) is a contraction in a suitable ball of the space X_T of functions defined on $[0,T]\times\mathbb{R}$, equipped with the norm

$$\|u\|_{X_T} = \|u\|_{L^\infty_T H^s_x} + \|\mathrm{D}^s_x\, u\|_{L^5_x\, L^{10}_T} + \|\mathrm{D}^{s/3}_t\, u\|_{L^5_x\, L^{10}_T} + \|\mathrm{D}^s_x\, u_x\|_{L^\infty_x\, L^2_T} + \|\mathrm{D}^{s/3}_t\, u_x\|_{L^\infty_x\, L^2_T}.$$

The argument relies on some methods from harmonic analysis (restriction phenomena, maximal function estimates, etc.). In the case $s = 0$ the argument breaks down. However, in that case we are able to insure the contraction property, if $\|u_0\|_{L^2}$ is small enough. Therefore, if $p = 5$, the equation (4) is L^2-critical.

[(2)]We refer to [56, 7, 25, 21] for earlier results on the well-posedness theory of (7).

Another very important aspect in the study of (7) is the Kato smoothing effect (see [25]). Let $\varphi \in C^3(\mathbb{R})$ be bounded with all its derivatives. If u is a solution of (4) then, multiplying (4) with φu and integrating by parts, we obtain the formal[(3)] identity

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2(t)\varphi = -3\int_{-\infty}^{\infty} u_x^2(t)\varphi' + \int_{-\infty}^{\infty} u^2(t)\varphi^{(3)} + \frac{2p}{p+1}\int_{-\infty}^{\infty} u^{p+1}(t)\varphi'. \tag{9}$$

Note that, if φ is increasing then the first term in the right-hand side of (9) is negative. This fact was used by Kato [25] to show a remarkable local smoothing effect for (7), if $p < 5$. Namely the solution turns out to be one derivative smoother than the data, locally in space. In [25] well-posedness results in weighted Sobolev spaces are also obtained. The article of Kato was a great source of inspiration for many further works on the subject. It is also the case in the papers by Martel-Merle. For example, the crucial monotonicity properties (see section 5 below) are strongly related to identity (9).

3. STABILITY AND INSTABILITY OF THE SOLITARY WAVES

The initial data giving rise to blow-up solutions in the work of Martel-Merle belong to a small neighborhood of the function $Q(x)$ which is the initial data for a solitary wave. Thus the question of long time stability (or instability) of the solution $Q(x-t)$ of (4) is closely related to Martel-Merle analysis. This question has a long history starting from the pioneering work of Benjamin [4]. The aim of this section is to briefly summarize the state of the art on the stability of $Q(x-t)$. Similar discussion is valid for the solitary wave $Q_c(x-ct)$ (recall that $Q = Q_1$).

Let us first notice that there exist data for (4) arbitrary close to $Q(x)$ such that the corresponding solution does not stay close to $Q(x-t)$ for long times. This is clearly the case of $Q_c(x)$ with c close but different from 1. Indeed, if c is close to 1 then $Q(x)$ is close to $Q_c(x)$, but, because of the different propagation speed, $Q(x-t)$ and $Q_c(x-ct)$ separate from each other for $t \gg 1$.

Notice however that in the previous example the solution issued from Q_c remains close to spatial translates of Q. Hence this example does not exclude orbital stability of Q (up to the action of the group of spatial translations). Indeed, it turns out that for $p < 5$ the solution $Q(x-t)$ is orbitally stable under small H^1 perturbations. Here is the precise statement.

Theorem 3.1. — *Let $p < 5$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if the initial data of* (7) *satisfies $\|u_0 - Q\|_{H^1} < \delta$ then there exists a C^1 function $x(t)$ such*

[(3)] The rigorous justification for sufficiently "nice solutions" u can be obtained by approximation arguments thanks to a propagation of regularity property of the local flow of (7).

that, for every $t \in \mathbb{R}$, the corresponding solution[4] *of* (7) *satisfies*

$$\|u(t, \cdot + x(t)) - Q(\cdot - t)\|_{H^1} < \varepsilon.$$

The proof of Theorem 3.1 can be found in [8] as an application of the general theory developed in [23]. See also [4, 6, 60, 61, 62] for earlier closely related results. Let us also mention the work of Cazenave-Lions [14] for an apparently different approach based on global variational arguments. In the case $p = 2$, we have L^2 solutions of (4) (see [12]) and it is a natural question whether the space H^1 in Theorem 3.1 could be replaced with L^2. The answer to that question is positive, as shown in the work by Merle-Vega [50].

Let us give the main idea of the proof of Theorem 3.1. Denote by $E(u)$ and $N(u)$ the functionals

$$E(u) := \frac{1}{2}\|u_x\|^2_{L^2(\mathbb{R})} - \frac{1}{p+1}\int_{-\infty}^{\infty} u^{p+1}(x)dx, \quad N(u) := \frac{1}{2}\|u\|^2_{L^2(\mathbb{R})}. \tag{10}$$

Recall that if u is a H^1 solution of (4) then the quantities $E(u(t))$ and $N(u(t))$ are time independent. Hence the functional $H(u) := E(u) + N(u)$ defines another conservation law. Since Q solves $(-\partial_x^2 + 1)Q = Q^p$, we infer that Q is a critical point of H. Using the implicit function theorem, we can find a C^1 function $x(t)$, defined a priori at least for small t such that $u(t, x + x(t)) = Q(x) + \mathcal{E}(t, x)$ and $\langle \mathcal{E}(t), Q' \rangle = 0$. Here and in the sequel $\langle \cdot, \cdot \rangle$ stands for the $L^2(\mathbb{R})$ inner product. Then clearly,

$$H(u(0)) = H(u(t)) = H(u(t, \cdot + x(t))) = H(Q + \mathcal{E}(t)),$$

and since Q is a critical point of H, by Taylor expansion, we deduce

$$H(u(0)) - H(Q) = H(Q + \mathcal{E}(t)) - H(Q) = \frac{1}{2}\langle L\mathcal{E}(t), \mathcal{E}(t) \rangle + o\big(\|\mathcal{E}(t)\|^2_{H^1}\big), \tag{11}$$

where $L = -\partial_x^2 - p\,Q^{p-1} + 1$. The next lemma is crucial.

Lemma 3.2. — *Let $p < 5$. Then there exists $C > 0$ such that for every $v \in H^1$ satisfying $\langle v, Q \rangle = \langle v, Q' \rangle = 0$, one has $\langle Lv, v \rangle \geqslant C\|v\|^2_{H^1}$.*

The proof of Lemma 3.2 uses the explicit form of Q. Actually we have a complete understanding of the spectrum of L, namely one simple negative eigenvalue associated to a positive eigenfunction, the simple eigenvalue zero associated to Q_x and all the rest of the spectrum is included in $[\gamma, \infty]$, for some $\gamma > 0$. The relevant fact related to the assumption $p < 5$ is that $d''(1) > 0$ where $d(c) = E(Q_c) + cN(Q_c)$. Unfortunately, the function $\mathcal{E}(t)$ involved in the decomposition of $u(t, x + x(t))$ is not orthogonal to Q and Lemma 3.2 alone does not suffice to complete the proof. However, by writing

[4] The existence is ensured from Theorem 2.1 and the Gagliardo-Nirenberg inequality as shown in the introduction.

$\mathcal{E}(t) = \alpha Q + \mathcal{E}_1(\mathrm{t})$ with $\mathcal{E}_1$ satisfying the assumptions of Lemma 3.2, we can apply the lemma to $\mathcal{E}_1$, the L^2 conservation law to evaluate α and to conclude that

$$\langle L\mathcal{E}(t), \mathcal{E}(t)\rangle \geqslant C\|\mathcal{E}(t)\|_{H^1}^2 + o\big(\|\mathcal{E}(t)\|_{H^1}^2\big),$$

provided $\|u\|_{L^2} = \|Q\|_{L^2}$. Hence Q is a local minimizer of the energy E on the sphere of L^2 centered at the origin and of radius $\|Q\|_{L^2}$. Finally using the continuity of E and N on H^1 allows one to complete the stability proof.

It turns out that the restriction $p < 5$ in Theorem 3.1 is sharp.

Theorem 3.3. — *Let $p \geqslant 5$. Then the solitary wave $Q(x-t)$ is not stable in the sense of Theorem 3.1.*

The proof of Theorem 3.3 for $p > 5$ can be found in [8] while the more involved analysis in the critical case $p = 5$ is performed in [33]. The argument is based on the construction of a suitable Lyapunov functional. If we set $\psi_c(x) := c^{1/(p-1)}Q(c^{2/(p-1)}x)$ then $\|\psi_c\|_{L^2} = \|Q\|_{L^2}$, and for $p > 5$, $\psi_1 = Q$ is a local maximum of $E(\psi_c)$ for c near 1. Thus, in contrast with the case $p < 5$, if $p > 5$ then Q is a saddle point of $E(u)$ subject to the constraint $\|u\|_{L^2} = \|Q\|_{L^2}$. Define a function y as

$$y(x) := \frac{1}{p-1}Q(x) + \frac{2x}{p-1}Q'(x).$$

The relevant fact about y is that $y = \frac{\partial \psi_c}{\partial c}|_{c=1}$. As in the stability proof, we "modulate" the solution of (4) with data close to Q as $u(t, x + x(t)) = Q(x) + \mathcal{E}(t, x)$ with $\langle \mathcal{E}(t), Q'\rangle = 0$. Such a decomposition of the solution is possible as far as it stays in a small H^1 neighborhood of the spatial translates of Q. For $p > 5$, the Lyapunov functional is

$$J(t) = \int_{-\infty}^{\infty} Y(x - x(t))u(t,x)dx,$$

where $Y(x) = \int_{-\infty}^{x} y(z)dz$. We refer to [23] for the general construction of Lyapunov functionals in the context of solitary waves for PDE's in the presence of symmetry. The discussion of [23] greatly clarifies the nature of the functional $J(t)$.

