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**CURVES AND THEIR FUNDAMENTAL GROUPS**  
[following Grothendieck, Tamagawa and Mochizuki]

by Gerd FALTINGS

**1. INTRODUCTION**

The fundamental group of a topological space is one of the most elementary invariants in algebraic topology. In fact many interesting topological spaces (for example negatively curved) are  $K(\pi, 1)$ 's, and thus their fundamental groups classify them completely up to homotopy. Similarly hyperbolic spaces (constant curvature  $-1$ ) of dimension at least three are up to isometry determined by their group (Mostow).

In SGA1, A. Grothendieck has defined the algebraic fundamental group of a scheme as the profinite group which classifies finite étale coverings. For a complex algebraic variety it is equal to the profinite completion of the topological fundamental group. However if the variety  $X$  is defined over a subfield  $K$  of  $\mathbb{C}$  there is an additional structure. Namely the geometric fundamental group fits into an exact sequence

$$0 \longrightarrow \pi_1(X \otimes_K \bar{K}) \longrightarrow \pi_1(X) \longrightarrow \text{Gal}(\bar{K}/K) \longrightarrow 0,$$

thus defining a homomorphism

$$\text{Gal}(\bar{K}/K) \longrightarrow \text{Out}(\pi_1(X \otimes_K \bar{K})),$$

which determines the class of the extension if  $\pi_1(X \otimes_K \bar{K})$  has trivial center.

We owe to A. Grothendieck the idea that this extension may encode many algebraic properties of  $X$ . Obviously one should restrict to varieties which classically would be  $K(\pi, 1)$ 's, so that for example projective spaces should be excluded. Grothendieck conjectures the existence of a "natural" class of anabelian schemes" (over fields  $K$ ), which should have the following properties:

- a)  $X$  is anabelian  $\iff X \otimes_K \bar{K}$  is anabelian.
- b) If  $X \rightarrow Y$  is proper and smooth (a fibration) and  $Y$  as well as the fibers are anabelian, so is  $X$ .

c) Hyperbolic curves are anabelian.

d) The moduli stacks  $\mathcal{M}_{g,n}$  (classifying curves of genus  $g$  with  $n$  punctures) are anabelian.

He then conjectures that for anabelian schemes  $X$  over a field  $K$  which is finitely generated over  $\mathbb{Q}$ , the functor which sends  $X$  to its fundamental group  $\pi_1(X)$  is fully faithful. Here maps  $\pi_1(X) \rightarrow \pi_1(Y)$  should be (and will be for the sequel of this talk)  $\pi_1(Y \otimes_K \bar{K})$ -conjugacy-classes of maps of extensions

$$0 \longrightarrow \pi_1(X \otimes_K \bar{K}) \longrightarrow \pi_1(X) \longrightarrow \text{Gal}(\bar{K}/K) \longrightarrow 0.$$

Especially rational points  $X(K)$  should correspond bijectively to conjugacy-classes of sections

$$s : \text{Gal}(\bar{K}/K) \longrightarrow \pi_1(X).$$

In many cases the Mordell-Weil theorem already implies that  $X(K)$  injects into the space of conjugacy-classes of sections. In this talk I want to report on recent progress about these conjectures. Firstly A.Tamagawa has shown that for (projective and irreducible) curves  $X$  over a finite field  $K$ , one can characterise the sections  $s$  coming from  $K$ -rational points entirely in terms of the groups. We present this here (in somehow modified form) to introduce a very important technique, namely to consider not only  $X$  but also an infinite number of coverings of it. This dates back to earlier work of H. Nakamura, and has been used by S. Mochizuki to show our main result:

Namely if  $K$  is a finitely generated extension of  $\mathbb{Q}_p$ , then for hyperbolic curves  $X_1$  and  $X_2$  over  $K$ , any open map of extensions

$$\pi_1(X_1) \longrightarrow \pi_1(X_2)$$

is induced from a (unique and dominant) map

$$X_1 \longrightarrow X_2.$$

This implies the corresponding result for number fields.

The proof uses  $p$ -adic Hodge theory, and we sketch the relevant facts. Also for simplicity we restrict ourselves to finite extensions of  $\mathbb{Q}_p$  (local fields) and projective curves. Finally we do not discuss the vast amount of literature dealing with the problem how to recover a field from its absolute Galois group (for example [P]). I thank S. Mochizuki for his help. The talk is based on his ideas. However in some arguments I have used different approaches, to further scientific diversity.

**2. PRELIMINARIES**

For a connected scheme  $X$  and a geometric point  $x$  of  $X$ , P. Grothendieck defined in SGA1 the algebraic fundamental group  $\pi_1(X, x)$ . It is a profinite compact topological group, such that the category of finite etale coverings is equivalent to that of finite sets with continuous  $\pi_1(X, x)$ -action. To a finite etale covering

$$f : Y \longrightarrow X$$

one associates the set  $f^{-1}(x)$ . Here connected coverings correspond to homogeneous  $\pi_1(X, x)$ -sets, and connected pointed coverings to open subgroups of  $\pi_1(X, x)$ . The fundamental group is a covariant functor for pointed maps. In general for different base-points the corresponding fundamental groups are isomorphic, the isomorphism being unique up to conjugation. Hence for arbitrary maps we obtain a functor into the category of groups with conjugacy classes of homomorphisms. If we neglect base-points we use the notation  $\pi_1(X)$ . For a scheme  $X$  of finite type over the complex numbers  $\mathbb{C}$  the algebraic fundamental group is the profinite completion of the topological fundamental group of  $X(\mathbb{C})$ . So for example for a smooth connected projective curve  $X$  of genus  $g$  the fundamental group  $\pi_1(X, x)$  is (profinutely) generated by  $2g$  elements

$$a_1, \dots, a_g, b_1, \dots, b_g,$$

subject to the single relation

$$a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdot \dots \cdot a_g \cdot b_g \cdot a_g^{-1} \cdot b_g^{-1} = 1.$$

It is known that one obtains the same group for any such curve over an algebraically closed field of characteristic zero. In positive characteristic the fundamental group is a quotient of the above. As a warmup we prove

1. *Lemma.*— *Suppose  $X$  is a smooth projective irreducible curve over an algebraically closed field  $K$ , of genus  $g \geq 2$ . Then  $\pi_1(X, x)$  has trivial center.*

*Proof.*— Suppose  $\sigma$  lies in the center. Then for any connected finite etale covering  $Y \rightarrow X$ ,  $\sigma$  defines an automorphism of  $Y$  which acts on the fundamental group  $\pi_1(Y, y)$  by inner automorphism. Especially  $\sigma$  acts trivially on the etale cohomology  $H^1(Y, \mathbb{Z}_\ell)$ ,  $\ell$  any prime invertible in  $K$ . As  $Y$  has also genus bigger than one  $\sigma$  must be the identity on  $Y$ , and as this holds for all  $Y$  we have  $\sigma = 1$ .

