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BROWNIAN MOTION IN A POISSON OBSTACLE FIELD

by **Tomasz KOMOROWSKI**

1. INTRODUCTION

The problem of a diffusion performed in a medium with randomly distributed static traps appeared in Physics literature some 80 years ago (cf. e.g. Smoluchowski (1916)). A common illustration is a motion of a Brownian particle among soft or hard core obstacles of finite radius with centers Poisson distributed in a d dimensional space, cf. e.g. Grassberger et al. (1982).

It can be shown, cf. Section 2, that on the average the behavior of the survival probability S_t of the particle in the field up to time t, the so called *annealed asymptotic*, is given by

$$S_t = \exp\left\{-\tilde{c}(d,\nu)t^{\frac{d}{d+2}}(1+o(1))\right\},\ t \gg 1.$$

Here ν is the intensity of the Poisson cloud and $\tilde{c}(d,\nu)$ is a constant given by (8). This result has been obtained non-rigorously in Kac and Luttinger (1973-74). The decay rate of the survival probability is much slower than exponential. The latter could be expected in light of the result of Kesten, Spitzer, Whitman, cf. the remark after Theorem 1.

The problem of determining the annealed asymptotic is related, as we shall explain in Section 5 below, to the question of finding a rigorous proof of the so called Lifshitz tail effect for the Integrated Density of States (IDS) function of the Schrödinger operator with a random Poisson potential. This question appeared in Physics literature in the context of quantum theory of a condensed state in the work of Lifshitz, cf. Lifshitz (1965).

The first rigorous argument giving the correct asymptotic of S_t via large deviation theory of Wiener sausage asymptotic has been presented by Donsker and Varadhan (1975).

This result has been obtained again via a different technique – the so called *Method of Enlargement of Obstacles* (MEO) in Sznitman (1990). The method later became a fundamental tool in understanding a rich collection of phenomena associated with Brownian motion among Poissonian traps such as: the asymptotic of the survival probability $S_{t,\omega}$

in a typical realization of the medium ω - the so called *quenched asymptotic*, the *Pinning Effect* and *Confinement Property* for the trapped Brownian Motion, cf. Sznitman (1998) and (1991), Povel (1998), the Lifshitz tails for the IDS function of the random magnetic Schrödinger operator with Poissonian random potential, cf. Erdösz (1998) to name just a few. Other results go in the direction of determining the behavior of the Brownian particle performing a long crossing between two distant points of the medium and consequently lead to the introduction of the so called Lyapunov exponents, cf. Sznitman (1994), (1995a,b) and also the Shape Theorem of Section 3.1 for the quenched case. The Lyapunov exponents turn out to be very convenient tools in deriving large deviation results for the quenched and annealed measures associated with the surviving Brownian process, cf. Sznitman (1995), (1995a,b).

In the context of Lyapunov exponent one can introduce also the notion of Crossing Brownian Motion (CBM), cf. Section 4. CBM measures appear naturally in certain physical models of growing interfaces, cf. Krug-Spohn (1991). One of the basic questions is to understand the relation between the size of transverse fluctuations of the CBM and the fluctuations of a certain natural random distance function defined by (17). This direction has been pursued by Wüthrich in Wüthrich (1998).

Lyapunov exponents have been also applied to the analysis of random walks in a random environment (RWRE) in higher dimensions. The appropriate Shape Theorem has been proven in this context by Zerner (1997). He used the resulting Lyapunov exponents in the derivation of the large deviation theorem for random walks in dimension $d \geq 2$ satisfying the so called *nestling property.*

To describe rigorously the obstacle field we consider $\mathbb{P}$ to be a probability measure given on Ω – the set of locally finite simple pure point measures on $\mathbb{R}^d$

$$\omega = \sum_i \delta_{x_i}, \; x_i \in \mathbb{R}^d \tag{1}$$

endowed with the canonical σ-algebra generated by the mappings

$$\omega \in \Omega \to \omega(A) \in \mathbb{Z}_+, \; A \in \mathcal{B}(\mathbb{R}^d) \text{ - the Borelian } \sigma\text{-algebra.}$$

We shall assume that the centers of the obstacles form a cloud of Poisson distributed points with intensity $\nu > 0$, i.e. that

$$\mathbb{E}\exp\left\{-\int f(y)\omega(dy)\right\} = \exp\left\{-\nu\int dy\left(1-e^{-f(y)}\right)\right\}$$

for any Borel measurable function f. Here $\mathbb{E}$ denotes the mathematical expectation of measure $\mathbb{P}$.

Let $Z_\cdot$ be the canonical d-dimensional Brownian Motion, P_x the Wiener measure corresponding to a Brownian particle starting at x and E_x its corresponding expectation. We shall assume that the particle interacts with the cloud ω via a potential representing the absorption rate of the random medium

$$V(x,\omega) := \sum_i W(x - x_i, \omega),$$

where x_i are determined from (1) and W is a nonnegative, measurable function whose support is contained in a ball of radius $a > 0$ centered at 0.

In what follows we shall also discuss the hard obstacle case. We define then the obstacle set as $A := \bigcup_{x_i}(x_i + K)$ for some nonpolar set K (i.e. of positive capacity) of finite diameter a. The particle is instantaneously killed upon entry into the region

$$\Theta := \mathbb{R}^d - A. \tag{2}$$

Informally speaking, in the hard obstacle case, we take $V(\cdot,\omega) = \infty$ on the obstacle set and 0 elsewhere.

The structure of our presentation is as follows. In the first four sections we give a brief overview of some of the results available in the area. Because of the scope of this paper we are unable to provide more than a short sketch of the techniques used in the field. However we try our best to give the reader some, mostly heuristic, explanations of the presented results while avoiding getting too technical in our presentation.

Section 6 presents the main ingredients of MEO presented here after Sznitman (1998), which, as we have already mentioned, is crucial in the theory. Here we try to be more precise than in the previous sections although we maintain our goal of keeping the presentation as simple as possible.

Finally we close the article with a short review of some open problems in the subject.

2. QUENCHED AND ANNEALED MEASURES

The Brownian motion among the obstacles can be described by the path measures

$$Q_{t,\omega} := \frac{1}{S_{t,\omega}} \exp\left\{ - \int_0^t V(Z_s,\omega)ds \right\} P_0 \qquad \textit{the quenched measure} \tag{3}$$

with $\omega \in \Omega$ or

$$Q_t := \frac{1}{S_t} \exp\left\{ - \int_0^t V(Z_s,\omega)ds \right\} P_0 \otimes \mathbb{P} \qquad \textit{the annealed measure.} \tag{4}$$

Here the normalizing constants are given by

$$S_{t,\omega} := E_0\left[\exp\left\{ - \int_0^t V(Z_s,\omega)ds \right\} \right]$$

and

$$S_t := E_0 \otimes \mathbb{E}\left[\exp\left\{-\int_0^t V(Z_s,\omega)ds\right\}\right].$$

Remark. Let us observe that in the hard obstacle case we have

$$S_t = E_0[\exp\{-\nu|W_t^a|\}], \tag{5}$$

where $W_t^a := \bigcup_{0\le s\le t} Z_s + \overline{B}(0,a)$ is the tubular neighborhood of radius a of the Brownian trajectory, the so called "Wiener sausage".