For $p > 5$, we have that $J'(t) \geqslant \kappa > 0$, if the initial data is ψ_c with c close but different from one, due to the property of the curve $\{\psi_c,\, c \sim 1\}$ described above. On the other hand, using some properties of the Airy function, by arguments in the spirit of the Cauchy problem analysis, we can obtain that $J(t) \leqslant C(t^{-2/3} + t^{2/3})$, $t > 0$, which in view of the lower bound for $J'(t)$ shows that instability holds.

In the critical case $p = 5$, the Lyapunov functional used in [33] is a suitable combination of $J(t)$ and the viriel functional

$$I(t) = \int_{-\infty}^{\infty} (x - x(t))u^2(t,x)dx.$$

The quantities $J(t)$ and $I(t)$ play a central role in Martel-Merle work. They are both measures for the loss of some mass during the time evolution. Notice that a priori these quantities do not make sense for H^1 solutions. The rigorous justification of the existence of $J(t)$ and $I(t)$ is one of the important analytic aspects in the analysis of (7) in a small neighborhood of the solitary waves.

Actually, the critical nature of the exponent $p = 5$ can be "predicted" from the spectral analysis of Pego-Weinstein [53]. Let us linearize (4) around the solution $Q(x-t)$. We set $u(t,x) = Q(x-t) + v(t, x-t)$. If we take $(t, x-t)$ as new variables that we note again with (t,x), we obtain that v solves the equation

$$v_t - \partial_x L\, v + R(v) = 0,$$

where $L = -\partial_x^2 - p\, Q^{p-1} + 1$ and the remainder $R(v)$ contains terms which are at least quadratic in v. Note that L is the same operator as in the expansion of the functional $H(u)$ (see (11) above). As usual, the spectral properties[(5)] of $\partial_x L$ are an indicator for the nonlinear stability of $Q(x-t)$ as a solution of (4). In [53], it is shown that $\partial_x L$, considered as an operator on L^2 with domain H^3, has the following spectrum :

– If $p \leqslant 5$ then the spectrum coincides with the imaginary axis.

– If $p > 5$ then the spectrum consists in the imaginary axis together with two simple, real eigenvalues, $-\lambda(p) < 0 < \lambda(p)$.

Therefore, if $p \neq 5$, the spectral analysis of [53] agrees with (and further clarifies) the stability theory presented above. The eigenvalue $\lambda(p)$ seems to be responsible for the instability in the case $p > 5$. This assertion *could* become rigorous by combining [53] with the ideas of [20].

Let us now turn to the concept of *asymptotic stability* which is at the heart of the work of Martel-Merle. The result of Theorem 3.1 says that the shape of the solitary wave is stable under small H^1 perturbations. But it is not clear whether $u(t, x + x(t))$ converges, in an appropriate sense, to a limit solitary wave. If it is indeed the case, we say that the family of solitary waves is asymptotically stable. Of course, the choice of the functional setting where one measures the convergence is crucial in that discussion. There are many results on asymptotic stability in the context of dissipative PDE's since in that case one can directly split the dynamics into noninteracting parts due to a spectrum of the linearized operator which does not quite meet the imaginary axis. It seems that the first results on asymptotic stability for Hamiltonian PDE's are those of M. Weinstein and collaborators. In the context of (4), Pego-Weinstein obtained asymptotic stability with convergence in the weighted Sobolev spaces H^1_a, where $a > 0$ and H^1_a is equipped with the norm $\|u\|_{H^1_a} = \|e^{ax}u(x)\|_{H^1}$. One can similarly define L^2_a. The following result is due to Pego-Weinstein.

Theorem 3.4 ([54]). — *Let $p = 2, 3$ and let a, b be two positive numbers such that $a^3 < 1/3$ and $b < a - a^3$. Then there exist $C > 0$ and $\varepsilon > 0$ such that if $u_0 \in H^2(\mathbb{R})$*

(5) This leads to the notion of *spectral stability*.

satisfies

$$\|u_0 - Q\|_{H^1} + \|u_0 - Q\|_{H^1_a} \leqslant \varepsilon$$

then there exist $x(t)$ *and* c_∞ *such that, for all* $t \geqslant 0$*, the solution of* (7) *satisfies*

$$\|u(t, \cdot + x(t)) - Q_{c_\infty}\|_{H^1} \leqslant C\varepsilon$$

and

$$\|u(t, \cdot + x(t)) - Q_{c_\infty}\|_{H^1_a} \leqslant C\varepsilon\, e^{-bt}.$$

The approach of Pego-Weinstein also works for some other $p < 5$ which are not integers (see [54] for a precise statement). The proof of Theorem 3.4 uses heavily the spectral analysis of $\partial_x L$ that we already discussed. In the case $p < 5$, the eigenvalue $\lambda(p)$ "becomes" a resonance (resolvent pole) and the corresponding mode plays an important role in the dynamics. If we consider $\partial_x L$ as an operator on L^2_a then the spectrum shifts from the imaginary axis and the situation becomes very similar to the case of a dissipative PDE. The number b involved in the statement of Theorem 3.4 is essentially the distance between the resonance and the imaginary axis. Let us also mention that the modulation parameter $x(t)$ in Theorem 3.4 is an *affine* function of t.

In [34], Martel-Merle prove an asymptotic stability result where the weighted norm H^1_a is replaced by a weak H^1 convergence. Here is the precise statement.

Theorem 3.5 ([34]). — *Consider* (7) *with* $p = 2, 3, 4$. *There exists* $\varepsilon > 0$ *such that if* $\|u_0 - Q\|_{H^1} \leqslant \varepsilon$ *then there exist* $x(t)$ *and* c_∞ *such that* $u(t, \cdot + x(t)) - Q_{c_\infty}$ *converges to zero, weakly in* H^1*, as* $t \to \infty$.

Note that in contrast with Theorem 3.4, the result of Theorem 3.5 does not give an estimate for the rate of convergence. On the other hand the assumptions on the data u_0 in Theorem 3.5 are less restrictive than in Theorem 3.4. Notice also that in Theorem 3.5 one can replace the weak H^1 convergence with strong $L^2_{\rm loc}$ convergence. In the proof of Theorem 3.5, we modulate the solution u as

$$\lambda^{2/(p-1)}(t)u(t, \lambda(t)x + x(t)) = Q(x) + \mathcal{E}(t, x), \tag{12}$$

where $\lambda(t)$ and $x(t)$ are chosen so that the remainder $\mathcal{E}$ satisfies the orthogonality conditions

$$\langle \mathcal{E}(t), Q\rangle = \langle \mathcal{E}(t), Q'\rangle = 0. \tag{13}$$

We note that, comparing to the stability analysis above, a new modulation parameter[(6)] $\lambda(t)$ appears in (12). That parameter is closely related to the *scaling* invariance of (4) which means that if $u(t, x)$ solves (4) then so does $\lambda^{2/(p-1)}u(\lambda^3 t, \lambda x)$. In that context, the modulation parameter $x(t)$ is related to the translation invariance of (4) which means that if $u(t, x)$ solves (4) then, for every $x_0 \in \mathbb{R}$, the equation (4) is also solved by $u(t, x + x_0)$. The orthogonality conditions (13) are clearly linked with

[(6)] The introduction of this parameter is in fact the main new point in the instability proof for $p = 5$.

Lemma 3.2 and can be achieved thanks to the implicit function theorem due to the following non degeneracy properties of Q,

$$\frac{d}{d\lambda}\langle Q_\lambda, Q\rangle\Big|_{\lambda=1} = \frac{5-p}{4(p-1)}\|Q\|_{L^2}^2, \quad \frac{d}{dx_0}\langle Q(\cdot+x_0), Q'(\cdot)\rangle\Big|_{x_0=0} = \|Q'\|_{L^2}^2. \tag{14}$$

Recall from the introduction that Q_λ is the initial data of a solitary wave of (4) propagating with speed λ. In view of (14), in the critical case $p = 5$, we are not able to modulate u so that $\langle \mathcal{E}(t), Q\rangle = 0$ which makes the analysis in that case quite different. To prove Theorem 3.5, one needs to show the convergence of $\lambda(t)$ as $t \to \infty$ and the convergence of $\mathcal{E}(t)$ to zero in H^1 weak. The proof of these two facts follows lines similar to the analysis of the critical case $p = 5$ to which the next two sections are devoted. In particular it heavily relies on a classification of L^2 compact solutions with data close to Q. It is important to mention that, in contrast with previous works on the subject, in the proof of Theorem 3.5 the bounds on $\mathcal{E}$ rely much less on solving the equation for $\mathcal{E}$ by an iteration scheme in a suitable functional setting.

