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HARMONIC FUNCTIONS ON LOOP GROUPS

par **Leonard GROSS**

1. INTRODUCTION

The loop space, $C(S^1, M)$, of a compact, finite dimensional, Riemannian manifold, M, supports some natural measures, w, associated to Brownian motion in M. There is also a gradient operator, ∇, on functions defined over $C(S^1, M)$. The Dirichlet form, $\int_{C(S^1,M)} |\nabla F(x)|^2 dw(x)$, associated to the pair, w, ∇, determines a natural "Laplacian", $\nabla^*\nabla$, on functions over the loop space. The harmonic functions for this Laplacian will be discussed and characterized in this lecture in case M is a compact Lie group and the loops start and end at the identity element. This is part of a long range goal of finding a theorem of Hodge–deRham type over these particularly interesting infinite dimensional manifolds. Although the objective of the investigations to be surveyed here is analysis over the infinite dimensional manifold $C(S^1, M)$, one of the most interesting byproducts has been the discovery of some surprising unitary transforms for functions over the finite dimensional compact Lie group M itself. In these "finite dimensional" theorems one typically operates in Hilbert spaces of the form $L^2(M, \rho_t(x)dx)$ where $\rho_t(x)dx$ is a heat kernel measure rather than Haar measure. Since these latter theorems can be stated without use of stochastic processes, I will describe them before the discussion of the infinite dimensional loop group theory that led to their discovery.

2. HEAT KERNELS ON LIE GROUPS

Denote by G a connected Lie group. Suppose that $\langle\ ,\ \rangle$ is an inner product on its Lie algebra, $\mathbf{g} := T_e(G)$. For each element $\xi \in \mathbf{g}$ denote by $\widetilde{\xi}$ the left invariant extension of ξ to G. For any orthonormal basis $e_1, \ldots, e_n$ of $\mathbf{g}$ the operator

$$\Delta = \sum_{j=1}^{n} (\widetilde{e}_j)^2 \tag{2.1}$$

is a second order differential operator on G. As an operator in $L^2(G$, right invariant Haar measure), Δ is a symmetric operator on the domain $C_c^\infty(G)$. Moreover it is essentially self–adjoint on this domain and $-\Delta$ is nonnegative. Consequently there is a semigroup, $e^{(t/2)\Delta}$, of bounded operators on L^2 whose infinitesimal generator is (the self–adjoint version of) $(1/2)\Delta$. It happens that if G is unimodular then Δ is the Laplace–Beltrami operator for the left invariant Riemannian metric on G which agrees with $\langle\ ,\ \rangle$ on $T_e(G)$.

The fundamental fact we will need is that the operators $e^{(t/2)\Delta}$ are given by convolution by a particularly nice family of functions, p_t, on G. Specifically, for each $t > 0$, there exists a unique function p_t on G such that

(i) $p_t(x) > 0$ for $t > 0$ and $x \in G$.

(ii) $p_t \in C^\infty(G)$ for $t > 0$.

(iii) $\int_G p_t(x)dx = 1$ where dx is right invariant Haar measure.

(iv) $(e^{(t/2)\Delta}f)(x) = \int_G f(xy^{-1})p_t(y)dy$ for $f \in L^2(G, dx)$.

The function p_t is the *heat kernel* for $(1/2)\Delta$ at time t. For a more detailed discussion of heat kernels on Lie groups and references to proofs see [23] or [73].

The following theorems concern the Hilbert spaces $L^2(G, p_t(x)dx)$. Thus the heat kernel measures $p_t(x)dx$ will replace Haar measure in the customary Hilbert spaces $L^2(G, dx)$. The resulting analysis has come to be called "heat kernel analysis" on G, as distinguished from harmonic analysis. For us, G will usually be either a compact Lie group or a complex Lie group.