Intuitively speaking the quenched measure describes the law of the particle conditioned on the event that it survives among the traps up to time t for a typical configuration of traps ω. On the other hand the annealed measure arises as a result of averaging over possible realisations of the random medium.

The question of interest is the asymptotic behavior of the path measures (3), (4) when $t \uparrow +\infty$. In the first step we describe the asymptotics of $S_{t,\omega}$ and S_t – the survival probabilities of the Brownian Motion among the obstacles – for large t. The following result holds.

THEOREM 1. — *When $t \uparrow +\infty$ $\mathbb{P}$ a.s. we have*

$$S_{t,\omega} = \exp\left\{-c(d,\nu)\frac{t}{(\log t)^{2/d}}(1+o(1))\right\} \text{ for soft obstacles} \tag{6}$$

and

$$S_t = \exp\left\{-\tilde{c}(d,\nu)t^{\frac{d}{d+2}}(1+o(1))\right\} \text{ for both soft and hard obstacles.} \tag{7}$$

Here

$$c(d,\nu) := \lambda_d\left(\frac{\nu\omega_d}{d}\right)^{2/d}$$

and

$$\tilde{c}(d,\nu) := \inf_{U \text{ open}}\{\nu|U| + \lambda(U)\} = (\nu\omega_d)^{2/d+2}\left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}}, \tag{8}$$

with $\lambda(U)$ denoting the principal Dirichlet eigenvalue of the Laplacian $-\frac{1}{2}\Delta$ corresponding to the region U. We have written λ_d and ω_d for the fundamental tone and volume of the unit ball in the d dimensional space respectively.

Remark. At this time it may be worthwhile to mention that motivated by the result of Kesten, Spitzer and Whitman (cf. Spitzer (1964), p. 40) stating that $\frac{|W_t^a|}{t}$ tends to the capacity of $B(0,a)$ (the ball of radius a centered at 0) when $t \uparrow +\infty$ and (5) one could mistakenly believe that S_t decays exponentially, which obviously contradicts (7).

The following argument gives some plausibility to the lower bounds on the asymptotic of the normalizing constants. In Section 6 we shall sketch the proof of the upper bound on S_t using MEO.

Informally speaking (when W is sufficiently smooth this is in fact accurate) we can write, using Feynman-Kac formula, that

$$S_{t,\omega} = u_\omega(t,0) \qquad \text{and} \qquad S_t = \mathbb{E}[u_\omega(t,0)],$$

where

$$u_\omega(t,x) := E_x \exp\left\{-\int_0^t V(Z_s,\omega)ds\right\}$$

satisfies the Cauchy problem

$$\partial_t u_\omega(t,x) = -H_\omega u_\omega(t,x), \tag{9}$$

$$u_\omega(0,x) \equiv 1$$

with

$$H_\omega := -\frac{1}{2}\Delta + V(x,\omega). \tag{10}$$

Using (9) we obtain that

$$S_{t,\omega} \approx \sum_i e^{-\lambda_i t} < \varphi_i, 1 > \varphi_i(0) \tag{11}$$

where λ_i, $i = 1, 2, \cdots$ denotes the complete set of eigenvalues of the self-adjoint operator H_ω and φ_i are the corresponding eigenfunctions. Consider now a sufficiently large box $\Lambda = (-l,l)^d$. With probability $\mathbb{P}$ close to one Λ contains a ball $B_l := B(x_l, R_l)$ of radius $R_l \sim R_0(\log l)^{1/d}$ on which $V(\cdot,\omega) \equiv 0$, $R_0 = \left(\frac{d}{\nu\omega_d}\right)^{1/d}$. In that case we have an easy upper bound $\lambda_1 \leq \lambda(B_l)$ – the Dirichlet principal eigenvalue of the Laplacian operator corresponding to the ball B_l. On the other hand we should expect then φ_1 to be localized in B_l and decay exponentially at a rate α away from x_l which is a consequence of e.g. formula (2.26) p. 276 of Sznitman (1998). Thus we can suppose that $\varphi_1(0) \sim e^{-\alpha|x_l|}$. Discarding the influence of higher eigenvalues we see that asymptotically $S_{t,\omega}$ should be greater than or equal to $\exp\{-(!!\alpha|x_l| + \lambda(B_l)t)\}$. Choosing now $l = o(\frac{t}{(\log t)^{2/d}})$ but such that $\log l \sim \log t$, we get

$$S_{t,\omega} \geq \exp\{-c(d,\nu)\frac{t}{(\log t)^{2/d}}\}.$$

Probabilistically this situation corresponds to the following particle survival strategy. To survive among the traps until time t the particle starting from the origin travels to a pocket B_l with no obstacles and stays there for the reminder of time t. This scenario minimizes the cost in terms of probability $\mathbb{P}$ associated with the creation of a pocket free of obstacles. One has every right to think that there should be no reason for this behavior of the Brownian particle to be typical. However, as it turns out, for large t the particle goes "extra mile" searching for some points x_l being "near minima" of a certain random functional and stays in their vicinity for the remainder of time t, cf. the so called *Pinning Effect* described in the following section.

By contrast the good survival strategy of a particle in the annealed case is to stay inside an obstacle free ball $B_t := B(0, \tilde{R}_t)$ of a certain radius $\tilde{R}_t = \tilde{R}t^{\frac{1}{d+2}}$ with $\tilde{R}$ to be determined later, where the chance of survival $\sim \exp\{-\lambda(B_t)t\} = \exp\{-\lambda(B(0,\tilde{R}))t^{\frac{d}{d+2}}\}$ is much larger than in the quenched case. The probability of an obstacle free hole of such a radius appearing in the Poisson cloud is $\sim e^{-\nu|B_t|}$. Hence asymptotically we obtain that

$$S_t \geq \exp\{-\left(\lambda_d/\tilde{R}^2 + \nu\omega_d\tilde{R}^d\right) t^{\frac{d}{d+2}}\}. \tag{12}$$

Optimizing over all $\tilde{R}$ we find that

$$S_t \geq \exp\{-\tilde{c}(d,\nu)t^{\frac{d}{d+2}}\}.$$

Here $\tilde{c}(d,\nu)$ is given by (8).

The optimal choice of $\tilde{R}$ in (12) is given by

$$\tilde{R}_0 = \left(\frac{2\lambda_d}{d\nu\omega_d}\right)^{1/(d+2)}. \tag{13}$$

Again as we shall see in the following section, cf. the so called *Confinement Property* this is not only the good strategy for survival but also the typical one in the annealed case.

3. PINNING EFFECT AND CONFINEMENT PROPERTY

3.1. Pinning Effect

We have described above a possible scenario under which the particle survives, in the quenched case, up to long time t. As we have already pointed out it is far from being clear that the strategy described there is typical. Furthermore it would be of some interest to find out which pockets of low potential are selected to stay in by the particle. To motivate the next result we present the following heuristic after Sznitman (1998).