**2. COROLLARY.**— *Suppose  $X$  is a smooth projective geometrically irreducible curve over an arbitrary field  $K$ , of genus  $g \geq 2$ . Consider the extension [SGA1]*

$$0 \longrightarrow \pi_1(X \otimes_K \bar{K}) \longrightarrow \pi_1(X) \longrightarrow \text{Gal}(\bar{K}/K) \longrightarrow 0.$$

*The isomorphism class of this extension is then uniquely determined by the induced map*

$$\text{Gal}(\bar{K}/K) \longrightarrow \text{Out}(\pi_1(X \otimes_K \bar{K})) \quad (= \text{automorphisms/inner automorphisms}).$$

More precisely for two such curves  $X$  and  $Y$ , any conjugacy-class of surjective homomorphisms  $\pi_1(X \otimes_K \bar{K}) \rightarrow \pi_1(Y \otimes_K \bar{K})$  respecting these Galois actions extends to a homomorphism of extensions  $\pi_1(X) \rightarrow \pi_1(Y)$ , unique up to conjugation by  $\pi_1(Y \otimes_K \bar{K})$ .

*Proof.*—  $\pi_1(X)$  is the fibered product of  $\text{Aut}(\pi_1(X \otimes_K \bar{K}))$  and  $\text{Gal}(\bar{K}/K)$ , over  $\text{Out}(\pi_1(X \otimes_K \bar{K}))$ . Here as topology on  $\text{Aut}(\pi_1(X \otimes_K \bar{K}))$  one uses uniform convergence.

### 3. CURVES OVER FINITE FIELDS

In this section we suppose that  $K$  is a finite field, thus  $\text{Gal}(\bar{K}/K)$  is free cyclic generated by Frobenius. Suppose  $X$  is a projective smooth geometrically irreducible curve over  $K$ . Then the number of  $K$ -rational points of  $X$  is equal to the alternating sum of the traces of (geometric) Frobenius on  $\ell$ -adic étale cohomology. Especially this number can be read off from the action of Frobenius on  $\pi_1(X \otimes_K \bar{K})$  (even the maximal abelian quotient suffices). Now suppose given a section

$$s : \text{Gal}(\bar{K}/K) \longrightarrow \pi_1(X)$$

of the projection. For any characteristic open subgroup  $H$  of  $G = \pi_1(X \otimes_K \bar{K})$  we obtain a subgroup  $H \cdot s(\text{Gal}(\bar{K}/K))$  of  $\pi_1(X)$ , which corresponds to an étale covering  $Y = Y(H) \rightarrow X$ . Here  $Y$  is also geometrically irreducible, and  $H \cdot s(\text{Gal}(\bar{K}/K))$  is equal to  $\pi_1(Y)$ .  $Y$  is a  $K$ -model of the Galois-cover of  $X \otimes_K \bar{K}$  defined by  $H$ . In fact the section  $s$  is defined by a rational point  $x \in X(K)$ , then choosing  $x$  as base point we see that  $Y$  is a pointed covering, that is  $x$  lifts to a point  $y \in Y(K)$ . There is a converse:

**3. THEOREM (A. Tamagawa).**— *Suppose each  $Y(H)$  has a  $K$ -rational point. Then  $s$  is defined by a unique point  $x \in X(K)$ .*

*Proof.*— We may assume that  $X$  has genus  $\geq 1$ . The non trivial part is to show that  $s$  comes from a section if each  $Y(H)(K)$  is non empty which we now assume. For each integer  $n$  let  $H_n \subset G$  the intersection of all kernels of homomorphisms from  $G$  to  $\mathbb{Z}/n \cdot \mathbb{Z}$ , and

$$f_n : Y_n = Y(H_n) \longrightarrow X$$

the corresponding etale covering of  $X$ . Then the pullback in etale cohomology

$$f_n^* : H^1(X \otimes_K \overline{K}, \mathbb{Z}/n \cdot \mathbb{Z}) \longrightarrow H^1(Y_n \otimes_K \overline{K}, \mathbb{Z}/n \cdot \mathbb{Z})$$

vanishes, and by flat duality the same holds for the pushforward (now in flat cohomology)

$$f_{n,*} : H^1(Y_n \otimes_K \overline{K}, \mu_n) \longrightarrow H^1(X \otimes_K \overline{K}, \mu_n).$$

That means the trace-map on Jacobians  $J_n \rightarrow J$  vanishes on  $K$ -valued  $n$ -torsion-points, and after composing with a high enough power of Frobenius the trace becomes divisible by  $n$ , as a homomorphism of abelian varieties defined over  $K$ . Now if  $y$  and  $z$  are two points in  $Y_n(K)$  their difference  $y - z$  is a divisor of degree zero and defines a point in  $J_n(K)$ . Its image under the trace-map is fixed by Frobenius, and thus lies in  $n \cdot J(K)$ . It is the line bundle defined by the divisor  $f_n(y) - f_n(z)$ . As  $J(K)$  is finite we can find an integer  $n$  which annihilates it. Then  $f_n(y)$  and  $f_n(z)$  define the same line bundle, thus must coincide. That is  $f_n(Y_n(K)) \subset X(K)$  contains at most one element.

Also any  $Y(H)$  is as good as  $X = Y(G)$  for that argument. Thus we find a characteristic  $H' \subset H$  with  $Y(H')(K) \rightarrow Y(H)(K)$  constant. As each  $Y(H)(K)$  was supposed to be non empty we derive that the projective limit of all  $Y(H)(K)$  contains precisely one element, given by a coherent system of  $K$ -points  $y(H)$  of  $Y(H)$ . In turn these define a system of sections

$$s(H) : \text{Gal}(\overline{K}/K) \longrightarrow H \cdot s(\text{Gal}(\overline{K}/K)),$$

well defined up to  $H$ -conjugation and (up to that) coherent. Especially for any  $H$ ,  $s(G)$  is modulo  $H$   $G$ -conjugate to  $s$ . By compactness of  $G$  it follows that  $s(G)$  is conjugate to  $s$ , so  $s$  comes from the (unique) point  $y(G)$ .