3. HEAT KERNEL ANALYSIS OVER LIE GROUPS

Let K be a compact, connected Lie group. Denote by $\langle\ ,\ \rangle$ an Ad K invariant inner product on its Lie algebra, $\mathbf{k} \equiv T_e(K)$. Define the Laplacian on K by Equ.(2.1) and denote the associated heat kernel on K by ρ_t. Next, consider the complexification, K_c, of K [49]. K_c is a complex Lie group which contains K as a closed subgroup. The Lie algebra of K_c is $\mathbf{k}_c \equiv \mathbf{k} \otimes_{\mathbb{R}} \mathbb{C}$. The prototypical example is $K = SU(2)$, in which case $K_c = SL(2, \mathbb{C})$. Now in general, the given inner product on $\mathbf{k}$ extends uniquely to a Hermitian inner product on the complex Lie algebra $\mathbf{k}_c$. Using the real part of this inner product, we may once more apply the procedure of Section 2 to construct a Laplacian, Δ_c, on K_c. For example if $e_1, \ldots, e_d$ is an orthonormal basis of $\mathbf{k}$ then, writing $i = \sqrt{-1}$, we have

$$\Delta_c = \sum_{j=1}^{d} (\tilde{e}_j^2 + (\widetilde{ie_j})^2).$$

Moreover, Section 2 assures us, taking $G = K_c$, that there is a heat kernel, μ_t, on K_c determined by this Laplacian. But we will choose the time parameter in μ_t so that

$$e^{(t/4)\Delta_c} = \mu_t * .$$

Write $\mathcal{H} = \mathcal{H}(K_c)$ for the space of holomorphic complex valued functions on K_c. As is well known, and easy to verify in local holomorphic coordinate charts, $\mathcal{H} \cap L^2(K_c, \mu_t)$ is a closed subspace of $L^2(K_c, \mu_t)$.

Theorem 3.1 (Hall's transform [49]).– Let $f \in L^2(K, \rho_t(x)dx)$. Then $\rho_t * f$ has a unique analytic continuation to all of K_c. Denote the analytic continuation by $\mathcal{H}_t f$. Then

$$\mathcal{H}_t : L^2(K, \rho_t dx) \to \mathcal{H} \cap L^2(K_c, \mu_t(y)dy)$$

is unitary.

Remark 3.2.– Since Hall's paper [49], there have been two other proofs of Hall's theorem. In [19], B. Driver removed the dependence of Hall's proof on the structure theory of semi–simple Lie algebras and otherwise simplified the proof. In [48] another proof was given which depends on stochastic analysis and on the ergodicity theorem of Section 5 of this survey. Driver's paper [19] also extends Hall's theorem to groups of compact type, thereby encompassing the classical case – the Segal-Bargmann transform, which transforms functions on $\mathbb{R}^n$ to holomorphic functions on $\mathbb{C}^n$. See [48] for an exposition of the extensive history of the Segal-Bargmann transform, which is the $\mathbb{R}^n$ predecessor of Theorem 3.1.

Remark 3.3.– All proofs of Theorem 3.1 depend heavily on the Ad K invariance of the given inner product $\langle\ ,\ \rangle$ on $\mathbf{k}$. It seems likely that this is an essential condition. The next theorem, however, which takes place entirely on the "complex side" does not require an Ad invariant inner product.

Let G be a connected complex Lie group and $\mathbf{g} \equiv T_e(G)$ its complex Lie algebra. Denote by $(\ ,\ )$ a Hermitian inner product on $\mathbf{g}$. As before, we denote by Δ_c the Laplacian on G defined as in Equ. (2.1), using the real inner product $\mathrm{Re}(\cdot, \cdot)$. μ_t will denote the heat kernel for $e^{(t/4)\Delta_c}$. G need not be the complexification of a real Lie group and $(\ ,\ )$ need not be Ad invariant under any particular subgroup. We wish to discuss the Taylor coefficients of functions in $\mathcal{H}(G) \cap L^2(G, \mu_t)$.