In a very recent paper of Martel-Merle [39], a different proof of Theorem 3.5 is presented. It is based on the use of a localized viriel type functional which provides a control on $\mathcal{E}$. Moreover, in [39] a strong H^1 convergence on the right of the solitary wave is obtained. In the proof of Theorem 3.5 it appears that the derivative of the modulation parameter $x(t)$ converges to c_∞. A natural question is whether, similarly to the result of Pego-Weinstein, in Theorem 3.5 one may take $x(t)$ an affine function of t. The answer to that question is negative, at least in the case $p = 2$, due to an example constructed in [39].

The approach of Theorem 3.5 is further extended in [40, 32] where the asymptotic stability in H^1 of suitable sums of N solitary waves is obtained (see [31] for earlier results). In the case $p = 2$, the result applies to the exact N-soliton solution of the KdV equation (see *e.g.* [52]). The N-solitons of KdV can be used to show that, for $p = 2$, in Theorem 3.5 strong L^2 convergence is impossible. The existence of such solutions also "explains" the use of weighted norms in the work of Pego-Weinstein.

4. CLASSIFICATION OF L^2-COMPACT GLOBAL SOLUTIONS WITH DATA CLOSE TO Q

From now on we only consider (4) in the case $p = 5$, *i.e.* we analyze the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \partial_x(u^5) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{15}$$

Let us notice that for $p = 5$, we have $E(Q_c) = 0$ and $\|Q_c\|_{L^2} = \|Q\|_{L^2}$ for all c. Hence in that case the conservation laws (5) and (6) do not present an obstruction for the existence of blow-up solutions of the form $Q_{c(t)}(x + x(t))$ with $c(t) \to \infty$.

The following rigidity result, called by the authors "Liouville property", is the main tool in the proof of existence of simple asymptotic objects for the dynamics close to $Q(x-t)$.

THEOREM 4.1 ([35]). — *Suppose that :*

(1) *The function $u(t,x)$ is a solution global in time of* (15) *such that*

$$c_1 \leqslant \|u_x(t)\|_{L^2} \leqslant c_2,$$

for some positive constants c_1 and c_2.

(2) *The solution $u(t,x)$ is L^2 compact which means that there exists a function $y(t)$ such that for every $\varepsilon > 0$ there exists $R > 0$ such that for every t,*

$$\|u(t,x)\|_{L^2(|x-y(t)|>R)} \leqslant \varepsilon.$$

Then there exists $\alpha > 0$ such that if $\|u_0 - Q\|_{H^1} \leqslant \alpha$ then there exist λ and x_0 such that

$$u(t,x) = \lambda^{1/2} Q(\lambda(x-x_0) - \lambda^3 t).$$

Remark 4.2. — It seems that Theorem 4.1 is the first result of this type for a Hamiltonian PDE. On the other hand, results in the spirit of Theorem 4.1 were previously known in the context of parabolic PDE's (see [51, 19] and the references therein).

Let us give the main lines of the proof of Theorem 4.1. We modulate the solutions $u(t,x)$ of (15) with data close to Q in H^1 as

$$\lambda^{1/2}(t) u(t, \lambda(t)x + x(t)) = Q(x) + \mathcal{E}(t,x), \tag{16}$$

where the geometric parameters $\lambda(t) > 0$ and $x(t)$ are defined so that $\mathcal{E}$ satisfies the orthogonality conditions

$$\langle \mathcal{E}(t), Q^3 \rangle = \langle \mathcal{E}(t), Q' \rangle = 0. \tag{17}$$

The reason to chose (17) as orthogonality conditions is that if $\mathcal{E}$ satisfies (17) then $\langle L\mathcal{E}(t), \mathcal{E}(t) \rangle \geqslant C\|\mathcal{E}(t)\|_{H^1}^2$, where $L = -\partial_x^2 - 5Q^4 + 1$ is the operator arising in the linearization of the energy functional around Q. It seems however that (17) is not the only possible choice of orthogonality conditions on $\mathcal{E}$ which makes the proof of Theorem 4.1 work. The assumption (1) and the conservation laws imply the smallness of $\mathcal{E}(t)$ in H^1, if u_0 is close to Q in H^1. If we change the time variable as

$$s = s(t) := \int_0^t \lambda^{-3}(\tau) d\tau \tag{18}$$

then we obtain that $\mathcal{E}(s,x)$ solves the following equation

$$\begin{aligned} \mathcal{E}_s - (L\mathcal{E})_x - \frac{\lambda_s}{\lambda}\Big(\frac{Q}{2} + xQ'\Big) - \Big(\frac{x_s}{\lambda} - 1\Big)Q' \\ = \frac{\lambda_s}{\lambda}\Big(\frac{\mathcal{E}}{2} + x\mathcal{E}_x\Big) + \Big(\frac{x_s}{\lambda} - 1\Big)\mathcal{E}_x - \big(10Q^3\mathcal{E}^2 + 10Q^2\mathcal{E}^3 + 5Q\mathcal{E}^4 + \mathcal{E}^5\big)_x. \end{aligned} \tag{19}$$

The new time variable s is clearly related to the scaling of the equation. An advantage of introducing it is that even if t ranges in a finite interval (blow-up regime) the variable s ranges in the whole real line. Indeed, from the H^1 well-posedness and a scaling argument one easily obtains that if a solution $u(t)$ of (15) blows up in finite time T, then

$$\|u_x(t,\cdot)\|_{L^2} \geqslant C|T-t|^{-1/3},$$

for $t \sim T$. In the context of the decomposition (16), $\lambda(t) \sim \|u_x(t,\cdot)\|_{L^2}^{-1}$ and therefore, in view of (18) the variable s takes all real values. Notice also that the blow-up of $u(t)$ can be simply expressed as $\lambda(t) \to 0$, while the assumption (1) of Theorem 4.1 can be seen as $\widetilde{c_1} \leqslant \lambda(t) \leqslant \widetilde{c_2}$.

The parameter $x(t)$ involved in the decomposition (16) is different from $y(t)$ in the assumption (2) of the theorem. But one easily obtains that $|x(t)-y(t)| \leqslant C$ and that $\mathcal{E}(s,x)$ is L^2-compact, *i.e.* for every $\varepsilon > 0$ there exists $R > 0$ such that for every s one has $\|\mathcal{E}(s,x)\|_{L^2(|x|>R)} \leqslant \varepsilon$. It turns out that the L^2-compactness together with the properties of the equation (19) give much stronger estimates on $\mathcal{E}(s,x)$.

Lemma 4.3. — *Let a and b be defined as*

$$a := \sup_{s\in\mathbb{R}} \|\mathcal{E}(s,\cdot)\|_{H^1(\mathbb{R})}, \quad b := \sup_{s\in\mathbb{R}} \|\mathcal{E}(s,\cdot)\|_{L^2(\mathbb{R})}. \tag{20}$$

Then there exist $a_0 > 0$, $C > 0$ and $\theta > 0$ such that if $a < a_0$, then $|\mathcal{E}(s,x)| \leqslant C\sqrt{ab}\, e^{-\theta|x|}$.

Remark 4.4. — The exponential decay displayed in Lemma 4.3 is related to a general property of L^2-compact solutions of (4) and is not restricted only to data close to Q (see [30] for more details).

To prove Lemma 4.3 one needs to observe that equation (19) is essentially a critical generalized KdV equation with exponentially decaying source term. Thus to prove Lemma 4.3, in addition to the L^2-compactness property one also has to use the L^2 small data scattering theory for (15) developed by Kenig-Ponce-Vega in [27] and the persistence of the decay by the linear KdV flow, a fact already observed in the work of Kato [25].

The next step is to use Lemma 4.3 to get the equivalence between the H^1 and L^2 norms of $\mathcal{E}(s)$.

Lemma 4.5. — *Let a and b be defined as in (20). Then there exist $a_1 > 0$ and $C > 0$ such that if $a < a_1$ then $a \leqslant Cb$.*

The exponential decay obtained in Lemma 4.3 allows us to use viriel type identities. Indeed, if we set

$$I(s) = \frac{1}{2}\int_{-\infty}^{\infty} x\mathcal{E}^2(s,x)dx$$

we can obtain the estimate

$$\frac{d}{ds}\big(\lambda(s)I(s)\big) \leqslant C_1 b^2 - C_2\|\mathcal{E}(s)\|^2_{H^1}$$

which easily provides the bound $a \leqslant Cb$.

With Lemma 4.3 and Lemma 4.5 in hand we can turn to the proof of Theorem 4.1. Let us first notice that if $\mathcal{E} = 0$ in (16) and u is a solution of (15) then, in view of the equation solved by $\mathcal{E}$, we deduce that there exist two constants α and β such that $\lambda(t) = \alpha$ and $x(t) = \alpha^{-2}t + \beta$ which in turn implies that $u(t,x)$ can be represented as claimed in the statement of Theorem 4.1.

The proof of Theorem 4.1 is indirect. Consider a sequence (u_n) of solutions of (15) with $\lim_{n\to\infty} \|u_n(0) - Q\|_{H^1} = 0$ such that u_n satisfies assumptions (1) and (2) of Theorem 4.1. We can thus represent u_n, for $n \gg 1$ as in (16) with corresponding modulation parameters $\lambda_n(t)$, $x_n(t)$ and remainder $\mathcal{E}_n$. We suppose that for every n the function $\mathcal{E}_n$ is not identically zero and we look for a contradiction.