Without uniqueness the proof simplifies, as the filtering projective limit of non empty finite sets is non empty itself.

## 4. GALOIS COHOMOLOGY AND DIFFERENTIALS

Suppose  $V$  is a complete discrete valuation ring with fraction field  $K$  of characteristic 0, uniformiser  $\pi$ , and residue field  $k = V/\pi$ .  $V$  perfect of characteristic  $p > 0$ . Suppose furthermore that  $R$  is a  $V$ -algebra of finite type etale over  $V[u, v]/(u \cdot v - \pi)$  (semistable singularity), or a strict henselisation of such, or an adic completion. We denote by  $\omega_{R/V} = R \cdot du/u$  its universal (finite) module of relative logarithmic differentials. For simplicity we also assume that  $R \otimes_V \bar{K}$  is an integral domain. This always holds if  $k$  is algebraically closed, as the integral closure of  $V$  in  $R$  is unramified over  $V$ .

We denote the fundamental group of  $\text{Spec}(R \otimes_V \bar{K})$  by  $\Delta$ . If  $\bar{R}$  denotes the normalisation of  $R$  in the union of all integral finite etale coverings of  $\text{Spec}(R \otimes_V \bar{K})$ , then  $\Delta$  acts continuously on  $\bar{R}$ , and  $\bar{R}$  contains the subextension  $R_\infty$  obtained by adjoining all  $p$ -power roots of  $u$  and  $v$  (the local parameters which were assumed to exist), and also of  $\pi$ . It is shown in [F] that  $\bar{R}$  is almost etale over this subextension, which means the following:

$R_\infty$  is the union of regular rings  $R_n$ . If  $S_\infty \subset \bar{R}$  is the normalisation of  $R_\infty$  in a finite etale covering of  $R_\infty \otimes_V K$ , then for big  $n$  this covering is defined over  $R_n \otimes_V K$ , and we denote the corresponding normalisation by  $S_n$ . As  $R_n$  is regular and  $S_n$  is normal of dimension  $\leq 2$  we know that  $S_n$  is a projective  $R_n$ -module. Thus the determinant of the trace form on  $S_n$  is an invertible ideal in  $R_n$  which divides a power of  $\pi$  (or  $p$ ). Then almost etale means that this power can be chosen arbitrarily small, provided  $n$  becomes big enough.

The notion of almost etale goes back to Tate, who used it to compute the continuous cohomology of  $\text{Gal}(\bar{K}/K)$  acting on Tate twists of  $\bar{K}^\wedge$  ( $p$ -adic completion). Namely if  $V_\infty$  denotes the normalisation of  $V$  in a ramified  $\mathbb{Z}_p$ -extension of  $K$ , then  $\bar{V}$  is almost etale over  $V_\infty$ . As a consequence one gets that  $H^i(\text{Gal}(\bar{K}/K), \bar{K}^\wedge(n))$  vanishes if  $i > 1$  or  $n \neq 0$ , and has dimension one over  $K$  if  $n = 0$ , and  $i = 0, 1$ . Similarly we can (almost) compute the Galois cohomologies  $H^*(\Delta, \bar{R}/p^n \cdot \bar{R})$  and  $H^*(\Delta, \bar{R}^\wedge)$  (continuous cohomology of  $p$ -adic completion). If for simplicity we denote by “=” equality up to  $p$ -torsion annihilated by any  $p$ -power with exponent  $> 1/(p-1)$ , then

$$\begin{aligned} H^0(\Delta, \bar{R}^\wedge) &= (R \otimes_V \bar{V})^\wedge, \\ H^1(\Delta, \bar{R}^\wedge) &\text{ “=” } \omega_{R/V} \otimes_V \bar{V}^\wedge(-1) \quad (\text{Tate-twist}) \\ H^i(\Delta, \bar{R}^\wedge) &\text{ “=” } (0) \quad \text{for } i > 1. \end{aligned}$$

As a consequence one can give a proof for the Hodge-Tate decomposition. Suppose  $X$  is a projective smooth curve over  $K$ . Then

$$H^1(X \otimes_K \bar{K}, \mathbb{Q}_p) \otimes \bar{K}^\wedge \cong H^1(X, \mathcal{O}_X) \otimes \bar{K}^\wedge \oplus H^0(X, \omega_{R/K}) \otimes \bar{K}^\wedge(-1).$$

Namely for the proof we may replace  $K$  by a finite extension and assume that  $X$  has a semistable model over  $V$ . Then for each small enough open affine  $\text{Spec}(R)$  in this model the Galois-cohomology  $H^*(\Delta, \bar{R}^\wedge \otimes_V K)$  is computed by a canonical bar-complex. Covering the model by such affines, one constructs out of these a double complex (with additional Čech-type differentials), whose cohomology  $\mathcal{H}^*(X)$  is independent of choices. Also from the “local” calculations (plus technical remarks, like that the cohomology of a projective formal scheme over  $\bar{V}^\wedge$  is equal to its algebraic cohomology, or that a spectral sequence degenerates because “different Tate-twists do not interact”), we have

**4. THEOREM.—**

$$\mathcal{H}^*(X) = H^*(X, \mathcal{O}_X) \otimes \bar{K}^\wedge \oplus H^{*-1}(X, \omega_{X/K}) \otimes \bar{K}^\wedge(-1).$$

Also as  $\mathbb{Z}_p \subset \bar{R}$ , there is a canonical map

$$H^*(X \otimes_K \bar{K}, \mathbb{Q}_p) \otimes \bar{K}^\wedge \longrightarrow \mathcal{H}^*(X),$$

which is then shown to be an isomorphism.

Finally for any smooth  $\mathbb{Q}_p$ -sheaf  $\mathbb{L}$  on  $X$  we can define  $\mathcal{H}^*(X, \mathbb{L})$  as above, replacing  $\bar{R}^\wedge$  by  $\bar{R}^\wedge \otimes \mathbb{L}$ , and again there is a natural map

$$H^*(X \otimes_K \bar{K}, \mathbb{L}) \otimes \bar{K}^\wedge \longrightarrow \mathcal{H}^*(X, \mathbb{L}).$$

One can show that this is always an isomorphism. For unipotent  $\mathbb{L}$ 's (that is repeated extensions of sheaves induced from the base) this follows already from the result for  $\mathbb{Q}_p$ , and this will be all we need in the sequel.