Let $T \equiv T(\mathbf{g})$ be the tensor algebra over the complex vector space $\mathbf{g}$. Denote by J the two sided ideal in T generated by $\{\xi\otimes\eta - \eta\otimes\xi - [\xi, \eta] : \xi, \eta \in \mathbf{g}\}$. The universal enveloping algebra of $\mathbf{g}$ is $\mathcal{U} \equiv T/J$. Any function $f \in \mathcal{H}(G)$ defines an element $\widehat{f}$ in the algebraic dual space $\mathcal{U}'$ as follows. If β is a left invariant differential operator on G and $f \in \mathcal{H}(G)$ then the map $\alpha : \beta \to (\beta f)(e)$ is complex linear (by the Cauchy–Riemann equations for f). As usual we may identify $\mathcal{U}$ with the left invariant differential operators on $\mathcal{H}(G)$. So the linear functional $\widehat{f} \equiv \alpha$ is in $\mathcal{U}'$. $\widehat{f}$ is the set of all (left invariant) derivatives of f, evaluated at e. The map $f \to \widehat{f}$ is the *Taylor map* from $\mathcal{H}(G)$ into $\mathcal{U}'$. It is convenient to identify $\mathcal{U}'$ with $\{\alpha \in T' : \alpha(J) = 0\}$ where T' denotes the algebraic dual space. We may use this to define a norm as follows. T' is the strong direct sum (i.e., direct product) $\sum_{n=0}^{\infty}(\mathbf{g}')^{\otimes n}$. Denote again by (,) the Hermitian inner product on $\mathbf{g}'$ induced by the given Hermitian inner product on $\mathbf{g}$. Then we may define, for any element $\alpha \in T'$,

$$\|\alpha\|_t^2 = \sum_{n=0}^{\infty} t^n (n!)^{-1} |\alpha_n|^2_{(\mathbf{g}')^{\otimes n}}, \; \alpha \in T', \tag{3.1}$$

where

$$\alpha = \sum_{n=0}^{\infty} \alpha_n \quad \alpha_n \in (\mathbf{g}')^{\otimes n} \quad n = 0, 1, 2, \dots .$$

Of course the annihilator, J^0, is exactly $\mathcal{U}'$. Let

$$J_t^0 = \{\alpha \in T' : \|\alpha\|_t < \infty, \; \alpha(J) = 0\}. \tag{3.2}$$

Then J_t^0 is a complex Hilbert space in the norm (3.1) and is contained properly in $\mathcal{U}'$.

Theorem 3.4 ([19, 23]).– Let $t > 0$. Then the Taylor map $f \to \widehat{f}$ is isometric from $\mathcal{H} \cap L^2(G, \mu_t)$ into J_t^0. I.e.,

$$\int_G |f(g)|^2 d\mu_t(g) = \|\widehat{f}\|_t^2.$$

Moreover if G is simply connected then the Taylor map is surjective (and therefore unitary).

This kind of theorem has also been proven recently for an infinite dimensional group, a natural subgroup of the complex orthogonal group on a Hilbert space, [39].

Let us combine Theorems 3.1 and 3.4: If K is a connected, compact Lie group with an Ad K invariant inner product on its Lie algebra then we may first apply $\mathcal{H}_t$ to a function f in $L^2(K, \rho_t dx)$ and then apply the Taylor map to the holomorphic function $\mathcal{H}_t f$. Since the complexification of the universal enveloping algebra of $\mathbf{k}$ is the universal enveloping algebra of $\mathbf{k}_c$ we have

Corollary 3.5.– Let $t > 0$. Let K be a compact, connected Lie group with an Ad K invariant inner product $\langle\ ,\ \rangle$ on its Lie algebra, $\mathbf{k}$. For the Hermitian extension of $\langle\ ,\ \rangle$ to $\mathbf{k}_c$, define J_t^0 as in (3.2). Then the map

$$\mathcal{D}_t f = (\mathcal{H}_t f)\widehat{\ }$$

is an isometry from $L^2(K, \rho_t dx)$ into J_t^0. It is unitary if K is simply connected.

It is useful to give an intrinsic characterization of the isometry $\mathcal{D}_t$ in terms of natural operators. For any element $\xi \in \mathbf{k}$ let R_ξ denote right multiplication by ξ in the tensor algebra $T(\mathbf{k}_c)$. Since $R_\xi J \subset J$ the adjoint $R_\xi^* : T' \to T'$ leaves the annihilator J^0 invariant. Fix $t > 0$ and denote by A_ξ the restriction of R_ξ^* to J_t^0. R_ξ raises rank by one and A_ξ therefore lowers rank by one. It is a naturally occurring operator in many contexts, including quantum field theory. The "annihilation operator" A_ξ is to be interpreted as the (actually unbounded) operator in J_t^0 with domain $\{\alpha \in J_t^0 : A_\xi \alpha \in J_t^0\}$.