It turns out that a suitably renormalized subsequence of $\mathcal{E}_n$ converges to a solution of a linear problem.

Lemma 4.6. — *Let $b_n := \sup_s \|\mathcal{E}_n(s)\|_{L^2}$. Then there exist a sequence (s_n) of real numbers and a subsequence $(\mathcal{E}_{n'})$ such that $b_{n'}^{-1}\mathcal{E}_{n'}(s_{n'}+s)$ converges in $L^\infty_{\mathrm{loc}}(\mathbb{R}\,;\,L^2(\mathbb{R}))$ to $w(s)$ which is not identically zero and satisfies $w \in C(\mathbb{R}\,;\,H^1(\mathbb{R}))\cap L^\infty(\mathbb{R}\,;\,H^1(\mathbb{R}))$. Moreover w solves the equation*

$$w_s - (Lw)_x = \alpha(s)\Big(\frac{Q}{2} + xQ'\Big) + \beta(s)Q' \tag{21}$$

for some continuous functions α and β. In addition w satisfies the orthogonality conditions

$$\langle w(s), Q^3\rangle = \langle w(s), Q'\rangle = 0 \tag{22}$$

and the exponential decay

$$|w(s,x)| \leqslant C\, e^{-\theta|x|} \tag{23}$$

with a suitable choice of the positive constants C and θ.

At a formal level one easily verifies that the linear equation (21) appears as a limit model for the nonlinear equation (19). We have the estimate

$$\Big|\frac{\lambda_s}{\lambda}\Big| + \Big|\frac{x_s}{\lambda} - 1\Big| \leqslant C \sup_s \|\mathcal{E}(s,\cdot)\|_{L^2} \tag{24}$$

as a very basic property of the decomposition (16). Indeed, it suffices to multiply (19) with Q^3 and Q' and to integrate on x. Since $\mathcal{E}_n$ solves (19), in view of the bound (24), we obtain that at the limit $b_n \to 0$ the limit equation for $b_n^{-1}\mathcal{E}_n$ is (21). Indeed, all terms in the right-hand side of (19) disappear either because of (24) or because $\mathcal{E}$ appears in higher powers.

In order to make the previous formal reasoning rigorous one heavily relies on the L^2 small data global existence theory of Kenig-Ponce-Vega [27]. The fact that $\|\mathcal{E}_n\|_{L^2}$ controls $\|\mathcal{E}_n\|_{H^1}$ is of great importance in the limit process.

The last step in the proof of Theorem 4.1 is the analysis of the linear equation (21). Theorem 4.1 follows from the following rigidity property of (21).

LEMMA 4.7. — *Let $w \in C(\mathbb{R}\,;\,H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}\,;\,H^1(\mathbb{R}))$ be a solution of (21) satisfying the orthogonality conditions (22) and the decay estimate (23). Then w is identically zero.*

Lemma 4.7 could be seen as a result of unique continuation *at infinity* for the equation (21) (the decay of w is essential for the proof). Let us give an outline of the proof of Lemma 4.7. It turns out that $w(s)$ satisfies an additional orthogonality condition. One can directly show that $\langle w(s), Q\rangle$ is a quantity independent of s. In order to show that it is zero, one appeals to the functional(7)

$$J(s) = \int_{-\infty}^{\infty} w(s,x)\Big(\int_0^x \Big(\frac{Q(y)}{2} + yQ'(y)\Big)dy\Big)dx.$$

A direct computation shows that $J'(s) = 2\langle w(0), Q\rangle$. Due to the exponential decay of $w(s)$, $|J(s)|$ is uniformly bounded and therefore $\langle w(s), Q\rangle = 0$. Thus, for all s, $w(s)$ is orthogonal to Q.

We next consider the viriel functional

$$I(s) = \frac{1}{2}\int_{-\infty}^{\infty} x\, w^2(s,x)dx.$$

A direct computation shows that

$$I'(s) = H(w(s), w(s)) + \alpha(s)\Big\langle x\Big(\frac{Q}{2} + xQ'\Big), w(s)\Big\rangle + \beta(s)\langle xQ', w(s)\rangle, \tag{25}$$

where $H(w,w) = \langle (Lw)_x, xw\rangle$. In view of (25), we slightly modify $w(s)$ by setting

$$\widetilde{w}(s) = w(s) + \gamma(s)\Big(\frac{Q}{2} + xQ'\Big) + \delta(s)Q'. \tag{26}$$

It turns out that with a suitable choice of $\gamma(s)$ and $\delta(s)$, $\widetilde{w}$ solves an equation of type

$$\widetilde{w}_s - (L\widetilde{w})_x = \widetilde{\alpha}(s)\Big(\frac{Q}{2} + xQ'\Big) + \widetilde{\beta}(s)Q',$$

satisfies the orthogonality conditions

$$\Big\langle \widetilde{w}(s), \Big(\frac{xQ}{2} + x^2Q'\Big)\Big\rangle = \langle \widetilde{w}(s), xQ'\rangle = \langle \widetilde{w}(s), Q\rangle = 0, \tag{27}$$

and, if we set $I_1(s) = \int_{-\infty}^{\infty} x\widetilde{w}^2(s,x)dx$, then $I_1'(s) = H(\widetilde{w}(s), \widetilde{w}(s))$.

(7)Recall that a *similar* functional is involved in the instability analysis of the previous section.

The next step is to use that due to the orthogonality conditions satisfied by $\widetilde{w}$, one has

$$-H(\widetilde{w}(s), \widetilde{w}(s)) \geqslant \frac{1}{10}\langle L\widetilde{w}(s), \widetilde{w}(s)\rangle. \tag{28}$$

Once again, in order to prove (28), one needs to make explicit calculations based on the very particular form of Q. Using the equation solved by $\widetilde{w}(s)$, one can easily check that $\langle L\widetilde{w}(s), \widetilde{w}(s)\rangle = \langle L\widetilde{w}(0), \widetilde{w}(0)\rangle$ and therefore $I_1'(s) \leqslant -\frac{1}{10}\langle L\widetilde{w}(0), \widetilde{w}(0)\rangle \leqslant 0$. Since $\widetilde{w}(s)$ is exponentially decaying we obtain that $\langle L\widetilde{w}(s), \widetilde{w}(s)\rangle = 0$.

But as a matter of fact, if $u \in H^1(\mathbb{R})$ is orthogonal to Q and satisfies $\langle Lu, u\rangle = 0$, then u is necessarily a linear combination of Q' and $\frac{Q}{2} + xQ'$. In view of (27), we directly conclude that $\widetilde{w}$ is identically zero. Finally, we obtain that w is identically zero thanks to (26) and the orthogonality conditions (22).

5. ASYMPTOTIC STABILITY IN THE REGULAR REGIME

A first consequence of Theorem 4.1 is an asymptotic stability result under the assumption that the solution with data close to Q is globally defined and uniformly bounded in $H^1(\mathbb{R})$.

THEOREM 5.1 ([35]). — *Suppose that $u(t)$ is a solution global in time of* (15) *such that $c_1 \leqslant \|u_x(t)\|_{L^2} \leqslant c_2$, for some positive constants c_1 and c_2. Then there exists $\alpha > 0$ such that if $\|u_0 - Q\|_{H^1} \leqslant \alpha$ then there exist $\lambda(t)$ and $x(t)$ such that $\lambda^{1/2}(t)u(t, \lambda(t)x + x(t))$ converges to $Q(x)$, weakly in $H^1(\mathbb{R})$, as $t \to \infty$.*

Notice that the result of Theorem 5.1 displays a weaker form of asymptotic stability compared with the case $p < 5$. Indeed, in contrast with the situation for $p < 5$, it is not clear whether the modulation parameter $\lambda(t)$, involved in the statement of Theorem 5.1, converges to some limit as $t \to \infty$.

Let us give the main ideas of the proof of Theorem 5.1. The two main ingredients are :

– Continuity of the flow of (15) with respect to the weak H^1 topology.

– An almost monotonicity property of the L^2 mass for solutions of (15) with data close to Q in H^1.

The continuity property of the flow with respect to the weak H^1 topology is a consequence of the well-posedness of the Cauchy problem below H^1 and of a viriel identity argument. This type of results seems to appear first in a paper by Glangetas-Merle [22]. The monotonicity property of the mass is closely related to the Kato identity discussed in section 2. This monotonicity property separates the dynamics into two noninteracting parts and it is related to the dispersion relation (the symbol) of the *linear part* of the equation. It is worth noticing that similar monotonicity properties hold for a fairly large class of equations such as the Benjamin-Bona-Mahony equation (see [17]), the Kadomtsev-Petviashvili II equation (see [57, 10]) but, it does not seem to hold for other models as the Kadomtsev-Petviashvili I equation or the nonlinear Schrödinger equation. We refer to [54, page 308] for a very clear explanation whether an equation in hand may enjoy the crucial separation of the dynamics property.