Following Mochizuki we consider (for a projective geometrically irreducible curve  $X$  over  $K$ ) the maximal pro- $p$  quotient  $\pi_1(X \otimes_K \bar{K})^{(p)}$  of the geometric fundamental group. It is (as all pro- $p$  groups) topologically nilpotent, and has over  $\mathbb{Q}_p$  an algebraic hull  $G$ . Here  $G$  is the projective limit of unipotent algebraic groups over  $\mathbb{Q}_p$ , and thus entirely determined by its Lie-algebra  $\mathfrak{g}$ . Abstractly  $G$  can be generated  $p$ -adically



by  $2g$  generators  $a_1, \dots, a_g, b_1, \dots, b_g$  subject to the usual relation, and  $\mathfrak{g}$  by their logarithms (also subject to one relation). We denote by  $Z^n(G)$  (respectively  $Z^n(\mathfrak{g})$ ) the descending central series, so

$$Z^0(G) = G, Z^0(G)/Z^1(G) = G^{ab} = T_p(J) \otimes \mathbb{Q}_p, \text{ etc.}$$

The Galois group  $\text{Gal}(\overline{K}/K)$  acts on  $G$  and  $\mathfrak{g}$  via outer automorphisms. These become a real action if we choose a section

$$s : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

of the projection, and anyway the quotients  $\text{gr}_Z^n(G) = \text{gr}_Z^n(\mathfrak{g})$  are canonical  $\text{Gal}(\overline{K}/K)$ -modules. After base extension to  $\overline{K}^\wedge$  these modules have a Hodge-Tate decomposition, with occurring Tate-twists by integers between 0 and  $n$ . As Galois linear extensions between vector spaces with different Tate-twists must split, this lifts for each  $n$  to a direct sum decomposition of  $(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes \overline{K}^\wedge$ , which depends on the choice of  $s$ . However it defines a  $G$ -invariant filtration

$$E^m(\mathfrak{g}/Z^n(\mathfrak{g})) = \text{sum of all Tate-twists by integers } \geq m,$$

which is independent of the choice of  $s$ . Thus  $\mathfrak{g} \otimes \overline{K}^\wedge$  and  $G(\overline{K}^\wedge)$  have canonical filtrations  $E^m(\mathfrak{g})$ , respectively  $E^m(G(\overline{K}^\wedge))$ . The quotient  $\mathfrak{h} = \text{gr}_E^0(\mathfrak{g})$  can be easily shown (for example by our next arguments) to be a free pro-unipotent Lie-algebra in  $g$  generators, and also  $H = \text{gr}_E^0(G)$  is a free pro-unipotent group in  $g$  generators over  $\overline{K}^\wedge$ , on which  $\text{Gal}(\overline{K}/K)$  acts (depending on  $s$ ). Mochizuki defines the section  $s$  to be Hodge-Tate if this action is base-extended from the trivial action (over  $K$ ), that is if  $\mathfrak{h}$  is generated by its Galois-invariants. More generally  $s$  is called Hodge-Tate up to level  $n$  if this holds module  $Z^n(\mathfrak{h})$ . He proves:

**5. THEOREM.**— *If  $s$  comes from a  $K$ -rational point, then it is Hodge-Tate.*

*Proof.*— Mochizuki uses a deformation argument to reduce to the ordinary case. Here we present an alternative, using the faithful representation of  $H$  (or  $\mathfrak{h}$ ) on the completed enveloping algebra of  $\mathfrak{h}$ . As  $\mathfrak{h}$  is free in  $g$  generators this enveloping algebra is a free tensor-algebra.

Let  $x \in X(K)$  induce  $s$ . Firstly  $H^1(X, \mathcal{O}_X)$  classifies vector bundles on  $X$  which are trivialised in  $x$ , and extensions of  $\mathcal{O}_X$  by  $\mathcal{O}_X$ . Thus if  $\Omega = H^0(X, \omega_{X/K})$  denotes its dual, there exists a universal such extension

$$0 \longrightarrow \Omega \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

One computes that the map

$$H^1(X, \mathcal{H}om(\mathcal{E}_1, \mathcal{O}_X)) \longrightarrow H^1(X, \mathcal{H}om(\Omega \otimes_K \mathcal{O}_X, \mathcal{O}_X)) = (\Omega^{\otimes 2})^{\text{dual}}$$

is an isomorphism, and gets a universal extension

$$0 \longrightarrow \Omega^{\otimes 2} \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow 0,$$

trivialised at  $x$ .

Continuing gives a projective system of extensions

$$0 \longrightarrow \Omega^{\otimes n} \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow 0.$$

Next assume that  $X$  admits a semistable model. We construct inductively  $\overline{K}^\wedge$ -valued smooth sheaves  $\mathbb{L}_n$  on  $X$ , trivialised along the section  $x$ , such that for a small enough affine  $\text{Spec}(R)$  of the semistable model we have functorially in  $R$

$$\mathbb{L}_n \otimes \overline{R}^\wedge \cong \mathcal{E}_n \otimes_R \overline{R}^\wedge.$$

The  $\mathbb{L}_n$  correspond to continuous representations of  $\pi_1(X, x)$  on finite dimensional  $\overline{K}^\wedge$ -vector spaces, whose restriction to  $s(\text{Gal}(\overline{K}/K))$  is the base-extension (from  $K$  to  $\overline{K}^\wedge$ ) of the direct sum of all  $\Omega^{\otimes m}$ ,  $0 \leq m \leq n$ . Furthermore they are extensions

$$0 \longrightarrow \Omega^{\otimes n} \otimes_K \overline{K}^\wedge \longrightarrow \mathbb{L}_n \longrightarrow \mathbb{L}_{n-1} \longrightarrow 0.$$

Start with the constant  $\mathbb{L}_0 = \overline{K}^\wedge$ . Given  $\mathbb{L}_{n-1}$ , the cohomology

$$H^1(X \otimes_K \overline{K}, \mathcal{H}om(\mathbb{L}_{n-1}, \overline{K}^\wedge)) = \mathcal{H}^1(X, \mathcal{H}om(\mathbb{L}_{n-1}, \overline{K}^\wedge))$$

has a Hodge-Tate decomposition, which (because of the isomorphisms  $\mathbb{L}_n \otimes \overline{R}^\wedge = \mathcal{E}_n \otimes_R \overline{R}^\wedge$ ) is equal to

$$H^1(X, \mathcal{H}om(\mathcal{E}_{n-1}, \mathcal{O}_X)) \otimes_K \overline{K}^\wedge \oplus H^0(X, \mathcal{H}om(\mathcal{E}_{n-1}, \omega_{X/K})) \otimes_K \overline{K}^\wedge(-1).$$