Corollary 3.6 ([42]).– Let $t > 0$ and assume that K is compact, connected and simply connected. Then $\mathcal{D}_t : L^2(K, \rho_t(x)dx) \to J_t^0$ is the unique unitary operator such that

(a) $\mathcal{D}_t 1 =$ the zero rank tensor $(= 1)$

and

(b) $\mathcal{D}_t \widetilde{\xi} = A_\xi \mathcal{D}_t$.

Here $\widetilde{\xi}$ is the left invariant vector field on K extending ξ and is to be interpreted as the closed operator in $L^2(K, \rho_t dx)$ with core $C^\infty(K)$.

4. THE GRADIENT OPERATOR OVER A PATH GROUP AND LOOP GROUP

Denote again by K a compact connected Lie group and by $\mathbf{k} := T_e(K)$ its Lie algebra. Once again we fix an Ad K invariant inner product on $\mathbf{k}$.

There are several groups consisting of K valued functions which will be of interest to us. The *path group* of K is the set

$$\mathcal{P} = \{k \in C([0,1]; K) : k(0) = e\}.$$

The *loop* group of K is the set

$$\mathcal{L} = \{k \in \mathcal{P} : k(1) = e\}.$$

$\mathcal{P}$ is clearly a topological group under pointwise multiplication and uniform convergence, while $\mathcal{L}$ is a closed subgroup of $\mathcal{P}$. In addition, there are the finite energy versions of these two groups. The *finite energy path group* is the set

$$\mathcal{P}_{fe} = \Big\{k \in \mathcal{P} : k \text{ is absolutely continuous and } \int_0^1 |k(s)^{-1}\dot{k}(s)|_{\mathbf{k}}^2 ds < \infty\Big\}.$$

Here, as in the following, we will use matrix notation $k(s)^{-1}\dot{k}(s)$ for the translate of the tangent vector $\dot{k}(s)$ back to the identity element of K. The *finite energy loop group* is

$$\mathcal{L}_{fe} = \mathcal{L} \cap \mathcal{P}_{fe}.$$

It is not hard to see that $\mathcal{P}_{fe}$ is a dense subgroup of $\mathcal{P}$ while $\mathcal{L}_{fe}$ is a dense subgroup of $\mathcal{L}$. We are going now to define tangent spaces to the two finite energy groups.

Let

$$\begin{aligned} H \;=\; & \Big\{h \in C([0,1]; \mathbf{k}) : h \text{ is absolutely continuous, } h(0) = 0 \text{ and} \\ & |h|^2 := \int_0^1 |\dot{h}(s)|_{\mathbf{k}}^2 ds < \infty\Big\}. \end{aligned}$$

Then H is a Hilbert space and $H_0 := \{h \in H : h(1) = 0\}$ is a closed subspace. Let $\exp : \mathbf{k} \to K$ denote the exponential map and define

$$(e^h)(s) = \exp h(s), \ 0 \le s \le 1, \ h \in H.$$

Then $e^h(s)$ is in K for each s and it is elementary that the function $s \mapsto e^h(s)$ is of finite energy. Moreover it can easily be shown, e.g. [40] [Lemma 2.1], that the map $h \to e^h(\cdot)$ takes a small neighborhood of 0 in H in a one to one way onto a neighborhood of the identity function, $e(\cdot) : s \mapsto e$, in $\mathcal{P}_{fe}$. We are justified, therefore, in regarding the Hilbert space H as the tangent space to the finite energy path group, $\mathcal{P}_{fe}$, at the identity element. Clearly $e^h \in \mathcal{L}_{fe}$ if $h \in H_0$. So we may similarly identify H_0 with the tangent space to $\mathcal{L}_{fe}$ at the identity function. It is clear that the function $\mathbb{R} \ni t \mapsto e^{th}$ is a one parameter group in $\mathcal{P}_{fe}$ (respectively $\mathcal{L}_{fe}$) if $h \in H$ (respectively H_0). The two finite energy groups are examples of Hilbert Lie groups.