The proof of Theorem 5.1 is again indirect. We modulate $u(t,x)$ as in (16)

$$\lambda^{1/2}(t)u(t,\lambda(t)x+x(t)) = Q(x)+\mathcal{E}(t,x). \tag{29}$$

Let us take a sequence $t_n \to \infty$ such that $\lambda^{1/2}(t_n)u(t_n,\lambda(t_n)x+x(t_n))$ converges, weakly in H^1, to $\widetilde{u}_0$ and $\lambda(t_n)$ converges to $\widetilde{\lambda}_0$. We suppose $\widetilde{u}_0 \neq Q$ and we seek for a contradiction by means of Theorem 4.1. Let $\widetilde{u}$ be the local solution of the critical generalized KdV equation with data $\widetilde{u}_0$. Since $\widetilde{u}_0$ is close to Q in H^1, we can modulate $\widetilde{u}$, at least for small times, with modulation parameters $\widetilde{\lambda}(t)$, $\widetilde{x}(t)$ and remainder $\widetilde{\mathcal{E}}(t,x)$ satisfying the same orthogonality conditions as $\mathcal{E}$. The continuity property of the flow with respect to the weak H^1 topology implies the following statement.

Lemma 5.2. — *The solution $\widetilde{u}(t)$ is defined for all $t\in\mathbb{R}$, and the sequence $\mathcal{E}(t_n+t)$ converges, weakly in $H^1(\mathbb{R})$, to $\widetilde{\mathcal{E}}(t)$. Moreover, for every $T>0$, $\lambda(t_n+t)-\widetilde{\lambda}(t)$ and $x(t_n+t)-\widetilde{x}(t)$ converge to zero in $C([-T,T]\,;\mathbb{R})$.*

Notice that one first proves the lemma for small times which implies a H^1 bound on $\widetilde{u}(t)$ thanks to the H^1 boundedness assumption on u. Then we extend $\widetilde{u}(t)$ for all times due to the H^1 local well-posedness of (15).

The asymptotic solution $\widetilde{u}$ being constructed, the aim is to show that it satisfies the assumptions of Theorem 4.1, *i.e.* we need to check that $\widetilde{u}$ is L^2-compact. Let us notice that the leading idea at this point is that the solution $\widetilde{u}$ enjoys more properties than the original solution u itself.

5.1. L^2-compactness of $\widetilde{u}$ on the right of the solitary wave

There are different ways to prove the L^2-compactness of $\widetilde{u}$ on the right. In this subsection, we discuss an argument which is based on a direct analysis of the limit solution $\widetilde{u}$. In the next subsection, we present a more involved method, using Lemma 5.2, providing compactness on both sides.

To get the L^2-compactness of $\widetilde{u}$ on the right, it is sufficient to show that small solutions of (15) can not travel too fast to the right. Such a result would imply the needed compactness since the main part of the solution is moving to the right with speed uniformly bounded from below. Let us state precisely a lemma giving the L^2-compactness on the right. We introduce a function $\psi\in C^\infty(\mathbb{R})$ by setting

$$\psi(x) = c_0\int_{-\infty}^{x} Q\Big(\frac{x}{K}\Big)dy,$$

where c_0 is chosen so that $\lim_{x\to\infty}\psi(x)=1$. Consider the functional

$$\mathcal{I}_\sigma(t) := \int_{-\infty}^{\infty} \psi(x-\sigma t)\, u^2(t,x)\, dx$$

which measures the distribution of the L^2-mass on the right with respect to a frame moving with speed σ. Since the quantity on the right of the solitary wave is essentially a "small solution" of (15), the next statement is the crucial point in the proof of the L^2-compactness of $\widetilde{u}$ on the right.

LEMMA 5.3. — *Let $\sigma>0$ and $K>\sqrt{2/\sigma}$. There exists a positive constant C_σ such that, if $\|u_0\|_{L^2}\leqslant C_\sigma$, then $\mathcal{I}_\sigma(t)$ is a non-increasing function on the trajectories of (15).*

The proof of Lemma 5.3 is an application of the Kato identity (9).

5.2. L^2-compactness of $\widetilde{u}$ on the left of the solitary wave

The analysis in that case is more delicate. The main point is to show that the loss of mass at the left is "irreversible". Notice that Lemma 5.3 does not hold for large solutions since there are solitary waves moving with arbitrary large speed. It turns out however that a weaker form of Lemma 5.3 survives for large data. Let $u(t)$ be a solution of (15) which is decomposed as in (16). For $(t_0,x_0)\in\mathbb{R}^2$ and $t_0\geqslant t$, we introduce the functional

$$\mathcal{I}_{x_0,t_0}(t) := \int_{-\infty}^{\infty} \psi(x-x(t)-x_0-\frac{3}{4}(x(t_0)-x(t)))\, u^2(t,x)\, dx,$$

where the function ψ is defined in the previous subsection. The following statement is now the substitute of Lemma 5.3.

LEMMA 5.4. — *Suppose that there exist two positive numbers c_1 and c_2 such that $c_1\leqslant\lambda(t)\leqslant c_2$. Then there exist $\delta>0$, $K>0$ and $C>0$ such that, if $\sup_t\|\mathcal{E}(t)\|_{H^1}<\delta$, then for every $x_0\geqslant 0$, and $0\leqslant t\leqslant t_0$,*

$$I_{x_0,t_0}(t_0)-I_{x_0,t_0}(t)\leqslant Ce^{-x_0/K}.$$

Remark 5.5. — The very particular structure of the functional $\mathcal{I}_{x_0,t_0}(t)$ is important for the proof. The term $\frac{3}{4}(x(t)-x(t_0))$ is strongly needed. The number $\frac{3}{4}$ can be replaced by any number between $\frac{1}{2}$ and 1.

We now explain how the L^2-compactness of $\widetilde{u}$ can be obtained by combining Lemma 5.4 and Lemma 5.2. Define the functional

$$m_r(u(t)) := \int_{-\infty}^{\infty} \psi(x-x(t)-x_0)\, u^2(t,x)\, dx$$

which measures the L^2-mass on the right of the solitary wave. It is easy to see that Lemma 5.4 implies that $m_r(t)$ is an "almost decreasing quantity". More precisely,

$$m_r(u(t))-m_r(u(t'))\leqslant Ce^{-x_0/K},\quad t\geqslant t'. \tag{30}$$

Indeed, we have that $m_r(u(t)) = \mathcal{I}_{x_0,t}(t)$ and due to the monotonicity of ψ and $x(t)$, $m_r(u(t')) \geqslant \mathcal{I}_{x_0,t}(t')$. Similarly, since if $u(t,x)$ solves (15) then so does $u(-t,-x)$, we deduce that the quantity

$$m_l(u(t)) := \int_{-\infty}^{\infty} (1 - \psi(x - (x(t) - x_0)))\, u^2(t,x)\, dx,$$

measuring the L^2-mass on the left of the solitary wave is "almost increasing". More precisely,

$$m_l(u(t)) - m_l(u(t')) \geqslant -Ce^{-x_0/K}, \quad t \geqslant t'. \tag{31}$$

Assume that $\widetilde{u}$ is not L^2-compact. It means that there exists $\delta > 0$ such that for every *bounded* interval $I \subset \mathbb{R}$, there exists t_0 such that

$$\int_I \widetilde{u}^2(t_0,x)dx \leqslant \|\widetilde{u}(0)\|_{L^2}^2 - \delta.$$

For a solution $u(t)$ of (15), we define the quantity $m_{\mathrm{loc}}(u(t))$, measuring the L^2-mass in a moving frame, via the identity

$$\|u(t)\|_{L^2}^2 = m_l(u(t)) + m_{\mathrm{loc}}(u(t)) + m_r(u(t)). \tag{32}$$

Taking $x_0 \gg 1$, we can assume

$$m_{\mathrm{loc}}(\widetilde{u}(0)) > \|\widetilde{u}(0)\|_{L^2}^2 - \frac{\delta}{4}. \tag{33}$$

The assumption of lack of L^2-compactness of $\widetilde{u}$ implies the existence of $t_0 \in \mathbb{R}$ such that

$$m_{\mathrm{loc}}(\widetilde{u}(t_0)) < \|\widetilde{u}(0)\|_{L^2}^2 - \frac{\delta}{2}, \tag{34}$$

if $x_0 \gg 1$. We suppose that $t_0 > 0$, the case $t_0 < 0$ being similar. From (33) and (34), we get the estimate

$$m_{\mathrm{loc}}(\widetilde{u}(0)) - m_{\mathrm{loc}}(\widetilde{u}(t_0)) > \frac{\delta}{4}.$$

Since the weak H^1-convergence implies the strong L^2_{loc}-convergence, using Lemma 5.2, we deduce that, if $x_0 \gg 1$ (independently of t_0) then there exists N such that for $n \geqslant N$,

$$m_{\mathrm{loc}}(u(t_n)) - m_{\mathrm{loc}}(u(t_n + t_0)) > \frac{\delta}{8}. \tag{35}$$

The L^2-conservation law for u can now be written as

$$m_l(u(t_n+t_0)) - m_l(u(t_n)) = m_r(u(t_n)) - m_r(u(t_n+t_0)) + m_{\mathrm{loc}}(u(t_n)) - m_{\mathrm{loc}}(u(t_n+t_0))$$

which, using (30) and (35), yields

$$m_l(u(t_n + t_0)) - m_l(u(t_n)) > \frac{\delta}{8} - Ce^{-x_0/K}. \tag{36}$$

We can clearly suppose that $t_{n+1} > t_n + t_0$. Therefore using (31) and (36), we get

$$m_l(u(t_{n+1})) - m_l(u(t_n)) = m_l(u(t_{n+1})) - m_l(u(t_n + t_0)) + m_l(u(t_n + t_0)) - m_l(u(t_n)) > \frac{\delta}{8} - 2Ce^{-x_0/K}.$$

Hence, if $x_0 \gg 1$, it follows that $m_l(u(t_n)) \to \infty$ which is clearly impossible in view of the L^2-conservation law. We obtain a contradiction coming from the assumption of lack of L^2-compactness of $\widetilde{u}$. Therefore $\widetilde{u}$ is L^2-compact.