We recall that according to what we have said in the previous section the presence of Poisson traps causes a localization effect for the particle expressed in the fact that most part of the support of any eigenfunction φ_i of H_ω is localized in a relatively small neighborhood of a certain point x_i and the function decays exponentially, say at the rate $\alpha|x - x_i|$ from its respective domain of localization. Thus the sequence $<\varphi_i, 1>$, $i \geq 1$ appearing in (11) is almost constant of positive value. Using further (11) we can see that the asymptotic behavior of $S_{t,\omega}$ is governed by the term

$$e^{-\min_i(\alpha|x_i|+t\lambda_i)}.$$

Thus the particle should go by time t to some pocket of low potential located near a minimum of a certain random functional of the form

$$\alpha|x| + t\lambda_\omega(B(x, R_t)) \tag{14}$$

with R_t being some random localization scale. Here $\lambda_\omega(U)$ denotes the principal Dirichlet eigenvalue of H_ω in an open set U.

This heuristic picture has some truth in it as it is shown by Theorem 3 below. Before presenting the result let us first introduce some notation. Let us denote by

$$e_\lambda(x, y, \omega) = E_x\left[\exp\left\{-\int_0^{H(y)} (\lambda + V(Z_s, \omega))ds\right\}, H(y) < +\infty\right]. \tag{15}$$

with $H(x)$ - the hitting time of the ball $\overline{B}(x) := \overline{B}(x, 1)$.

The strong Markov property of Brownian Motion implies that

$$e_\lambda(x, y, \omega) \geq e_\lambda(x, z, \omega) \inf_{\overline{B}(z)} e_\lambda(\cdot, y, \omega), \ \lambda \geq 0, \ x, y, z \in \mathbb{R}^d$$

which in turn guarantees the supermultiplicative property of $e_\lambda(x, y, \omega)$. By Liggett's ergodic theorem, cf. e.g. Liggett (1985), it is possible to obtain then the following, cf. Sznitman (1994),

THEOREM 2. — (The shape theorem.) *There exists a deterministic nontrivial norm $\alpha_\lambda(\cdot)$ on $\mathbb{R}^d$ satisfying for any $M > 0$*

$$\lim_{|x| \to +\infty} \sup_{0 \leq \lambda \leq M} \frac{1}{|x|}| - \log e_\lambda(0, x, \omega) - \alpha_\lambda(x)| = 0 \ \mathbb{P} \ \text{a.s.}. \tag{16}$$

Remarks.

1. The name appearing by the above theorem deserves a brief comment. The functions e_λ can be related to site "passage times" considered in the first time percolation theory cf. Kesten (1986), Hammersley-Welsh (1965). The theorem carries then some analogy to the shape theorem proven for the first passage percolation in Kesten (1986). The word "shape" refers here to the closed unit balls of the norm α_λ.
2. One can introduce also a random distance function given by

$$d_\lambda(x, y, \omega) := \max\{-\inf_{\overline{B}(x)} \log e_\lambda(y, \cdot, \omega), -\inf_{\overline{B}(y)} \log e_\lambda(\cdot, x, \omega)\}. \tag{17}$$

It can be proven, cf. Proposition 5.2.2 of Sznitman (1998) that there exists a positive constant C depending only on the dimension d such that for all ω, $\lambda \geq 0$, $x, y \in \mathbb{R}^d$ with $|x - y| > 4$ one has

$$|d_\lambda(x, y, \omega) + \log e_\lambda(x, y, \omega)| \leq C(1 + F_\lambda(x) + F_\lambda(y)). \tag{18}$$

Here F_λ is a certain nonnegative function depending on ω, which is bounded if $d \geq 3$ or $\lambda > 0$ and otherwise of sublogarithmic growth $\mathbb{P}$ a.s., cf. Lemma 1.1 of Wüthrich

(1998). In consequence one can also claim the Shape Theorem with d_λ in place of e_λ. The theorem has actually even broader range of applicability and extends also to Green's function in place of e_λ, cf. Theorem 5.2.5 of Sznitman (1998) for more details.

One can introduce two random scales, $R_{t,\omega}$ satisfying

$$R_{t,\omega} = o(\exp\{(\log t)^{1-\chi}\}). \tag{19}$$

for some $\chi > 0$ and $S_\omega(t)$, which roughly speaking measures the distance of the locus of the farthest minimum of the random functional

$$F_t(x,\omega) := \alpha_0(x) + t\lambda_\omega(B(x, R_{t,\omega})) \tag{20}$$

from the origin. There exists such a $\chi > 0$ for which

$$\frac{t}{(\log t)^{\frac{2}{d}+\chi}} \geq S_\omega(t) \geq \exp\{(\log t)^{1-\chi}\}, \tag{21}$$

cf. Section 6.3 of Sznitman (1998).

The scale $R_{t,\omega}$ is larger than the macroscopic unit scale $(\log t)^{1/d}$ relevant to the quenched case but is much smaller than the scale $S_\omega(t)$ where the farthest near minima sites of $F_t(\cdot,\omega)$ occur. The importance of these scales is highlighted by Theorem 3 below. Before formulating the result let us introduce a bit more notation. For $\omega \in \Omega$ we define the event

$$C := [Z_. \text{ comes before time } t \text{ within distance 1 of some } x \in \mathcal{L}_{c,t,\omega}$$

$$\text{from which then it does not move any further than distance } R_{t,\omega} \text{ up to time } t].$$

Here $\mathcal{L}_{c,t,\omega}$ stands for the skeleton of "near mininima" of $F_t(\cdot,\omega)$, i.e.

$$\mathcal{L}_{c,t,\omega} := [x \in \frac{1}{\sqrt{d}}\mathbb{Z}^d : F_t(\cdot,\omega) \leq \mu_{t,\omega} + cS_\omega(t)]$$

where $\mu_{t,\omega} := \min F_t(\cdot,\omega)$ and $c \in (0,1)$ is a constant. Since $\mu_{t,\omega} \sim c(d,\nu)\frac{t}{(\log t)^{2/d}}$ for $t \gg 1$, cf. (6.3.15) of Sznitman (1998) we can conclude thanks to (21) that $S_\omega(t) \ll \mu_{t,\omega}$ for large t. Hence we can see that the term "near minimum" is indeed well founded.

The statement of the pinning effect is then as follows.

THEOREM 3. — *For a sufficiently small $\chi > 0$ and $\mathbb{P}$ a.s. ω we have*

$$\lim_{t \uparrow +\infty} Q_{t,\omega}(C) = 1.$$

3.2. Confinement Property

We consider here only the hard obstacle case. The entire theory works also for soft obstacles (cf. Povel (1998a)). In this situation, as we recall, the particle gets instantaneously killed upon the contact with the obstacle field. It is convenient to rescale the entire problem so that the true obstacles have size $\varepsilon = t^{-\frac{1}{d+2}}$. The Brownian Motion, after rescaling, is given by $\varepsilon Z_{\frac{s}{\varepsilon^2}}$, $s \geq 0$. We can restrict our attention to those trajectories which do not leave, up to time $s := t^{\frac{d}{d+2}}$, the box $\mathcal{T} := [-t, t]^d$.