If we identify the étale  $H^1$  with isomorphism classes of smooth sheaves over  $X \otimes_K \overline{K}$ , the second direct summand measures the “local” extension obtained after tensoring with  $\overline{R}^\wedge$ . Thus to the first direct summand there corresponds a universal “locally trivial” extension  $\mathbb{L}_n$  of the type as above, trivialised at  $x$ . It is unique up to unique isomorphism, thus  $\text{Gal}(\overline{K}/K)$ -invariant, and we are done. Now the  $\mathbb{L}_n$

correspond to representations of the algebraic group  $G$  which must factor over  $H$ , and the corresponding representation of the Lie-algebra  $\mathfrak{h}$  on the projective limit is isomorphic to that on the completed enveloping algebra, hence faithful. As we know how  $s(\text{Gal}(\overline{K}/K))$  acts on it we derive that  $\mathfrak{h}$  is generated by its Galois invariants, so  $s$  is Hodge-Tate.

6. *Remark.*— a) Suppose

$$s' : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

is another section, and consider the extension

$$0 \longrightarrow \text{gr}_Z^1(\mathfrak{h}) \longrightarrow \mathfrak{h}/Z^2(\mathfrak{h}) \longrightarrow \text{gr}_Z^0(\mathfrak{h}) \longrightarrow 0,$$

with  $\text{Gal}(\overline{K}/K)$  acting via  $s'$ . As canonically

$$\begin{aligned} \text{gr}_Z^0(\mathfrak{h}) &= \text{Hom}_K(\Omega, \overline{K}^\wedge), \\ \text{gr}_Z^1(\mathfrak{h}) &= \Lambda^2 \text{gr}_Z^1(\mathfrak{h}) = \text{Hom}_K(\Lambda^2 \Omega, \overline{K}^\wedge), \end{aligned}$$

this corresponds to a class

$$c(s, s') \in \text{Hom}_K(\Lambda^2 \Omega, \Omega) \otimes_K H^1(\text{Gal}(\overline{K}/K, \overline{K}^\wedge)),$$

which must vanish if  $s'$  is also Hodge-Tate.

One can compute  $c(s, s')$  as follows:

Computing modulo  $Z^1(G)$  the difference of  $s$  and  $s'$  defines a cohomology-class in

$$H^1(\text{Gal}(\overline{K}/K), \text{gr}_Z^0(G)) = H^1(\text{Gal}(\overline{K}/K, T_p(J) \otimes \mathbb{Q}_p)).$$

By the Hodge-Tate decomposition of  $T_p(J) \otimes \mathbb{Q}_p$  this maps to a class

$$c^*(s, s') \in \Omega^{\text{dual}} \otimes_K H^1(\text{Gal}(\overline{K}/K, \overline{K}^\wedge)).$$

Mochizuki shows (by an easy calculation) that up to a normalising factor two the second class maps to the first if we let  $\Omega^{\text{dual}}$  act on  $\Lambda^2 \Omega$  by interior multiplication. Especially if  $g \geq 2$  then  $s'$  is Hodge-Tate up to level 2 if and only if  $c^*(s, s')$  vanishes.

7. *Remark.*— b) For two such curves  $X$  and  $Y$ , and an open homomorphism of extensions

$$\alpha : \pi_1(X) \longrightarrow \pi_1(Y)$$

the composition  $\alpha \circ s$  of  $\alpha$  with a Hodge-Tate section

$$s : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

is again Hodge-Tate. This follows because the induced map on Lie-algebras  $\mathfrak{h}_X \rightarrow \mathfrak{h}_Y$  is surjective.

8. *Remark.*— c) With similar methods one can show that for semistable  $X$  the Lie-algebra  $\mathfrak{g}$  is (with the action *via* a section  $s$  coming from a point) the projective limit of log-crystalline representations of  $\text{Gal}(\overline{K}/K)$ , as defined by J.-M. Fontaine.

## 5. COHOMOLOGICAL COMPUTATIONS

Suppose  $X$  is a projective smooth geometrically irreducible curve over a field  $K$  of characteristic zero (or at least different from  $p$ ). Assume also that the genus of  $X$  is at least one. The  $p$ -adic étale cohomologies  $H^*(X, \mathbb{Z}_p)$  and  $H^*(X \otimes_K \overline{K}, \mathbb{Z}_p)$  then coincide with the continuous cohomologies of the corresponding fundamental groups. They are related to the projective limits of cohomologies modulo  $p^n$  via short exact sequences

$$0 \rightarrow \text{proj.lim}^{(1)} H^{m-1}(X, \mathbb{Z}_p/p^n \cdot \mathbb{Z}) \rightarrow H^m(X, \mathbb{Z}_p) \rightarrow \text{proj.lim} H^m(X, \mathbb{Z}/p^n \cdot \mathbb{Z}) \rightarrow 0.$$

Here the derived projective limit  $\text{proj.lim}^{(1)}$  vanishes if the projective system satisfies a Mittag-Leffler condition, which will be the case in all our applications. As usual  $\mathbb{Q}_p$ -adic cohomology is defined by tensoring with that field.

Let  $J^{(1)}$  denote the homogeneous space (under the Jacobian  $J$  of  $X$ ) which classifies isomorphism classes of line-bundles of degree one on  $X$ . Note that a  $K$ -rational point of  $J^{(1)}$  does not necessarily correspond to an actual line bundle of degree 1. (See below for some remarks about that topic.)

The fundamental group  $\pi_1(J^{(1)})$  is the quotient of  $\pi_1(X)$  under the commutator subgroup of  $\pi_1(X \otimes_K \overline{K})$ , with the surjection induced by the canonical map from  $X$  to  $J^{(1)}$ . As before denote by  $\pi_1(J^{(1)})^{(p)}$  its quotient where the geometric fundamental group is replaced by its maximal pro- $p$ -quotient. This is an extension of  $\text{Gal}(\overline{K}/K)$  by  $T_p(J)$ . Let us study sections  $s_{ab}$  of the projection  $\pi_1(J^{(1)})^{(p)} \rightarrow \text{Gal}(\overline{K}/K)$ . Assume that  $X$  has genus  $g \geq 2$ , and note that the adjoint action of  $\pi_1(X)$  on  $H/Z^2(H)$  and  $\mathfrak{h}/Z^2(\mathfrak{h})$  factors through  $\pi_1(J^{(1)})^{(p)}$ . Thus we define  $s_{ab}$  to be Hodge-Tate if  $\mathfrak{h}/Z^2(\mathfrak{h})$  is generated by its invariants under  $s_{ab}(\text{Gal}(\overline{K}/K))$ . This is the case if  $s$  is defined

by a  $K$ -rational point of  $X$ . In fact it suffices if  $s_{ab}$  comes from a  $K$ -rational point of  $J^{(1)}$ :