The right action of $\mathcal{P}_{fe}$ on $\mathcal{P}$ allows one to define the directional derivative of functions on $\mathcal{P}$ as follows. Let $F : \mathcal{P} \to \mathbb{R}$ be any function. For any element $h \in H$ define

$$(\partial_h F)(k) = \frac{d}{dt} F(ke^{th})\big|_{t=0} \quad k \in \mathcal{P}, \ h \in H, \tag{4.1}$$

if the derivative exists. We are only going to be interested in such directional derivatives in the finite energy directions $h \in H$ because these directional derivative operators, ∂_h, relate well to the integration theory over $\mathcal{P}$ to be described later. Specifically, an integration by parts formula holds for ∂_h if $h \in H$ but not if h is merely continuous. Note that for each $h \in H$ the operator ∂_h defines a left invariant vector field on the path group $\mathcal{P}$.

Now if $F : \mathcal{P} \to \mathbb{R}$ is such that $h \to (\partial_h F)(k)$ is linear and continuous on H then the *gradient* of F at k is the element $\nabla F(k) \in H$ defined by

$$(\nabla F(k), h)_H = (\partial_h F)(k) \quad h \in H. \tag{4.2}$$

As an example, suppose that $u \in C^\infty(K^n)$ and that $0 < s_1 < s_2 < \cdots < s_n \le 1$. Define

$$F(k) = u(k(s_1), k(s_2), \ldots, k(s_n)). \tag{4.3}$$

One verifies easily that $\nabla F(k)$ exists for all $k \in \mathcal{P}$. This class of smooth cylinder functions plays an important technical role, but fails to separate some interesting sets, as we will see in the Example 4.1 below. But first let us observe that these differentiation notions make sense on the loop space $\mathcal{L}$ also, if one replaces H by H_0. Thus if $F : \mathcal{L} \to \mathbb{R}$ and $k \in \mathcal{L}$ and $h \in H_0$ then (4.1) is a meaningful definition of $\partial_h F(k)$ because ke^{th} is in $\mathcal{L}$ for all t. We may now define $\nabla F(k) \in H_0$ by (4.2) with h restricted, of course, to be in H_0.

Example 4.1.– Let $K = SO(3)$. $\mathcal{L}$ is now a disjoint union of two closed sets, the two homotopy classes, because the fundamental group of K is Z_2. Let $F(k) = 1$ if k is homotopic to the constant function and let $F(k) = 0$ on the other homotopy class. F is not a cylinder function. But it is infinitely differentiable. In fact $\partial_h F(k) = 0$ for all $k \in \mathcal{L}$ and $h \in H_0$

because ke^{th} is in the same homotopy class as k for all t. Thus $\nabla F = 0$. The main theorem of the next section asserts that the only harmonic functions on $\mathcal{L}$ are the functions which are constant on each homotopy class in $\mathcal{L}$.

5. BROWNIAN MOTION MEASURE OVER THE PATH GROUP AND LOOP GROUP

Continuing the notation of Section 4, denote by ρ_t the associated heat kernel on K. There exists a unique probability measure, P, on the Borel field of the path group $\mathcal{P}$ with the following properties.

(i) for $0 \le s < t \le 1$ and any Borel set $B \subset K$

$$P(\{k \in \mathcal{P} : k(s)^{-1}k(t) \in B\}) = \rho_{t-s}(B)$$

(ii) if $0 = s_0 < s_1 < s_2 < \cdots < s_n \le 1$ then the K valued functions $\mathcal{P} \in k \mapsto k(s_{i-1})^{-1}k(s_i), i = 1, .., n$, are independent.

Actually such a measure exists even if K is not compact and the inner product is not Ad K invariant. In his fundamental paper, [75], Wiener proved the existence of this measure (Wiener measure) when $K = \mathbb{R}$. For a general Lie group the reader could consult [22, 26, 54, 62, 64, 72] for existence and properties of P.

In so far as we may regard $\mathcal{P}$ as an infinite dimensional manifold, it is the measure P which will play for us the role of "Riemann–Lebesgue" measure. In order to carry out our analysis of the desired Laplacian on $\mathcal{P}$ it is necessary, here as in finite dimensions, to understand integration by parts. To this end it is essential to understand first the properties of P under translation.