Remark 5.6. — Notice that Theorem 5.1 gives an important information about negative energy solutions with data close to Q. Namely, such solutions cannot be uniformly bounded in $H^1(\mathbb{R})$. Indeed, let u be a negative energy solution of (15) with data close to Q which is uniformly bounded in $H^1(\mathbb{R})$. Then similarly to above, we can construct an asymptotic solution $\widetilde{u}$ which satisfies the assumptions of Theorem 4.1. Thus $\widetilde{u}$ is necessarily a rescaled and translated version of Q. In particular, $E(\widetilde{u}) = 0$ since in the critical case $p = 5$, $E(Q_c) = 0$ for all c. On the other hand $\widetilde{u}$ is obtained as a weak H^1 limit from solutions close to $Q(x - t)$ with negative energies which implies that $E(\widetilde{u}) < 0$. We thus get a contradiction with the assumption of uniform H^1-boundedness of u.

Let us finally remark that since $E(Q) = 0$ and $\nabla E(Q) = -Q$, we obtain that there exists a large set of negative energy initial data for (15) which is close to Q. An example of such data is clearly $u_0(x) = \pm(1 + \varepsilon)Q$, where $0 < \varepsilon \ll 1$.

6. BLOW-UP IN FINITE OR INFINITE TIME

In this section, we present a second consequence of Theorem 4.1 which is the existence of solutions of (15) blowing up in finite or infinite time.

THEOREM 6.1 ([44]). — *There exists $\alpha > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies*

$$\|u_0\|_{L^2} \leqslant \|Q\|_{L^2} + \alpha \tag{37}$$

and $E(u_0) < 0$ (negative energy) then the solution of (15) blows up in finite or infinite time which means that there exists $T \in]0, \infty]$ such that $\lim_{t\to T} \|u(t,\cdot)\|_{H^1} = \infty$.

Remark 6.2. — Under the assumptions of Theorem 6.1, the initial data $u_0(x)$ is close in $H^1(\mathbb{R})$ to $\pm\lambda^{1/2}Q(\lambda(x + x_0))$ for some constants λ_0 and x_0 (see [41, 42, 60]).

Notice that if u_0 satisfies the assumptions of Theorem 6.1, then so does $-u_0$. We also remark that the new point in Theorem 6.1 with respect to Theorem 5.1 is that we have the existence of the limit as t goes to T of $\|u(t,\cdot)\|_{H^1}$ and not only the existence of a sequence (t_n) such that $\|u(t_n,\cdot)\|_{H^1}$ goes to infinity.

The approach of Theorem 6.1 is similar to that of Theorem 5.1. The new ingredients are :

– Extension of the L^2-compactness of the limit solution to the case when there is no lower bound on the scaling modulation parameter $\lambda(t)$.

– The use of a third conservation law of (15) which provides a control on the size of $\lambda(t)$.

The starting point in Theorem 6.1 is to modulate a negative energy solution of (15), with data satisfying (37) for α small, as

$$\lambda^{1/2}(t)u(t,\lambda(t)x+x(t)) = \pm Q(x) + \mathcal{E}(t,x) \tag{38}$$

with $\mathcal{E}(t,x)$ satisfying the orthogonality conditions (17). Without loss of generality, we may assume that the sign in front Q is plus. The decomposition (38) is the same as in (16) but the proof of the control on the modulation parameters $\lambda(t)$, $x(t)$ and the remainder $\mathcal{E}(t,x)$ is different under the assumptions of Theorem 6.1. Using the variational nature of Q one can show that the decomposition (38) holds with smallness estimates(8) on $\mathcal{E}(t,x)$ and bounds on $\lambda(t)$ and $x(t)$, as far as the solution u exists. Here, "the variational nature of Q" means that if $u \in H^1(\mathbb{R})$ is such that $E(u)=0$, $\|u\|_{L^2} = \|Q\|_{L^2}$ and $\|u'\|_{L^2} = \|Q'\|_{L^2}$ then $u(x) = \pm Q(x+x_0)$ for some constant $x_0 \in \mathbb{R}$ (see [41, 42, 60]).

The proof is by contradiction. Take a sequence (u_n) of negative energy global solutions of (15) such that $\|u_n(0)\|_{L^2}$ tends to $\|Q\|_{L^2}$ as $n\to\infty$. We suppose that for each n there exist a sequence $(t_{n,m})$ and a constant c_n so that $\|\partial_x u_n(t_{n,m})\|_{L^2} \leqslant c_n$, uniformly in m. We seek for a contradiction under this assumption. Similarly to the previous section, we define a limit object $\widetilde{u_n}(0)$ from the sequence $(t_{n,m})$ and the decomposition (38) applied to u_n. We denote by $\widetilde{u_n}(t)$ the local solution of (15) with initial data $\widetilde{u_n}(0)$, defined on a time interval $(-T_1(n),T_2(n))$. The closeness to $Q(x-t)$ and the weak H^1-convergence yield

$$E(\widetilde{u_n}) < 0. \tag{39}$$

Similarly to the considerations on the L^2-compactness of $\widetilde{u}$ in the previous section, one can show that $\widetilde{u_n}$ is L^2-compact and satisfies a crucial exponential decay property. This allows one to use a third conservation law of (15) applied to $\widetilde{u_n}$. Namely,

$$\int_{-\infty}^{\infty} \widetilde{u_n}(t,x)dx = \int_{-\infty}^{\infty} \widetilde{u_n}(0,x)dx, \tag{40}$$

if $t\in(-T_1(n),T_2(n))$. The conservation law (40) shows that the scaling modulation parameter $\widetilde{\lambda}_n(t)$, involved in the decomposition of $\widetilde{u_n}(t)$, is uniformly bounded from below. Using that $\widetilde{\lambda}_n(t) \sim \|\partial_x \widetilde{u_n}(t,\cdot)\|_{L^2}^{-1}$, we obtain a uniform bound on $\|\widetilde{u_n}(t,\cdot)\|_{H^1}$ which, thanks to the H^1 well-posedness of (15) implies that $\widetilde{u_n}(t)$ is globally defined. Since $\widetilde{u_n}$ is L^2-compact, using Theorem 4.1, we conclude that $\widetilde{u_n}$ is a rescaled and translated Q. In particular, $E(\widetilde{u_n})=0$ which is in contradiction with (39).

(8) Depending on the smallness of $(\|u_0\|_{L^2} - \|Q\|_{L^2})$.

7. BLOW-UP PROFILE

In this section, we present an extension of the asymptotic stability result of Theorem 5.1 to the singular regime of Theorem 6.1. It turns out that the blow-up solutions with data close to Q converge in H^1 weak, after a suitable *singular renormalization* to the profile Q. Thus the concept of asymptotic stability naturally extends to the singular regime as shows the next statement.

THEOREM 7.1 ([36]). — *There exists $\alpha > 0$ such that if $\|u_0\|_{L^2} \leqslant \|Q\|_{L^2} + \alpha$ and if the solution $u(t)$ of* (15) *blows up in finite or infinite time $T \in]0, \infty]$ then there exist $\lambda(t) > 0$ and $x(t)$ such that $\lambda^{1/2}(t)u(t, \lambda(t)x + x(t))$ converges as $t \to T$, weakly in $H^1(\mathbb{R})$, either to $Q(x)$ or to $-Q(x)$.*

Notice that, if $E(u_0) < 0$ then the result of Theorem 7.1 applies to the blow-up solutions considered in the previous section. It is also worth noticing that the energy conservation and the weak H^1 convergence imply the strong convergence in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R})$.

The proof of Theorem 7.1 follows a similar strategy to that of Theorem 5.1. However, the classification result of Theorem 4.1 alone is not sufficient to conclude because the asymptotic solution may be singular ($\lambda(t) \to 0$). For that purpose a new rigidity result adapted to the singular regime has to be established. Again, the viriel functional $I(t)$ and the functional $J(t)$, appeared already several times in our discussion, are the key for the argument.