We may extend  $K$  and assume that  $s_{ab}$  is defined by a divisor of degree one on  $X$ , and that this divisor is a linear combination of  $K$ -rational points of  $X$ . But each such point defines a section  $s_{ab}$  which is Hodge-Tate. Also for one such Hodge-Tate section  $s_{ab}$ , another section is Hodge-Tate if and only if the difference is a Hodge-Tate class in  $H^1(\text{Gal}(\overline{K}/K), T_p(J))$ , that is lies in the kernel of the map to  $H^1(\text{Gal}(\overline{K}/K), \Omega \otimes_K \overline{K}^\wedge)$ . This implies our claim, and also the well known fact that line-bundles of degree zero define Hodge-Tate classes in  $H^1(\text{Gal}(\overline{K}/K), T_p(J))$ .

One may ask for a converse. Some approximation of such a converse holds if  $K$  is a local field, that is a finite extension of  $\mathbb{Q}_p$ . Namely if  $J^{(m)}$  denotes the homogeneous  $J$ -space classifying line-bundles of degree  $n$ , multiplication by  $n$  induces

$$n. : J^{(1)} \longrightarrow J^{(n)}.$$

The induced map on geometric fundamental-groups is also multiplication by  $n$ .

**9. PROPOSITION.**— *Assume that  $K$  is a finite extension of  $\mathbb{Q}_p$ ,  $X$  a curve of genus  $g \geq 2$ , and*

$$s_{ab} : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m)})^{(p)}$$

*a section which is Hodge-Tate. Then there exists a positive integer  $n$  prime to  $p$ , such that*

$$n. s_{ab} : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m \cdot n)})^{(p)}$$

*is induced by a point in  $J^{(m \cdot n)}(K)$ .*

*Proof.*— The key fact is of course the theorem of Bloch and Kato ([BK], example 3.11) that up to torsion  $J(K)$  is isomorphic to the Hodge-Tate classes in  $H^1(\text{Gal}(\overline{K}/K), T_p(J))$ . If  $J^{(m)}$  has a  $K$ -rational point this implies the assertion, except for the fact that  $n$  can be chosen prime to  $p$ . However replacing  $m$  by a multiple we can find such a rational point. If  $n$  is divisible by  $p$ , we note that

$$p. : J^{(n/p)} \longrightarrow J^{(n)}$$

is a principal homogeneous space with group  $J[p]$  ( $p$ -division-points). The obstruction to lift a  $K$ -rational point  $x \in J^{(n)}(K)$  to  $J^{(n/p)}$  is a class in

$$H^1(\text{Gal}(\overline{K}/K), J[p]) = H^1(\text{Gal}(\overline{K}/K), T_p(J)/p \cdot T_p(J)).$$

It coincides with the obstruction to lift the corresponding section

$$\text{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(n)})$$

to  $\pi_1(J^{(n/p)})$  and lifts of sections correspond uniquely to lifts of points (the geometric fibre over  $x$  is  $\pi_1(J^{(n)})/\pi_1(J^{(n/p)})$ , with  $\text{Gal}(\overline{K}/K)$ -action *via* the section corresponding to  $x$ ). Thus if we can lift sections we can lift points, and this allows to remove factors  $p$  from  $n$ .

The conclusion of the proposition merits its own name. It works for an arbitrary ground field  $K$  of characteristic zero.

**10. DEFINITION.**— *A section*

$$s_{ab} : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m)})^{(p)}$$

is called *geometric up to torsion* if there exists a positive integer  $n$  such that

$$n \cdot s_{ab} : \text{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m \cdot n)})^{(p)}$$

comes from a point in  $J^{(m \cdot n)}(K)$ .

From the previous it follows that we may always choose  $n$  prime to  $p$ , and that it suffices if the condition holds over a finite extension of  $K$  (use a trace argument).

In fact we would like this point in  $J^{(m \cdot n)}(K)$  to be represented by an actual line-bundle  $\mathcal{L}$  on  $X$ . There is an obstruction in the Brauer-group of  $K$  which has to vanish. Namely there exists over a Galois-extension of  $K$  a line-bundle  $\mathcal{L}$  which represents the point in  $J^{(m \cdot n)}$  and which is isomorphic to all its Galois-conjugates. However it may happen that these isomorphisms cannot be arranged to form a descent-datum, thus the obstruction. Nevertheless one checks that it is annihilated by the dimensions of cohomology-groups  $H^i(X, \mathcal{L})$  and also by the Euler characteristic, that is by  $m \cdot n + 1 - g$ . Especially if  $p$  divides  $g - 1$  and if  $m$  is prime to  $p$ , we may replace  $n$  by a further multiple (still prime to  $p$ ) to get that  $n \cdot s_{ab}$  is represented by an actual line-bundle  $\mathcal{L}$ , of degree  $m \cdot n$ .

The rest of the section becomes a little bit technical. We assume given two curves  $X_1$  and  $X_2$ , of genus  $\geq 2$ , and an open map of extensions

$$\alpha : \pi_1(X_1) \longrightarrow \pi_1(X_2),$$

inducing

$$\alpha_{ab} : \pi_1(J_1^{(m)})^{(p)} \longrightarrow \pi_1(J_2^{(m)})^{(p)}.$$

The base field will be an extension  $K^+$  of a local  $p$ -adic field  $K$ , and the curves as well as the morphism  $\alpha$  will be definable over a finite extension  $K_1$  of  $K$ . In fact  $K^+$  will also be a  $p$ -adic field with integers,  $V^+$  a complete discrete valuation-ring dominating  $V$  but with residue field  $k^+$  a function-field in one variable over  $k$ .

Recall that sections have the uniform  $p$ -adic topology.