Theorem 5.1.– Let $k_0 \in \mathcal{P}$. The translated measure $P(\cdot k_0)$ is absolutely continuous with respect to P if and only if $k_0 \in \mathcal{P}_{fe}$. If $k_0 \in \mathcal{P}_{fe}$ one has a Radon–Nikodym derivative

$$dP(gk_0)/dP(g) = J_{k_0}(g).$$

Moreover if $h \in H$ then the function $t \to J_{e^{th}}(\cdot)$ from $\mathbb{R}$ into $L^p(P)$ is differentiable for all $p < \infty$. Its derivative

$$j_h(g) = \frac{d}{dt} J_{e^{th}}(g)\big|_{t=0} \quad \text{is in} \quad L^p(P)\ \forall\, p < \infty.$$

This theorem has a long history. For a proof and some variations of the theorem see [67].

Remark 5.2.– Since $\mathcal{P}_{fe}$ is a group, the theorem shows that $P(\cdot k_0)$ is equivalent to $P(\cdot)$ for any element $k_0 \in \mathcal{P}_{fe}$. One refers to this as *quasi-invariance* of P under $\mathcal{P}_{fe}$. But P is never invariant under such translations. It is because of this quasi-invariance theorem that the finite energy subgroup $\mathcal{P}_{fe}$ plays a central role in analysis over $\mathcal{P}$. It should be emphasized, however, that one cannot do away with the rest of $\mathcal{P}$. $\mathcal{P}_{fe}$ is a set of P measure zero! Thus one needs $\mathcal{P}$ to carry the measure and $\mathcal{P}_{fe}$ to determine the allowed translations. This is typical of infinite dimensional integration theory.

The quasi-invariance of P under right translation by $\mathcal{P}_{fe}$ has its counterpart at the infinitesimal level. This is the basis of integration by parts.

Corollary 5.3.– If F_1 and F_2 are smooth cylinder functions (cf. Equ. (4.3)) and $h \in H$, then

$$\int_{\mathcal{P}} (\partial_h F_1)(k) F_2(k) dP(k) = \int_{\mathcal{P}} F_1(k)[-\partial_h F_2(k) + j_h(k) F_2(k)] dP(k). \tag{5.1}$$

Remark 5.4.– Denote by $\mathcal{C}^\infty$ the smooth cylinder functions. $\mathcal{C}^\infty$ is dense in $L^2(P)$. Equation (5.1) shows that in the Hilbert space $L^2(\mathcal{P}, P)$, the operator $\partial_h \mid \mathcal{C}^\infty$ has an adjoint ∂_h^* given on $\mathcal{C}^\infty$ by

$$\partial_h^* \mid \mathcal{C}^\infty = -\partial_h + j_h.$$

In particular the adjoint of $\partial_h \mid \mathcal{C}^\infty$ is densely defined. Therefore $\partial_h \mid \mathcal{C}^\infty$ has a closed extension, which I will simply denote by ∂_h again. It is an important, though seemingly technical matter, to understand the domains of these closed operators ∂_h, first on $\mathcal{P}$ and later on $\mathcal{L}$. For example the function given in Example 4.1 is far from being a cylinder function; one can't determine the homotopy class of a curve just from a knowledge of the curve at finitely many time points. Yet it is precisely these functions (indicator functions of homotopy classes) which will be the harmonic functions for our Laplacian. Functions of this kind are in the domain of the closed operator ∂_h.

Let $h_1, h_2, \ldots$ be an orthonormal basis of H. Define

$$N = \sum_{j=1}^{\infty} \partial_{h_j}^* \partial_{h_j}.$$

The quadratic form of N is thus the Dirichlet form

$$(NF, F) = \sum_{j=1}^{\infty} \int_{\mathcal{P}} |\partial_{h_j} F(k)|^2 dP(k).$$

The sum on the right is independent of the orthonormal basis of H. Actually this operator is quite well understood because of its role in quantum field theory. Its spectrum is $\{0, 1, 2, \ldots\}$ and N is referred to as the number operator. The version of this operator of interest for this survey is the analog of this Dirichlet form operator over the loop group $\mathcal{L}$.

We will now describe the simple modifications necessary to define the corresponding "Laplacian" over $\mathcal{L}$.