We have already mentioned that one possible survival strategy of the particle is to stay inside a ball B_0 centered at 0 of radius $\tilde{R}_0$. This ball minimizes the functional

$$\tilde{c}(d, \nu) := \inf_{U \text{ open}} (\nu |U| + \lambda(U)). \tag{22}$$

It can be proven, cf. Povel (1998) Proposition 1, that with large probability there exist random, open "clearing sets" $\mathcal{U}^{cl}_{\varepsilon,\omega}$ such that they are "almost" obstacle free and their complements are contained in the regions having high concentration of obstacles. In addition by virtue of Proposition 1 of Povel (1998) the clearing sets are "almost" optimal with respect to the variational problem (22) i.e. for some $\chi > 0$ we have

$$\mathbb{P}_\varepsilon \otimes P_0(\nu |\mathcal{U}^{cl}_{\varepsilon,\omega}| + \lambda(\mathcal{U}^{cl}_{\varepsilon,\omega}) \leq \tilde{c}(d, \nu) + \varepsilon^\chi \mid T_{\Theta_\varepsilon} > s) \to 1, \text{ when } t \uparrow +\infty$$

where T_{Θ_ε} is the exit time from the spatially rescaled obstacle free region Θ_ε (cf. (2)) and, as we recall, $s := t^{\frac{d}{d+2}}$. $\mathcal{U}^{cl}_{\varepsilon,\omega}$ can be defined as $\mathcal{T} \cap \mathcal{O}_\varepsilon(\omega) - D_\varepsilon(\omega)$, cf. Section 6 for the definition of $\mathcal{O}_\varepsilon(\omega)$ and $D_\varepsilon(\omega)$.

By Faber-Krahn inequality (cf. e.g. Chavel (1984)) we can conclude that

$$\nu |\mathcal{U}^{cl}_{\varepsilon,\omega}| + \lambda_d (\omega_d / |\mathcal{U}^{cl}_{\varepsilon,\omega}|)^{2/d} \leq \tilde{c}(d, \nu) + \varepsilon^\chi.$$

In consequence $\mathcal{U}^{cl}_{\varepsilon,\omega}$ must have both the volume and principal Dirichlet eigenvalue close to the corresponding quantities for the optimal ball. Using a strengthening of the Faber-Krahn result due to Hall (1992) one can choose a ball B_ω with radius $\approx \tilde{R}_0$ and such that $|\mathcal{U}^{cl}_{\varepsilon,\omega} - B_\omega|$ is small. This ball contains only a tiny portion of obstacles but it is surrounded by a dense forest of traps. Hence the rescaled particle must stay inside the ball if it is to survive up to time s. Along these lines, Sznitman (1991) for $d = 2$ and Povel (1998) have proven that:

THEOREM 4. — *For $d \geq 2$ there exists $\chi_1 > 0$, $1 > \chi_2 > 0$ such that for any ω there is a ball B_0^ω with center in $B(0, (R_0 + \chi_1 t^{-\chi_2/(d+2)}) t^{1/(d+2)})$ and a radius in $[R_0 t^{1/(d+2)}, (R_0 + \chi_1 t^{-\chi_2/(d+2)}) t^{1/(d+2)}]$ for which*

$$\lim_{t \uparrow +\infty} Q_t [T_{B_0^\omega} > t] = 1.$$

Here $T_{B_0^\omega}$ denotes the exit time from the ball B_0^ω.

At this point let us also mention the following result due to Schmock (1990) in $d = 1$, Sznitman (1991) for $d = 2$ and Povel (1998) when $d \geq 3$.

THEOREM 5. — *As $t \uparrow +\infty$, the process $\left\{t^{-\frac{1}{d+2}} Z_{\tau t^{\frac{2}{d+2}}}\right\}_{\tau \geq 0}$ converges in law under Q_t to the mixture with weight $\varphi(x)/\int\varphi$ of the laws of Brownian motion starting from 0 and conditioned not to exit the ball $B(x, \tilde{R}_0)$. φ is the principal eigenfunction of $-\frac{1}{2}\Delta$ in $B(0, \tilde{R}_0)$.*

4. CROSSING BROWNIAN MOTION (CBM)

In this section we assume that the shape function W is rotationally invariant and the resulting random potential V is truncated on a certain level $M > 0$. We shall discuss the quenched case only. To our knowledge there are no corresponding results available for the annealed case.

The law of CBM on $\left(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}_{H(y)}\right)$ is defined by

$$d\hat{P}_x^y := \frac{1}{e_\lambda(x,y,\omega)} \exp\left\{-\int_0^{H(y)} (\lambda + V)(Z_s, \omega)ds\right\} 1_{[H(y)<+\infty]} dP_x,$$

where, as we recall $H(y)$ is the hitting time of the ball $\overline{B}(y)$, $\mathcal{F}_{H(y)}$ its σ-algebra and e_λ is given by (15).

The measure describes the behavior of the Brownian path in the potential $\lambda + V$ conditioned on the event that it performs a crossing from x to y. One of the questions of considerable physical interest related to CBM is an interplay between the strength of the transversal path fluctuation whose magnitude is supposed to be of a typical size $\sim |x-y|^\chi$ and the shape fluctuations of the random sphere given by the distance function defined in (17). The latter are assumed to be of order $|d_\lambda(0,y,\omega) - \alpha_\lambda(y)| \sim |y|^\xi$. ξ must belong to interval $[0,1]$ by virtue of Theorem 2. It has been conjectured (cf. ibid) that the transverse fluctuations should satisfy the scaling identity

$$2\xi - 1 = \chi. \tag{23}$$

More specifically to measure the size of the aforementioned fluctuations let us define

$$\xi_1 := \inf\left\{\gamma \geq 0 : \lim_{|y|\uparrow+\infty} \mathbb{E}\left[\hat{P}_0^y[A(y,\gamma)]\right] = 1\right\},$$

where $A(y,\gamma)$ denotes the event that the trajectory starting at the origin and performing the crossing to $y \in \mathbb{R}^d$ stays inside a truncated cylinder of radius $|y|^\gamma$ centered on the axis l_y: αy, $\alpha \in \mathbb{R}$ and defined as

$$C(y,\gamma) := [z : \mathrm{dist}(z, l_y) \leq |y|^\gamma \text{ and } -|y|^\gamma \leq \langle z, y/|y| \rangle \leq |y| + |y|^\gamma].$$