In the proof of Theorem 7.1, we write once again

$$\lambda^{1/2}(t)u(t, \lambda(t)x + x(t)) = \pm Q(x) + \mathcal{E}(t, x), \tag{41}$$

for t near the blow-up time T and α small enough. The difference is that now the modulation parameters $\lambda(t)$ and $x(t)$ are chosen so that $\mathcal{E}$ satisfies the orthogonality conditions

$$\Big\langle \mathcal{E}(t), \Big(\frac{xQ}{2} + x^2 Q'\Big)\Big\rangle = \langle \mathcal{E}(t), xQ' \rangle = 0. \tag{42}$$

Notice that we already considered orthogonality conditions of type (42) in the linear analysis of section 4 (see (27)). The orthogonality conditions (42) are used to cancel some second order terms in variation of the viriel functional $I(t) = \frac{1}{2}\int_{-\infty}^{\infty} x\,\mathcal{E}^2(t,x)dx$. More precisely, at least formally,

$$\begin{aligned}\lambda^3(t)\dot{I}(t) = \lambda^2(t)\dot{\lambda}(t)\Big\langle \mathcal{E}(t), \Big(\frac{xQ}{2} + x^2 Q'\Big)\Big\rangle + (\lambda^2(t)\dot{x}(t) - 1)\langle \mathcal{E}(t), xQ' \rangle \\ + H(\mathcal{E}(t), \mathcal{E}(t)) + R(\mathcal{E}(t)),\end{aligned}$$

where $R(\mathcal{E}(t))$ contains only higher order terms in $\mathcal{E}(t)$ and $H(\mathcal{E}, \mathcal{E}) = \langle (L\mathcal{E})_x, x\mathcal{E} \rangle$ is the bilinear form which already appeared in the proof of Lemma 4.7. The choice of (42) as orthogonality conditions is possible thanks to the implicit function theorem, in view of another non degeneracy property of Q. Despite the "loss of sign" of $\langle L\mathcal{E}, \mathcal{E}\rangle$ with the new orthogonality conditions, one is still able to get smallness bounds on $\mathcal{E}$ and a control on the modulation parameters λ and x of type (24). The variational nature of Q is again used in the smallness estimates on $\mathcal{E}$.

In order to prove Theorem 7.1, one has to show that $\mathcal{E}(t)$ converges to zero in H^1 weak, as $t \to T$. One first proves that $\mathcal{E}(t_n)$ converges to zero in H^1 weak, as $n \to \infty$, for a specific choice of the sequence (t_n). Namely, t_n is so that $\lambda(t_n) = (1.1)^{-n}$ and $\lambda(t) \leqslant \lambda(t_n)$ for $t \in [t_n, T[$. The case of an arbitrary sequence (t_n) then can be treated by using the monotonicity of the L^2-mass.

Let us describe the argument for the specific sequence (t_n). The proof is by contradiction. We suppose that there exists a subsequence of (t_n) still denoted by (t_n) such that $\mathcal{E}(t_n)$ converges weakly in H^1 to $\widetilde{\mathcal{E}}(0)$ which is not zero and we look for a contradiction. Let $\widetilde{u}(0) := \pm Q + \widetilde{\mathcal{E}}(0)$. We denote by $\widetilde{u}(t)$ the local solution of (15) subject to initial data $\widetilde{u}(0)$. Let $\widetilde{\lambda}(t)$, $\widetilde{x}(t)$ and $\widetilde{\mathcal{E}}(t)$ be the modulation parameters and the remainder in a decomposition of type (41) applied to $\widetilde{u}(t)$. The solution $\widetilde{u}(t)$ may develop singularities in finite time and this is the new feature in the analysis.

Notice that $\widetilde{\lambda}(0) = 1$. Thanks to the special choice of (t_n) one has $\widetilde{\lambda}(t) \leqslant 1$ and we can define a maximal $\tau \in]0, \infty]$ such that $(1.1)^{-1} < \widetilde{\lambda}(t) \leqslant 1$ for every $t \in [0, \tau)$. Two possibilities appear, either $\tau = \infty$ or $\tau < \infty$. In the case $\tau = \infty$, the solution $\widetilde{u}$ is global, uniformly bounded in H^1, and one can show similarly to before that $\widetilde{u}$ is L^2-compact. Thus Theorem 4.1 applies and gives a contradiction as in Theorem 5.1. In the case $\tau < \infty$ a fairly new argument is needed. Introduce the new time variable s as in (18) with $\widetilde{\lambda}(t)$ instead of $\lambda(t)$. Set $\tau_1 := s(\tau)$. The contradiction arises from a lower and upper bound on the quantity Λ, defined as

$$\Lambda := \int_0^{\tau_1} \int_{-\infty}^{\infty} \widetilde{\mathcal{E}}^2(s,x)\, e^{-|x|/2} dx ds.$$

One can prove an exponential decay of $\widetilde{\mathcal{E}}$ to the left by the monotonicity properties considered in section 5. This allows to consider the functional

$$J(s) = \int_{-\infty}^{\infty} \widetilde{\mathcal{E}}(s,x) \Big(\int_x^{\infty} \Big(\frac{Q(y)}{2} + yQ'(y) \Big) dy \Big) dx - \frac{1}{4} \|Q\|_{L^2}^2. \tag{43}$$

The second term in (43) is of course not essential. Using direct computations and the basic properties of the decomposition (41), one can show the bound

$$\Big| J'(s) + \frac{\lambda_s}{2\lambda} J(s) + 2\langle \widetilde{\mathcal{E}}(s), Q\rangle \Big| \leqslant C \int_{-\infty}^{\infty} \widetilde{\mathcal{E}}^2(s,x)\, e^{-|x|/2} dx. \tag{44}$$

Estimate (44) is now the key for the proof of the lower bound

$$\Lambda \geqslant C\Lambda_1, \tag{45}$$

where $C > 0$ is independent of α,

$$\Lambda_1 := 1 + \int_0^{\tau_1} \int_{-\infty}^{\infty} \widetilde{\mathcal{E}}_x^2(s,x) dx ds + |E| \int_0^{\tau_1} (\widetilde{\lambda}(s))^2 ds$$

and E is the energy of $\widetilde{u}$.

For the upper bound on Λ, a localized[9] viriel type functional is used. For $A > 0$, we consider the function $\psi_A(x) = A\psi(A^{-1}x)$, where $\psi(x)$ is a smooth odd function such that $\psi(x) = x$ for $|x| < 1$, $\psi'(x) = e^{-x}$ for $x > 2$, and for $x \in [1, 2]$, ψ is increasing and concave. We consider the following localized viriel functional

$$I_A(s) := \int_{-\infty}^{\infty} \psi_A(x)\widetilde{\mathcal{E}}^2(s,x)dx.$$

Thanks to the new orthogonality conditions one obtains that there exist $A > 2$, $\gamma > 0$ such that

$$I'_A(s) \leqslant -\gamma \int_{-\infty}^{\infty} e^{-|x|/A}\big(\widetilde{\mathcal{E}}^2(s,x) + \widetilde{\mathcal{E}}_x^2(s,x)\big)dx + \frac{1}{\gamma}\langle \widetilde{\mathcal{E}}(s), Q\rangle^2, \tag{46}$$

provided u_0 is close enough to Q. The bilinear form H is naturally involved in the proof of (46). More precisely, it turns out that

$$-H(\mathcal{E}(s), \mathcal{E}(s)) \geqslant C\|\mathcal{E}(s)\|_{H^1}^2,$$

if $\langle \mathcal{E}(s), Q\rangle = \big\langle \mathcal{E}(s), \big(\frac{xQ}{2} + x^2Q'\big)\big\rangle = 0$. We have the second term in the right-hand side in (46) because the orthogonality with respect to Q is "forbidden" for $\widetilde{\mathcal{E}}$ (see (14)).

Using (46) one can get the upper bound

$$\Lambda \leqslant C_{u_0}\Lambda_1 \tag{47}$$

where the constant C_{u_0} is tending to zero, if $\|u_0\|_{L^2}$ is tending to $\|Q\|_{L^2}$. In view of (45) and (47), we get a contradiction for $\|u_0\|_{L^2}$ close enough to $\|Q\|_{L^2}$.

Remark 7.2. — A corollary of Theorem 7.1 is a lower bound on the blow-up rate which excludes the existence of self-similar blow-up solutions for data in H^1. However, in [9], Bona-Weissler construct solutions of (7) with self-similar blow-up which are missing the space H^1.

8. BLOW-UP IN FINITE TIME

In this section, we present a result showing that under an additional assumption on the initial data, the blow-up solutions of Theorem 6.1 develop their singularities in *finite time.*

THEOREM 8.1 ([37]). — *There exists $\alpha > 0$ such that, if $u_0 \in H^1(\mathbb{R})$ satisfies*

$$\|u_0\|_{L^2} \leqslant \|Q\|_{L^2} + \alpha, \quad E(u_0) < 0, \quad \forall\, x_0 > 0,\ \|u_0(x)\|_{L^2(x \geqslant x_0)} \leqslant C\,|x_0|^{-3},$$

then the solution of (15) *blows up in finite time, i.e. there exists $T \in\,]0, \infty[$ such that $\lim_{t\to T} \|u(t,\cdot)\|_{H^1} = \infty$.*

[9] This localization is needed because we do not have available a decay of $\widetilde{\mathcal{E}}(s,x)$ for $x \to +\infty$.

A detailed presentation of all steps in the proof of Theorem 8.1, as written is [37], would be quite technical. For that reason, we will only give an informal presentation of the main idea.

Consider the usual decomposition

$$\lambda^{1/2}(t)u(t,\lambda(t)x+x(t)) = \pm Q(x) + \mathcal{E}(t,x), \tag{48}$$

for a blow-up solution u with t near the blow-up time T. We suppose that the sign in front of Q in the right hand side of (48) is plus. In order to prove that T is finite, the *ideal situation* would be to have the bound

$$\dot\lambda(t) \leqslant -C < 0. \tag{49}$$

Estimate (49) is not known to hold in the context of (15) but one is able to prove a weaker version of (49) as we explain below.