**11. PROPOSITION.**— *Suppose*

$$s_{ab} : \text{Gal}(\overline{K}^+ / K^+) \longrightarrow \pi_1(J_1^{(1)})^{(p)}$$

*is a section which is geometric up to torsion. Then*

$$\alpha_{ab} \circ s_{ab} : \text{Gal}(\overline{K}^+ / K^+) \longrightarrow \pi_1(J_2^{(1)})^{(p)}$$

*is the  $p$ -adic limit of such sections.*

*Proof.*— We may pass to finite extensions of  $K$  and  $K^+$  and assume that  $\alpha$ ,  $X_1$ ,  $X_2$  are defined over  $K$ , and the curves have semistable reduction. Furthermore the assertion holds if we replace  $K^+$  by  $K$ , and as we may assume that  $X_1$  has a  $K$ -rational point, we have to show the following claim:

$\alpha_{ab}$  preserves up to torsion the images of

$$J(L) \otimes \mathbb{Z}_p \longrightarrow H^1(\text{Gal}(\overline{K}^+ / K^+), T_p(J)), \quad (J = J_1, J_2).$$

For this we may replace  $J_i$  by the connected component of its Neron-model and then by the Raynaud-extensions  $G_i$  [see for example [C], Ch. II,1]. These are the extensions of abelian varieties by tori which define the same  $p$ -adic formal schemes as  $J_i$ . It is known that  $\alpha_{ab}$  preserves  $T_p(G_i) \subset T_p(J_i)$  (the quotient is maximal with the property that  $\text{Gal}(\overline{K}/K)$  acts *via* a finite unramified quotient), so it is enough to give a good characterisation of the images of  $G_i(V^+) \otimes \mathbb{Z}_p$  in  $H^1(\text{Gal}(\overline{K}^+ / K^+), T_p(G_i))$ .

So suppose  $G$  is such a semiabelian variety, with associated  $p$ -divisible group  $G_\infty$ . There is a map

$$\lambda : G(V^+) \longrightarrow \text{Ext}_{V^+}(\mathbb{Q}_p/\mathbb{Z}_p, G_\infty) \quad (= \text{extensions of } p\text{-divisible groups over } V^+)$$

“which sends  $g \in G(V^+)$  to its  $p^n$ -division-points”. Also there is a variant of  $\lambda$  with  $V^+$  replaced by  $k^+$ , and a reduction map (on both sides). It is classical that we get an isomorphism on the kernels of the reduction maps, that is extensions trivial

over  $k^+$  correspond to elements of  $G(V^+)$  which are trivial modulo the maximal ideal (these are also  $R$ -points of the associated formal group to  $G_\infty$ ). Thus the image of  $G(V^+)$  consists of extensions lifting elements of  $\lambda(G(k^+))$ . Similarly the  $\mathbb{Z}_p$ -submodule generated by  $G(V^+)$  consists of lifts of  $\lambda(G(k^+) \otimes \mathbb{Z}_p)$ . Now apply this to our situation. By a theorem of Tate  $\alpha$  extends to a morphism of  $p$ -divisible groups (denoted for simplicity by the same name)

$$\alpha : G_{1,\infty} \longrightarrow G_{2,\infty}.$$

Its reduction modulo the maximal ideal is a homomorphism of  $p$ -divisible groups over  $k$ . By a variant of a (different) theorem of Tate it lies in  $\text{Hom}_k(G_1, G_2) \otimes \mathbb{Z}_p$ . Thus

$$\alpha(G_1(k^+) \otimes \mathbb{Z}_p) \subset G_2(k^+) \otimes \mathbb{Z}_p,$$

and  $\alpha$  also respects the  $\mathbb{Z}_p$ -lattices generated by  $\lambda(G_i(V^+))$ . Finally apply the obvious maps from  $\text{Ext}_W(\mathbb{Q}_p/\mathbb{Z}_p, G)$  to  $H^1(\text{Gal}(\overline{K}^+/K^+), T_p(G))$  to obtain the assertion.

Let us sketch a proof for the mentioned variant of Tate's theorem:

**12. THEOREM** (Tate).— *Suppose  $k$  is a finite field of characteristic  $p$ ,  $G_1$  and  $G_2$  semiabelian varieties over  $k$ . Then the map*

$$\text{Hom}_k(G_1, G_2) \otimes \mathbb{Z}_p \longrightarrow \text{Hom}_k(G_1(p^\infty), G_2(p^\infty))$$

*is an isomorphism.*

*Proof.*— One can check that any homomorphism of  $p$ -divisible groups respects the Tate-modules of the subtori (eigenvalues of Frobenius on crystalline cohomology), and that the result holds for tori. Also extensions of an abelian variety  $A$  by  $\mathbb{G}_m$  are classified by  $A^t(k)$  (dual abelian variety) which is a finite abelian group so that any such extension splits up to isogeny. These two remarks allow us to reduce to the abelian case, so that from now on  $G_1$  and  $G_2$  are abelian varieties over  $k$ . If

$$\alpha_n : G_1(p^n) \longrightarrow G_2(p^n)$$

denotes the  $n$ -th stage of  $\alpha$  (a map of  $p$ -divisible groups over  $k$ ), consider the quotient

$$H_n = (G_1 \times G_2)/\text{graph}(\alpha_n)$$

which is also an abelian scheme over  $k$ . One then knows that infinitely many  $H_n$ 's are isomorphic and derives the result (see for example [CF], Ch. V, proof of th. 4.7). One can even adapt that proof to extend theorem 12 to function-fields).



**6. PROOF OF THE MAIN RESULT**

As before  $K$  is a finite extension of  $\mathbb{Q}_p$ , (with  $V, \pi, k$ , etc.),  $K^+$  an extension, with integers  $V^+$  etc., such that  $k^+$  is a function field in one variable over  $k$ . The universal (finite) module of differentials of  $V^+$  is denoted by  $\omega_{V^+}$ , and let  $\omega_{K^+} = \omega_{V^+} \otimes_{V^+} K^+$ . Furthermore  $X$  should be a geometrically irreducible projective curve over  $K$ , of genus  $g \geq 2$ . A point  $x \in X(K^+)$  is called non-constant if it does not lie in  $X(\overline{K})$ . An equivalent condition is the following:

The pullback (by  $x$ ) of any global differential form on  $X$  defines an element of  $\omega_{K^+}$ , thus a linear map (the derivative of  $x$ ) from  $\Omega = H^0(X, \omega_{X/K})$  to  $\omega_{K^+}$ .  $x$  is non-constant precisely if this map is not identically zero.