To measure the shape fluctuation of the distance function $d_\lambda(x,y,\omega)$ around its median $M_\lambda(x,y)$ we introduce

$$\chi^{(1)} := \inf\left\{\kappa \geq 0 : \lim_{|y|\uparrow+\infty} \mathbb{P}[|d_\lambda(0,y,\omega) - M_\lambda(0,y)| \leq |y|^\kappa] = 1\right\}$$

or alternatively

$$\chi^{(2)} := \inf\left\{\kappa \geq 0 : \lim_{r\uparrow+\infty} \mathbb{P}[\sup_{x\in\overline{B}(0,r)} |d_\lambda(0,x,\omega) - M_\lambda(0,x)| \leq r^\kappa] = 1\right\}. \tag{24}$$

The second coefficient measures the fluctuation size in a uniform manner thus obviously $\chi^{(1)} \leq \chi^{(2)}$. The following result is a corollary of Sznitman (1996):

THEOREM 6. — *For all $d \geq 2$ we have*

$$\chi^{(2)} \leq 1/2.$$

We can obtain also, cf. (23), that

$$2\xi_1 - 1 \leq \chi_2.$$

Indeed, in consequence of the rotational invariance and spatial homogeneity of the shape function W we get

$$M_\lambda(x,y) = M_\lambda(0, |y-x|e_1), \tag{25}$$

where $e_1 = (1, 0, \cdots, 0)$. From the Shape Theorem 2 we conclude easily that there exists a certain constant $C > 0$ for which

$$M_\lambda(0,x) \geq C|x| \qquad \text{for a sufficiently large } |x|. \tag{26}$$

Both here and in the sequel C shall stand for any generic constant independent of x, y.

We define C_y a finite set of points on $\partial C(y,\gamma)$ whose cardinality is less than or equal to $C|y|^d$ and such that $\partial C(y,\gamma) \subseteq \bigcup_{z\in C_y} \overline{B}(z)$.

Using the definition (24) of χ_2 we can write that

$$\mathbb{E}[\hat{P}_0^y[A(y,\gamma)^c]] \approx \tag{27}$$

$$\mathbb{E}\left\{A(\varepsilon), \hat{P}_0^y\left[\bigcup_{z\in C_y}[0 < H(z) < H(y) < +\infty]\right]\right\}$$

with

$$A(\varepsilon) := [\omega : \text{ the exponent } \kappa \text{ of the fluctuation of } d_\lambda \text{ does not exceed } \chi^{(2)} + \varepsilon].$$

Applying the strong Markov property of Wiener measure to stopping times $H(z) < H(y)$ we can estimate the probability $\hat{P}_0^y$ on the right hand side of the approximate equality (27) by

$$\frac{C}{e_\lambda(0,y,\omega)}|y|^d \sup_{z\in C_y} e_\lambda(0,z,\omega)e_\lambda(z,y,\omega). \tag{28}$$

Using a version of Harnack principle, cf. (5.2.22) of Sznitman (1998) and the definition (17) we can estimate (28) by

$$(29)\qquad C|y|^d \sup_{z\in C_y} \exp\{d_\lambda(0,y,\omega) - d_\lambda(0,z,\omega) - d_\lambda(z,y,\omega)\} \le$$

$$C|y|^d \sup_{z\in C_y} \exp\left\{M_\lambda(0,y) - M_\lambda(0,z) - M_\lambda(z,y) + 5|y|^{\chi^{(2)}+\varepsilon}\right\}.$$

The last line follows from the fact that $\omega \in A(\varepsilon)$.

Using relation (25) we can write that

$$(30)\qquad M_\lambda(0,z) + M_\lambda(z,y) = M_\lambda(0,|z|e_1) + M_\lambda(|z|e_1,(|y-z|+|z|)e_1) \ge M_\lambda(0,(|y-z|+|z|)e_1).$$

The following lemma is a fairly straightforward consequence of the strong Markov property of Wiener measure and the aforementioned Harnack's principle, cf. Lemma 2.1 in Wüthrich (1998) for the proof.

LEMMA 1. — *There exists a constant $C > 0$ such that*

$$(31)\qquad M_\lambda(0,(|y-z|+|z|)e_1) \ge M_\lambda(0,|y|e_1) + M_\lambda(|y|e_1,(|y-z|+|z|)e_1) - C|y|^{\chi^{(2)}+\varepsilon},$$

for z, y, $y-z$ all sufficiently large in magnitude.

In consequence we can write that the right hand side of (29) can be estimated by

$$(32)\qquad C|y|^d \sup_{z\in C_y} \exp\left\{M_\lambda(0,(|y-z|+|z|-|y|)e_1) + C|y|^{\chi^{(2)}+\varepsilon}\right\}.$$

Applying now an elementary inequality

$$|z| + |z-y| \ge |y| + C|y|^{2\gamma-1}$$

for all $z \in \partial C(y,\gamma)$ and $|y| > 1$ together with (26) we can see from (32) that

$$\mathbb{E}[\hat{P}_0^y[A(y,\gamma)^c]] \le C\exp\{C_1|y|^{\chi_2+\varepsilon} - C_2|y|^{2\gamma-1}\},$$

for some positive constants C, C_1, C_2. The right hand side of the above inequality vanishes for large $|y|$ provided that $2\gamma - 1 > \chi_2 + \varepsilon$. Hence we have concluded the first part of the following result due to Wüthrich (1998).

THEOREM 7. — *For all $d \ge 2$ we have*

1) $\frac{\chi^{(2)}+1}{2} \ge \xi_1$
2) $\xi_1 \ge \frac{\chi_1+\chi_1/\chi_2}{2}$.

Remarks.

1. In order to obtain (23) we need the equality $\chi^{(1)} = \chi^{(2)}$, which is an open problem at this time.
2. It would be of considerable interest to prove a lower bound on $\chi^{(2)}$. It is believed that in two dimensions $\chi^{(2)} = \frac{1}{2}$.

5. INTEGRATED DENSITY OF STATES OF THE RANDOM SCHRÖDINGER OPERATOR

Let us consider a Random Schrödinger operator H_ω given by (10). By $N_{\Lambda,\omega}(\lambda)$ we denote the number of those Dirichlet eigenvalues of H_ω in Λ, counting multiplicities, which are less than or equal to λ. Here $\Lambda = (-l, l)^d$. We define then the thermodynamic limit

$$N(\lambda) := \lim_{l\uparrow+\infty} \frac{1}{|\Lambda|} N_{\Lambda,\omega}(\lambda)$$

whose existence $\mathbb{P}$ a.s. follows from the classical subadditive ergodic theorem, cf. Krengel (1985) or Carmona-LaCroix (1991). $N(\lambda)$ is a deterministic and increasing function called integrated density of states (IDS). In Quantum Physics $N(\lambda)$ describes a volume density of energy states lying below a fixed energy level λ for a Schrödinger operator with random Poissonian impurities. We are interested in determining the low energy level asymptotic of IDS, i.e. the behavior of $N(\lambda)$ when $\lambda \downarrow 0$. We define by $L(t)$ the Laplace transform of the measure $N(d\lambda)$ corresponding to IDS.