First, the probability measure P_0 on $\mathcal{L}$ which will replace P is the so called pinned Brownian motion measure. By definition this is the conditional measure $P(\ \mid k(1) = e)$ on

$\mathcal{L}$. Moreover, as in the unpinned case, an integration by parts formula is a consequence of the following quasi–invariance theorem.

Theorem 5.5 [63].– P_0 is quasi-invariant under right translations by elements of $\mathcal{L}_{fe}$.

Corollary 5.6 [42].– Let $h \in H_0$. Then $\partial_h \mid \mathcal{C}^\infty$, as a densely defined operator in $L^2(\mathcal{L}, P_0)$, has a unique closed extension, which we also denote by ∂_h. Let $h_1, h_2, \dots$ be an orthonormal basis of H_0. The operator

$$N_0 \equiv \sum_{j=1}^{\infty} \partial_{h_j}^* \partial_{h_j}$$

is a densely defined self–adjoint operator in $L^2(\mathcal{L}, P_0)$ and is independent of the choice of orthonormal basis.

Theorem 5.7 (Main Theorem) [42].– Assume that K is a compact, connected Lie group with Ad K invariant inner product on Lie K. Then the null space of N_0 is spanned by the indicator functions of homotopy classes in $\mathcal{L}$. [Cf. Example 4.1.] In particular, if K is simply connected then the null space of N_0 is spanned by the constant functions.

In order to understand the position of this theorem in the rest of mathematics it is convenient to reformulate it as an ergodicity theorem. Suppose, again, that K is simply connected. $\mathcal{L}_{fe}$ is dense in $\mathcal{L}$ and so are all the orbits of $\mathcal{L}_{fe}$. But, as is well known, this in itself does not imply that the right action of $\mathcal{L}_{fe}$ on $\mathcal{L}$ is ergodic.

Theorem 5.8 (Main Theorem′) [42].– The right action of $\mathcal{L}_{fe}$ on $(\mathcal{L}, P_0)$ is ergodic. That is, if $F : \mathcal{L} \to \mathbb{R}$ is measurable and, for all $k_0 \in \mathcal{L}_{fe}$, $F(kk_0) = F(k)$ a.e. $[P_0]$, then there is a constant c such that $F(k) = c$ a.e. on $\mathcal{L}$.

It should be clear from the previous discussion that the very existence of N_0 (and of N) as a self-adjoint operator depends on having an integration by parts formula, which itself depends on having a quasi-invariance theorem. For this reason any attempt to develop a harmonic analysis over the path or loop space of a general compact Riemannian manifold will require a quasi-invariance theorem as the first step. The prototypes of such theorems are the quasi-invariance theorems of Cameron and Martin, developped in the 1940s for the case in which the target manifold is just the real line, and in particular, Cameron's integration by parts formula, [10]. For a general compact target manifold, a breakthrough in this direction was made by B. Driver [16]. This has been followed by extensive development in the past six years. For a small sample of the work that has been inspired by [16] see [1, 2, 3, 4, 6, 7, 11, 12, 13, 14, 17, 20, 21, 24, 25, 29, 30, 31, 53, 59, 74] and many other works by these authors and their coauthors. Some other important works related to analysis of Dirichlet form operators over loop spaces are listed in the Bibliography.

The abstract machinery for a Hodge-de Rham type theorem in finite or infinite dimensions is well understood [9]. However a key input to that machinery is the existence of a spectral gap at eigenvalue zero. (See the survey [45] for further discussion of this.) In the present, loop group, case, the desired objective is the proof that zero is an isolated eigenvalue of our "Laplacian" N_0. In spite of much effort, this has not been settled at the present time. Theorem 5.7 may be regarded as only a small step in that direction. Some of the previous references focus on the spectral gap problem over $\mathcal{P}$ itself (e.g. [31]) or on the more difficult problem of proving a logarithmic Sobolev inequality [3, 4, 11, 55, 57]. Usually the context for this work has been a path space rather than a loop space. However, for a different and equally natural measure on loop groups, a logarithmic Sobolev inequality has recently been established [24].

6. PROOF OF MAIN THEOREM

The objective in this section is to sketch how the finite dimensional theorems of Section 3 are linked to the proof of Theorems 5.7 and 5.8. For this purpose we will take K to be simply connected. It is only Corollary 3.6 which is needed in the proof. This corollary was proved by stochastic methods in [42] and stimulated the subsequent work [15, 19, 23, 39, 43, 44, 46, 47, 48, 49, 50, 51, 52, 65], much of which is not stochastic in nature.