Substituting (48) in (15), we obtain that the equation solved by $\mathcal{E}(t,x)$ is

$$\begin{aligned} \lambda^3(t)\mathcal{E}_t - (L\mathcal{E})_x - \lambda^2(t)\dot\lambda(t)\Big(\frac{Q}{2}+xQ'\Big) - (\lambda^2(t)\dot x(t)-1)Q' \\ = \lambda^2(t)\dot\lambda(t)\Big(\frac{\mathcal{E}}{2}+x\mathcal{E}_x\Big) + (\lambda^2(t)\dot x(t)-1)\mathcal{E}_x - \big(10Q^3\mathcal{E}^2+10Q^2\mathcal{E}^3+5Q\mathcal{E}^4+\mathcal{E}^5\big)_x. \end{aligned} \tag{50}$$

Let us impose that $\mathcal{E}$ satisfies the orthogonality conditions

$$\int_{-\infty}^{\infty}\mathcal{E}(t,x)w(x)dx = \int_{-\infty}^{\infty}\mathcal{E}(t,x)xw'(x)dx = 0, \tag{51}$$

where

$$w(x) := \int_{-\infty}^{x}\Big(\frac{Q(y)}{2}+yQ'(y)\Big)dy.$$

The choice (51) is formally possible, again due to the implicit function theorem via an explicit calculation on Q. Let us notice that $w(x)$ does not tend to zero as $x\to\infty$. Hence one needs to ensure that $\mathcal{E}(t,x)$ decays sufficiently fast to the right and this is one of the major analytical problems in the proof of Theorem 8.1.

Imposing (51) as orthogonality conditions is natural, in view of the equation (50) and the identities satisfied by w

$$Lw' = -2Q, \quad \langle w', Q\rangle = 0. \tag{52}$$

The verification of (52) is straightforward. Under the orthogonality conditions (51), using (52) and integration by parts, we easily get the following (at least formal) identities

$$\langle\lambda^3(t)\mathcal{E}_t, w\rangle = 0,\ \langle (L\mathcal{E})_x, w\rangle = 2\langle\mathcal{E},Q\rangle,\ \Big\langle\frac{Q}{2}+xQ', w\Big\rangle = c_0,\ \langle Q',w\rangle = \langle x\mathcal{E}_x, w\rangle = 0,$$

where c_0 is a *positive* constant which can be easily written explicitly in terms of Q. Therefore, multiplying (50) with w, gives the identity

$$\lambda^2(t)\dot\lambda(t) = -c_1\langle\mathcal{E}(t),Q\rangle + R_1(\mathcal{E}(t)), \tag{53}$$

where $c_1 := 2/c_0$ and $R_1(\mathcal{E}(t))$ is an explicit expression containing terms of quadratic and higher order with respect to $\mathcal{E}(t)$.

On the other hand, if we denote by $E < 0$ the energy of u, then we get easily from (48), the identity

$$E = E(u(t)) = \lambda^{-2}(t)E(Q + \mathcal{E}(t)). \tag{54}$$

Substituting $Q + \mathcal{E}(t)$ in the energy functional yields

$$E(Q + \mathcal{E}(t)) = -\langle \mathcal{E}(t), Q\rangle + \frac{1}{2}\|\mathcal{E}_x(t)\|_{L^2}^2 + R_2(\mathcal{E}(t)), \tag{55}$$

where $R_2(\mathcal{E}(t))$ is another explicit expression of $\mathcal{E}(t)$ containing only quadratic and higher order terms.

Using (53), (54) and (55), we directly obtain

$$\lambda^2(t)E = -\langle \mathcal{E}(t), Q\rangle + \frac{1}{2}\|\mathcal{E}_x(t)\|_{L^2}^2 + R_2(\mathcal{E}(t)), \tag{56}$$

and

$$\dot{\lambda}(t) = -c_2 - \frac{c_1\lambda^{-2}(t)}{2}\|\mathcal{E}_x(t)\|_{L^2}^2 + \lambda^{-2}(t)\, R(\mathcal{E}(t)), \tag{57}$$

where $c_2 := -c_1 E > 0$ and $R(\mathcal{E}(t)) = R_1(\mathcal{E}(t)) - c_1 R_2(\mathcal{E}(t))$.

Notice that if we neglect the third term in the right-hand side of (57), we get an estimate of type (49). Thus one needs to bound the third term in the right-hand side of (57). This can be achieved, with a viriel inequality of type (46). Recall that $ds = \lambda^{-3}(t)dt$. Thus an estimate of type (46) for $\mathcal{E}(t)$ together with (56) provides a bound for $\lambda^3(t)\, R(\mathcal{E}(t))$ in terms of $I'_A(t)$, *i.e.* it is realistic to expect that the singularity of $\lambda^{-2}(t)$ can be compensated by the smallness of $R(\mathcal{E}(t))$ (quadratic in $\mathcal{E}(t)$). Since the bound is in terms of $I'_A(t)$, the relevant estimates one can get are only for averages of $R(\mathcal{E}(t))$ on time intervals where $\lambda(t)$ does not vary much.

In [37], Martel-Merle are able to make the previous formal discussion rigorous. More precisely, let us define a sequence (t_n) such that $t_n \to T$ and such that

$$\|u_x(t_n, \cdot)\|_{L^2} = 2^n\|Q'\|_{L^2}$$

and, for $t \in]t_n, T[$, one has $\|u_x(t, \cdot)\|_{L^2} > 2^n\|Q'\|_{L^2}$. Notice that the existence of (t_n) follows from Theorem 6.1. It turns out that for $n \gg 1$,

$$t_{n+1} - t_n \leqslant C(\lambda(t_n) - \lambda(t_{n+1})) \tag{58}$$

which is an integrated form of an estimate of type (49). We can deduce directly from (58) that $T < \infty$.

The main point in the proof of (58) is of course the estimate of $\int_{t_n}^{t_{n+1}} |R(\mathcal{E}(t))|dt$. For that purpose, two modulations of the solutions with different orthogonality conditions are used. The first one is very similar to the one considered in the previous section and enjoys the viriel type estimates of Theorem 7.1. One uses the assumption

$$\int_{x_0}^{\infty} u_0^2(x)dx \leqslant C\,|x_0|^{-6}, \quad x_0 > 0$$

to get a decay to the right of the solution. This decay allows one to use a second decomposition with the orthogonality conditions (51). One then needs to compare the remainders of the two decompositions. It turns out that one gets cancelations up to second order which is the crucial point in the comparison between the two key quantities $\lambda^2(t)\dot{\lambda}(t)$ and $\langle \mathcal{E}(t), Q\rangle$ which in turn provides the key estimate (58).

Remark 8.2. — In Theorem 8.1 the L^2-mass accumulated in the blow-up time is $\|Q\|_{L^2}^2$ (see also [28]). A natural question is whether one may construct blow-up solutions that do not disperse any mass[10] at the blow-up time, *i.e.* such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$. It turns out that, due to the result in [38], the answer of that question is negative. Therefore the blow-up solutions of Theorem 8.1 necessarily lose some mass on the left of the "main core" during the time evolution.

9. FINAL REMARKS

The work of Martel-Merle has already been quite influential. In a remarkable series of recent papers, using many of Martel-Merle ideas, Merle-Raphaël [48, 45, 46, 47, 49, 55] obtained a number of new results on the understanding of the blow-up phenomena for the L^2-critical nonlinear Schrödinger equations (NLS). The literature on blow-up for NLS is enormous and we refer to the recent books [11, 13, 59] for an introduction to that domain. In [17, 18], the ideas of [34] are successfully used to get the asymptotic stability for the family of solitary waves for the BBM equation which is an alternative to the KdV model in the theory of water waves (see [5]).

Let us point out that the existence of blow-up solutions in the case $p > 5$ remains an open problem. It seems that the approach of Martel-Merle meets serious difficulties in this case. Notice that for $p = 5$, the solution $Q(x-t)$ is *spectrally stable*, *i.e.* there is no eigenvalue of $\partial_x L$ with positive real part. Therefore the dynamics for solutions with data close to Q can be successfully parameterized by the modulation parameters $\lambda(t)$ and $x(t)$. It seems that in the case $p > 5$, the eigenfunction of $\partial_x L$ with positive real part is also involved in the long time dynamics, even for data close to Q.

Let us finally notice that it would be interesting to extend the asymptotic stability analysis for (4) to the generalized Benjamin-Ono equation

$$u_t + \partial_x(-Hu_x + u^p) = 0 \tag{59}$$

[10] Notice that such solutions exist in the case of the L^2-critical NLS (see [59]). Moreover they are completely classified (see [41]).

which is another important model in the water waves theory (see [3]). In (59), H is the Hilbert transform and $p \geqslant 2$ is an integer. The equation (59) has a lower order dispersion compared with the KdV equation and $p = 3$ is the smallest value of p such that one may expect blow-up. In the context of (59), the natural energy space is $H^{1/2}(\mathbb{R})$ and therefore a first difficulty is that at the present moment the space $H^{1/2}(\mathbb{R})$ is not covered by the well-posedness theory for (59) (see [26] and the references therein). However, at least at a formal level, equation (59) shares many of the properties of (4) used in the work of Martel-Merle.

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Nikolay TZVETKOV

Université de Lille 1
Département de Mathématiques
F-59655 Villeneuve d'Ascq Cedex
E-mail : `nikolay.tzvetkov@math.univ-lille1.fr`