The classical Tauberian theorem (cf. e.g. Bingham et al. (1987), p. 254) tells us that the behavior of $N(\lambda)$ for small λ can be determined from the asymptotic behavior of $L(t)$ for large t. To characterize the latter we write

$$L(t) = \int_0^{+\infty} e^{-t\lambda} N(d\lambda) = \lim_{l\uparrow+\infty} \frac{1}{|\Lambda|} \int_0^{+\infty} e^{-t\lambda} N_{\Lambda,\omega}(d\lambda), \tag{33}$$

where $N_{\Lambda,\omega}(d\lambda)$ is the appropriate random measure associated with $N_{\Lambda,\omega}(\lambda)$. Using the trace formula and ergodic theorem we can represent the left hand side of (33) as

$$\mathbb{E} r_\omega(t, 0, 0) \tag{34}$$

with $r_\omega(t, x, y)$ the random kernel associated with an $L^2(R^d)$ strongly continuous semigroup

$$\left(e^{tH_\omega} f\right)(x) := E_x \left[f(Z_t) \exp\left\{ -\int_0^t V(Z_s, \omega) ds \right\} \right] = \int_{R^d} f(y) r_\omega(t, x, y) dy.$$

Using the expectation with respect to the Brownian Bridge measure $P_t^{x,y}$ we obtain that

$$r_\omega(t, x, y) = (2\pi t)^{-d/2} p(t, x, y) E_t^{x,y} \exp\left\{ -\int_0^t V(Z_s, \omega) ds \right\}.$$

The asymptotic of $L(t)$ and S_t for large t can be shown to be identical.

Indeed, thanks to (34) we can write that

$$L(t) = \mathbb{E} \int_{R^d} r_\omega(t-1, 0, y) r_\omega(1, y, 0) dy \le c S_{t-1}.$$

On the other hand using again (34) and denoting by U^a the a-neighborhood of U we obtain

$$L(t) = \frac{1}{|U|}\mathbb{E}\int_U r(t,x,x)dx \geq \frac{1}{|U|}\mathbb{E}[e^{-\lambda_\omega(R^d)t}, \text{ no obstacle falls into } U^a] = \frac{1}{|U|}e^{-\lambda(U)t-\nu|U^a|}$$

and the last quantity is bounded below by

$$e^{-\tilde{c}(d,\nu)t^{\frac{d}{d+2}}(1+O(1))}.$$

for a suitable choice of U. Here $\tilde{c}(d,\nu)$ is given by (8). We recall here that $B(0,a)$ contains the support of W and $\lambda_\omega(\cdot)$, $\lambda(\cdot)$ stand for the local Dirichlet eigenvalues of H_ω, $-\frac{1}{2}\Delta$ respectively. Applying now Theorem 1 we obtain the following.

Theorem 8. — *The integrated density of states satisfies*

$$N(\lambda) = e^{-\nu|B(0,\sqrt{\frac{\lambda_d}{\lambda}})|(1+o(1))} \text{ for } \lambda\downarrow 0. \tag{35}$$

Remarks.

1. It can be proven, cf. Pastur et al. (1992), that in the case when the shape function W decays at infinity at the rate $\sim \frac{C}{|x|^\alpha}$, $|x|\gg 1$ (the so called classical regime), with $\alpha\in(d,d+2)$, the tails of IDS behave like $N(\lambda)\sim\exp\{-C(d,\nu,W)\lambda^{-\frac{d}{\alpha-d}}\}$, $\lambda\ll 1$. The constant $C(d,\nu,W)$ depends then on the shape function which is in sharp contrast with the situation discussed in Theorem 8 where the corresponding constant depended only on the field intensity ν and the dimension d, cf. (35). The result of the theorem can be extended to cover also the case of the shape function having $\alpha > d+2$ (nonclassical regime).
2. Let us also mention here that an analogous result to Theorem 8 has been obtained in the dimension two for the magnetic Schrödinger operator

$$H_\omega := (-i\nabla + A(\cdot))^2 + V(\cdot,\omega)$$

in Erdösz (1998). Here $A(x_1,x_2) = (-\frac{Bx_2}{2},\frac{Bx_1}{2})$. Erdösz has proved that the integrated density of states defined for the bottom of the L^2-spectrum of H_ω, i.e. $\frac{B}{2}$, is given by

$$N(B/2+\lambda)\sim\lambda^{-\frac{2\pi\nu}{B}}$$

for small $\lambda > 0$.

6. THE METHOD OF ENLARGEMENT OF OBSTACLES (MEO)

The problems described in Sections 2, 3.1 and 5 highlight the importance of controls over the local Dirichlet eigenvalues $\lambda_\omega(U)$ of the open set U. Actually the upper bounds of the eigenvalues, which correspond to the lower bounds on the quenched or annealed asymptotics, are quite easy to derive thanks to the monotonicity property of local eigenvalues with respect to their respective domains. This explains relative simplicity of our "derivation" of the lower bounds on both quenched and annealed asymptotics given in Section 2. It is considerably more difficult to obtain lower bounds on $\lambda_\omega(U)$. They allow to control S_t, $S_{t,\omega}$ of Section 2 from above, thanks to the estimates of the type

$$(36)\qquad 0\sup_x E_x\left[\exp\left\{-\int_0^t V(Z_s,\omega)ds\right\}, T_U>t\right]\le c[1+(\lambda_\omega(U)t)^{d/2}]e^{-\lambda_\omega(U)t},$$

cf. (3.1.9) of Sznitman (1998).

In the first step we rescale for convenience space and time by $x'=\varepsilon x$, $t'=\varepsilon^2 t$ with ε corresponding to the inverse of the typical spatial scale of the problem on hand, e.g. the heuristics of Section 2 suggest that ε equals $(\log t)^{-1/d}$ in the quenched and $t^{-1/(d+2)}$ in annealed case. The true obstacles in the rescaled situation are of size $\sim\varepsilon$. We wish to obtain lower bounds of $\lambda^\varepsilon_\omega(U)$ -the Dirichlet principal eigenvalue of the scaled Schrödinger operator $-\frac{1}{2}\Delta+\frac{1}{\varepsilon}V(\frac{\cdot}{\varepsilon})$ consider in a region U.

In U we can distinguish three disjoint subregions corresponding to various degrees of the concentration of obstacles. In the close neighborhood of the first region there is a high density of obstacles so that the particle entering it "feels" them very quickly and thus gets trapped in short time. This fact allows to "solidify" those obstacles by imposing there Dirichlet zero boundary condition, or equivalently to remove this subregion of U from the consideration without significantly influencing the local eigenvalue.

In the second region the concentration of obstacles is too small to perform the kind of "surgery" just described without significant distortion of the magnitude of the local eigenvalue. However it can be proven, cf. the "Volume Control" Theorem 11 below, that this part of U occupies asymptotically vanishing fraction of volume, as $\varepsilon\downarrow 0$, so we can discard it entirely without significantly changing the upper bound on the eigenvalue coming from the Faber-Krahn isoperimetric theorem, cf. Chavel (1984).

The remaining third part of region U receives no obstacles.