At the present time it is unavoidable to use a substantial portion of Itô's stochastic calculus in the proof of Theorems 5.7 and 5.8, even with the use of Corollary 3.6. For an exposition of Itô's stochastic calculus that is well adapted to the present Lie group context see [22, 26, 58, 62, 64, 71, 72]. For details of the following assertions see [42].

Let $b(s)$, $0 \le s \le 1$, denote $\mathbf{k}$ valued Brownian motion with covariance determined by the given Ad K invariant inner product $\langle\ ,\ \rangle$. The solution, $g(\cdot)$, to the stochastic differential equation

$$dg(s) = g(s) \circ db(s) \text{ (Stratonovich)}, \quad g(0) = e \tag{6.1}$$

maps the Wiener process $b(\cdot)$ to the space $\mathcal{P}$ and induces the measure P on $\mathcal{P}$. In order to show that the only functions on $\mathcal{L}$ that are invariant under the right action of $\mathcal{L}_{fe}$ are the constants, it suffices to show that the only functions on $\mathcal{P}$ that are invariant under the right action of $\mathcal{L}_{fe}$ are functions of the endpoint, $g(1)$. Similarly, to show that the only harmonic functions on $\mathcal{L}$ are constants, it suffices to show that the only functions on $\mathcal{P}$ for which $\partial_h F = 0$ for all $h \in H_0$ are the functions of the endpoint. To this end it suffices to consider a function F in $L^2(\mathcal{P}, P)$ and investigate its structure under the assumption that $F(gk) = F(g)$ a.e. $[P]$ for all k in $\mathcal{L}_{fe}$. Now any (say real valued) function in $L^2(\mathcal{P}, P)$ has a unique Itô multiple integral representation:

$$F(g) = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle v_n(\mathbf{s}), db(s_1) \otimes \cdots \otimes db(s_n) \rangle \quad \text{(Itô)} \tag{6.2}$$

where $\Delta_n = \{(s_1, s_2, \ldots, s_n) : 0 < s_1 < s_2 < \cdots < s_n < 1\}$ and $v_n : \Delta_n \to (\mathbf{k}')^{\otimes n}$ satisfy standard square integrability conditions. This is the expansion which plays for us the same role as the double Fourier series expansion in the classical proof of ergodicity of the irrational flow on the torus.

The first step in the analysis of the $\mathcal{L}_{fe}$ invariant function F is a characterization of the expansion coefficients $\{v_n(\cdot)\}$.

Proposition 6.1 (Theorem 5.1 of [42]).– F is right $\mathcal{L}_{fe}$ invariant if and only if each $v_n(\cdot)$ is constant a.e on Δ_n, say $v_n(s) = \alpha_n \in (\mathbf{k}')^{\otimes n}$, and $\alpha \equiv (\alpha_0, \alpha_1, \alpha_2, \dots)$ (which is in $T(\mathbf{k})')$ is in J_t^0, with $t = 1$.

The proof of Theorems 5.7 and 5.8 may now be concluded with the help of Corollary 3.6. Any function u in $L^2(K, \rho_t(x)dx)$ defines a right $\mathcal{L}_{fe}$ invariant function F in $L^2(\mathcal{P}, P)$ by means of the equation

$$F(g) = u(g(1)). \tag{6.3}$$

It is to be shown that all $\mathcal{L}_{fe}$ invariant functions F in L^2 have this form. But Proposition 6.1 shows that any $\mathcal{L}_{fe}$ invariant L^2 function has an Itô expansion whose expansion coefficients $\{v_n(\cdot)\}$ are given by an element α in J_1^0. Moreover the characterization given in Corollary 3.6 can easily be used to show that the map $L^2(K, \rho_1(x)dx) \ni u \to F \to \alpha$ is exactly the map $\mathcal{D}_1$ of Corollary 3.6. Since $\mathcal{D}_1$ is surjective, every $\mathcal{L}_{fe}$ invariant function F in $L^2(\mathcal{P}, P)$ has the form (6.3), thereby concluding the proof of the Main Theorems of Section 5.

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