Let us explain now the method by getting an upper estimate of the annealed asymptotic of S_t. For simplicity we consider here only the case when $d\ge 3$. First we wish to construct a coarse grained picture of the region where $V>0$. Unfortunately the use of a fine scale, say of order of magnitude of obstacles, will result in too high a number of possible region shapes. In effect we will not be able to obtain any meaningful control of the survival probability. This fact provides motivation to enlarge the true obstacles, which are of size ε, say to the size ε^γ, $0<\gamma<1$ in order to lower the combinatorial complexity of the problem. The *density set* $D_\varepsilon(\omega)$ corresponding to the first region is defined then as the set having a lot of obstacles in its immediate neighborhood.

More specifically for any fixed integer $L>1$ we consider the lattice $\mathbb{L}_{n_\gamma(\varepsilon)}:=\frac{1}{L^{n_\gamma(\varepsilon)}}\mathbb{Z}^d$, with $n_\gamma(\varepsilon):=[\gamma\frac{\log\frac{1}{\varepsilon}}{\log L}]$. We define $D_\varepsilon(\omega)$ as the union of those boxes C from the lattice for which

$$\sum_{n_\alpha(\varepsilon)<k<n_\gamma(\varepsilon)} L^{k(d-2)}\mathrm{cap}_k \geq \delta(n_\gamma(\varepsilon)-n_\alpha(\varepsilon)). \tag{37}$$

Here $\delta > 0$, $0 < \alpha < \gamma$ are some parameters and cap_k denotes the capacity of the obstacles lying in the box from the lattice $\mathbb{L}_k$ containing C. The parameter $n_\alpha(\varepsilon)$ determines the "testing spatial scale" for condition (37) and equals $n_\alpha(\varepsilon) := [\alpha\frac{\log\frac{1}{\varepsilon}}{\log L}]$. Notice that $\varepsilon^\alpha \gg \varepsilon^\gamma \gg \varepsilon$ when $\varepsilon \downarrow 0$.

We have then two crucial spectral controls, cf. Theorems 4.2.3 and 4.2.6 of Sznitman (1998).

THEOREM 9. — *There exists a positive constant $C(d,W)$ depending only on the dimension d and the shape function W such that for any $M > 0$ and $\varrho \in (0, \delta C\frac{\gamma-\alpha}{(d+2)\log L})$ we have*

$$\limsup_{\varepsilon\downarrow 0}\sup_{U,\omega} \varepsilon^{-\varrho}\left(\lambda^\varepsilon_\omega(U-\overline{D}_\varepsilon(\omega))\wedge M - \lambda^\varepsilon_\omega(U)\wedge M\right) = 0$$

where the sup *is taken over any $\omega \in \Omega$ and open $U \subseteq \mathbb{R}^d$.*

To formulate the second result we need a notion of a clearing set. For a given parameter $r > 0$ we declare a box C of the unit lattice $\mathbb{Z}^d$ to be a *clearing box* if

$$|C - D_\varepsilon(\omega)| \geq r^d.$$

If otherwise holds we call C a *forest box.*

Let $\mathcal{A}_\varepsilon(\omega)$ be the union of all clearing boxes. Suppose that $\mathcal{O}_\varepsilon(\omega)$ is the R-neighborhood of $\mathcal{A}_\varepsilon(\omega)$. We have then the following.

THEOREM 10. — *Let $M > 0$. There exist $C(d) > 0$, $C'(d,M) > 1$, $r_0(d,M) \in (0,\frac{1}{4})$ such that*

$$\limsup_{\varepsilon\downarrow 0}\widetilde{\sup}\exp\left\{C\left[\frac{R}{4r}\right]\right\}(\lambda^\varepsilon_\omega(U\cap\mathcal{O}_\varepsilon(\omega))\wedge M - \lambda^\varepsilon_\omega(U)\wedge M) = 0$$

provided that $\varepsilon^\alpha < r < r_0$, $R/(4r) > C'$. $\widetilde{\sup}$ is taken over all configurations ω and open sets U.

For the second region $B_\varepsilon(\omega)$ we take boxes of size $\sim \varepsilon^\beta$ for some $\gamma < \beta < 1$ from the appropriate lattice $\mathbb{L}_{n_\beta(\varepsilon)}$ which are contained in the complement of density boxes and receive some points of the obstacle cloud. We have then

THEOREM 11. — *Let $d \geq 3$. There exists $L_0(d)$ such that for $L \geq L_0$ we have $\delta_0(d,L) > 0$ for which for any $\delta < \delta_0$*

$$\limsup_{\varepsilon\downarrow 0}\sup_C \varepsilon^{-\kappa_0}|B_\varepsilon(\omega)\cap C| < +\infty$$

where sup *is taken over all unit size boxes C, $\kappa_0 > 0$ depends on all the parameters involved except for ε and ω.*

The third and last region is $R^d - (D_\varepsilon(\omega) \cup B_\varepsilon(\omega))$. Of course, as we have already mentioned before, it contains no points of the cloud. Observe that for any unit box C the sets $D_\varepsilon(\omega) \cap C$ and $B_\varepsilon(\omega) \cap C$ can take no more than $2^{\varepsilon^{-d\gamma}}$ and $2^{\varepsilon^{-d\beta}}$ different possible shapes respectively. This should be contrasted with the number of possible shapes of the true obstacle set which is of order of complexity $2^{\varepsilon^{-d}} \gg 2^{\varepsilon^{-d\beta}} \gg 2^{\varepsilon^{-d\gamma}}$, when $\varepsilon \downarrow 0$.

We apply the above framework to obtain an upper bound for the annealed norming constant S_t. We start by choosing a large box $\mathcal{T} := (-t,t)^d$.

Let us define now the "essential" event E consisting of those clouds for which $\lambda_\omega^\varepsilon(\mathcal{T}) \leq 2\tilde{c}(d,\nu)$ and $\mathcal{A}_\varepsilon(\omega) \cap \mathcal{T}$ consists of at most n_0 clearing boxes where

$$\frac{\nu}{2} n_0 r^d \geq 2\tilde{c}(d,\nu).$$

We can easily observe that

$$\mathbb{P}_\varepsilon[E^c] \leq (4t)^{dn_0} \mathbb{P}_\varepsilon[[0,1)^d \subseteq \mathcal{A}_\varepsilon(\omega)]^{n_0}. \tag{38}$$

From the definition of the clearing set and Theorem 11 we can conclude that

$$\mathbb{P}_\varepsilon[[0,1)^d \subseteq \mathcal{A}_\varepsilon(\omega)] = \mathbb{P}_\varepsilon[|[0,1)^d - D_\varepsilon(\omega)| \geq r^d] \leq$$

$$\mathbb{P}_\varepsilon[|[0,1)^d - (D_\varepsilon(\omega) \cup B_\varepsilon(\omega))| \geq r^d - \varepsilon^{\kappa_0}].$$

Since the set $[0,1)^d - (D_\varepsilon(\omega) \cup B_\varepsilon(\omega))$ contains no point of the cloud we can estimate the left hand side of (38) by

$$2^{2\varepsilon^{-d\beta}} \exp\left\{-\nu\varepsilon^{-d} r^d\right\} \leq \exp\left\{-\frac{\nu}{2}\varepsilon^{-d} r^d\right\}$$

for sufficiently small ε (or equivalently for sufficiently large t).

From (38) and (6) we obtain that S_t is, up to term of order $e^{-2\tilde{c}(d,\nu)s}$ ($s = t^{\frac{d}{d+2}}$), less than or equal to

$$\mathbb{E}_\varepsilon \otimes E_0 \left[\exp\left\{-\int_0^s V_\varepsilon(Z_u)du\right\}, E\right].$$

We are ready now to apply Theorems 9, 10 and 11. Set $M := 2\tilde{c}(d,\nu)$. Define

$$U(\omega) := \mathcal{T} \cap \mathcal{O}_\varepsilon(\omega) - D_\varepsilon(\omega)$$

$$V(\omega) := \mathcal{T} \cap \mathcal{O}_\varepsilon(\omega) - (D_\varepsilon(\omega) \cup B_\varepsilon(\omega)).$$

From Theorem 11 we obtain that

$$|U(\omega)| \leq |V(\omega)| + |\mathcal{T} \cap \mathcal{O}_\varepsilon(\omega)|\varepsilon^{\kappa_0} \leq |V(\omega)| + (2R+1)^d n_0 \varepsilon^{\kappa_0}. \tag{39}$$

Observe that the number of possible shapes of $U(\omega)$, $V(\omega)$ cannot exceed $2^{C\varepsilon^{-d\beta}}$, for some $C > 0$. For a given deterministic U_0 and V_0 we consider the event $\mathcal{G}_{U_0,V_0}$ consisting of those ω for which $U(\omega) = U_0$, $V(\omega) = V_0$. Thanks to (36) we can estimate as follows

$$\mathbb{E}_\varepsilon \otimes E_0 \left[\mathcal{G}_{U_0,V_0} \cap E, \exp\left\{ - \int_0^s V_\varepsilon(Z_u)du \right\} \right] \leq$$

$$c(d)(1 + (2\tilde{c}s)^{d/2})\mathbb{E}_\varepsilon \otimes E_0 \left[\mathcal{G}_{U_0,V_0} \cap E, \exp\left\{ -(\lambda_\omega^\varepsilon(\mathcal{T}) \wedge M)s \right\} \right] \leq$$

$$c(d)(1 + (2\tilde{c}s)^{d/2})\mathbb{P}_\varepsilon[\mathcal{G}_{U_0,V_0}] \exp\left\{ -\lambda_\omega^\varepsilon(\mathcal{T} \cap \mathcal{O}_\varepsilon(\omega)) \wedge Ms \right\}.$$

In the last line we used Theorem 10. Applying Theorem 9 we obtain then from (39) that

$$S_t \leq 2^{C\varepsilon^{-d\beta}} \exp\{- \inf_{\text{all admissible } U_0} (\lambda(U_0) + \nu|U_0|) \wedge Mt^{\frac{d}{d+2}} + \text{ smaller order terms}\} \leq$$

$$\exp\{-\tilde{c}(d,\nu)t^{\frac{d}{d+2}} + \text{ smaller order terms}\}.$$

This proves the upper bound in part 2) of Theorem 1.

7. OPEN PROBLEMS

Let us start our review of some of the open questions with the problem of defining a "right" notion of random geodesics for Brownian motion moving among random traps. We have already mentioned in Section 4 that $d_\lambda(\cdot,\cdot,\omega)$ defined by (17) is a metric. However it is not a geodesic type of a distance. One can verify that

$$d_\lambda(x,y,\omega) + d_\lambda(y,z,\omega) = d_\lambda(x,z,\omega)$$

implies that at least two of x, y, z must be identical. In this context it is worthwhile to mention the existence of a continuous analogue of the first passage time geodesic given by

$$\varrho_\lambda(x,y,\omega) := \inf_{\gamma \in \mathcal{P}(x,y,1)} \left\{ \int_0^1 \sqrt{2(\lambda + V)(\gamma_s,\omega)}|\gamma'(s)|ds \right\}.$$

Here $\mathcal{P}(x,y,1)$ is the set of Lipschitz paths γ leading from x to y in time 1. It can be verified that the analogue of the Shape Theorem 2 holds for ϱ_λ with the limiting deterministic norm denoted by $\mu_\lambda(\cdot)$. On the other hand one can obtain, cf. Wüthrich (1998a) that for any Euclidean norm unit vector e

$$\mu_\lambda(e) = \lim_{\beta \uparrow +\infty} \frac{\alpha_{\beta,\lambda}(e)}{\sqrt{\beta}}$$

where $\alpha_{\beta,\lambda}$ is the norm resulting from a shape theorem for the quantities

$$e_{\beta,\lambda}(x,y,\omega) := E_x\left[\exp\left\{-\beta\int_0^{H(y)}(\lambda+V(Z_s,\omega))ds\right\}, H(y)<+\infty\right], \tag{40}$$

cf. Theorem 2. One can view then $-\log e_{\beta,\lambda}(x,y,\omega)/\sqrt{\beta}$ for large β as some sort of distance function. In determining the path integral one uses, in light of the fact that the potential becomes infinite for $\beta\uparrow+\infty$, only the most "optimal" paths which so to say perform the fastest possible crossings. The proof that the appropriate limit exists is not available yet. It leads also to the question of determining random "pseudo" geodesics, i.e. tubes of width small in comparison with the crossing distance which contains with large probability the paths of the CBM. This problem has also connections with better formulation of the scaling identity (23) in terms of the intrinsic geodesic instead of Euclidean cylinders.

In the context of the results presented in Section 5, it is worthwhile to mention that the approach to the asymptotic of IDS via Tauberian theory is not entirely satisfactory. A direct proof of the Lifshitz tail could be constructed via counting the number of energy states lying below certain λ for the Dirichlet problem in a sufficiently large "a priori" box. It is clear that the typical (unit) size associated with the problem is of order of magnitude $1/\sqrt{\lambda}$. Using MEO it should be possible to show that one can in fact put a Dirichlet boundary on the forest part of the box without distorting significantly the behavior of IDS for small λ. The expected result would probably yield asymptotic of IDS up to higher order terms, cf. Theorem 8.

Finally let us also mention that the random scales $S_\omega(t)$ and $R_{t,\omega}$ introduced in the subsection discussing the pinning effect are rather poorly understood at this time. They characterize the location of the farthest near minimum of the random functional (20) and the size of the pocket in which a pinned Brownian Motion is located after arriving in the vicinity of a relevant near minimum. It would be desirable to get better asymptotic of those scales than those provided by (19) and (21). In addition it would be of considerable interest to obtain some better understanding of the magnitude of times at which the particle arrives at the pinning location. The results in that direction concern mostly the one dimensional case. A reader is encouraged to consult Section 6.5 of Sznitman (1998) for details.

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