If this is the case, it determines a  $K^+$ -point in the projective space  $\mathbb{P}(\Omega) = \mathbb{P}^{g-1}$ , which is the image of  $x$  under the canonical map (embedding if  $X$  is not hyperelliptic)

$$X \longrightarrow \mathbb{P}(\Omega).$$

We can recover the derivative from Galois-cohomology, as follows: Let

$$\Delta = \text{Gal}(\overline{K}^+ / L) \subset \text{Gal}(\overline{K}^+ / K^+)$$

denote the subgroup fixing the compositum  $L$  of  $\overline{K}$  and the maximal unramified extension of  $K^+$ . Note that  $L$  contains the maximal tame extension of  $K^+$ , so that  $\Delta$  is a pro- $p$ -group. Then the Galois-cohomology of  $\Delta$  with values in the  $p$ -adic completion of  $\overline{V}^+$  is (up to a finite  $p$ -power) equal to the tensor product of  $\omega_{V^+}$  with  $W^\wedge(-1)$  ( $X = \text{integers in } L$ ). Furthermore the induced map on fundamental groups  $\text{Gal}(\overline{K}^+ / K^+) \rightarrow \pi_1(X)$  sends  $\Delta$  into  $\pi_1(X \otimes_K \overline{K})$ , and induces by pullback a map

$$H^1(X \otimes_K \overline{K}, \overline{K}^\wedge) = \Omega^{\text{dual}} \otimes_K \overline{K}^\wedge \oplus \Omega \otimes_K \overline{K}^\wedge(-1) \longrightarrow \omega_{V^+} \otimes_{K^+} L^\wedge(-1).$$

As it is Galois-invariant it must vanish on the first direct summand  $\Omega^{\text{dual}} \otimes_K \overline{K}^\wedge$ , and on the second it is the derivative of  $x$ . Thus we can recover the derivative of any  $L$ -valued point from its induced map on fundamental groups. We also remark that both sides have natural integral structures, and then the  $p$ -powers in the denominators of these maps have an upper bound only depending on  $X$ , and not even that if  $X$  has semistable reduction. Also the association

$$\{\text{maps of extensions } \text{Gal}(\overline{K}^+ / K^+) \longrightarrow \pi_1(X)\} \longrightarrow \text{Hom}_K(\Omega, \omega_{V^+} \otimes_{V^+} L^\wedge)$$

is continuous, where on the left the topology is uniform convergence, and on the right the  $p$ -adic one. Especially if for a sequence of points  $x_n \in X(L)$  the induced maps

on fundamental groups converge, and the limit has non-trivial derivative, then the images of the points in  $\mathbb{P}(\Omega)$  points converge  $p$ -adically to a non-constant limit, and so do the points themselves if  $X$  is not hyperelliptic.

**13. Remark.**— If  $X \rightarrow Y$  is an étale Galois-covering with group  $G$ , and if  $X$  is hyperelliptic, then so is  $Y$ , and  $G$  is abelian of exponent 2:

We have to show that any element in  $G$  has exponent 2, thus may assume that  $G$  is cyclic and thus abelian. The hyperelliptic involution on  $X$  is unique, thus commutes with  $G$  and induces a hyperelliptic involution on  $Y$ . However this involution acts as  $-\text{id}$  on the Jacobian of  $Y$  and thus on any covering group of an abelian cover. Hence  $\text{id} = -\text{id}$  on  $G$ .

This remark will be helpful to exclude hyperelliptic curves.

Finally we can formulate and prove the main result:

**14. THEOREM.**— Suppose  $K$  is a finite extension of  $\mathbb{Q}_p$ ,  $X$  and  $Y$  smooth geometrically irreducible projective curves over  $K$ ,

$$\alpha : \pi_1(X) \longrightarrow \pi_1(Y)$$

an open homomorphism of extensions (of  $\text{Gal}(\overline{K}/K)$ ). Then  $\alpha$  is induced from a unique dominant morphism of curves  $X \rightarrow Y$ .

*Proof.*— We start with some general remarks. Obviously we may assume that  $\alpha$  is surjective. Then  $\alpha$  is uniquely determined by its restriction to the geometric fundamental group  $\pi_1(X \otimes_K \overline{K})$ , as the geometric  $\pi_1$ 's have trivial centers. Hence we can always pass to finite extensions of  $K$ , and for example assume that  $X$  and  $Y$  have semistable models. Furthermore passing to (two disjoint) cyclic étale coverings of  $Y$  we may assume that  $Y$  is not hyperelliptic, and that  $g(Y) - 1$  is divisible by  $p$ . Also Hodge-Tate theory defines an injective “pullback”

$$\alpha^* : \Omega_Y \longrightarrow \Omega_X$$

(which will be the differential of our map  $X \rightarrow Y$ ).

Choose an extension  $V^+$  as before, and a non-constant point  $x \in X(K^+)$  such that the pullback of some differential on  $Y$  does not vanish in  $x$ . If  $X^+ = X \otimes_K K^+$  and  $Y^+$  denote the base-changes, then the point  $x$  defines a section

$$\text{Gal}(\overline{K}^+/\overline{K}) \longrightarrow \pi_1(X^+) = \text{fibered product of } \pi_1(X) \text{ and } \text{Gal}(\overline{K}^+/K^+),$$

and its composition with  $\alpha$  a section  $s$  into  $\pi_1(Y^+)$ . If we project further to  $\pi_1(J_Y^{(1)})^{(p)}$  the corresponding section is geometric up to torsion. It thus follows that  $Y^+$  admits a line-bundle of degree prime to  $p$ . Of course this already follows from the existence of a semistable model.

But any characteristic open subgroup  $H \subset \pi_1(Y \otimes_K \overline{K})$  extends to an open subgroup  $H \cdot s(\text{Gal}(\overline{K}^+/K))$  of  $\pi_1(Y^+)$ , which is the fundamental group of a cover  $Y^+(H)$  of  $Y^+$ . Then  $s$  defines a section  $s_{ab}(H)$  for  $J^{(1)}$  which is limit of sections geometric up to torsion :

If  $X^+(H)$  denotes the covering of  $X^+$  corresponding to the subgroup  $\alpha^{-1}(H) \cdot s(\text{Gal}(\overline{K}^+/K^+))$  of  $\pi_1(X^+)$ , then  $x$  lifts to a rational point  $x(H) \in X^+(H)(K^+)$  (which in turn defines  $s$ ). Furthermore everything is definable over a finite extension of  $K$ , thus the assertion by proposition 11.

Especially each  $Y(H)$  admits a line-bundle  $\mathcal{L}(H)$  of degree prime to  $p$ , and thus contains a rational point in an extension field of  $K^+$  of degree prime to  $p$  which is contained in  $L$ . Thus a point  $y^*(H) \in Y^+(H)(L)$ , and a corresponding section (well defined up to  $H$ -conjugation)

$$s^*(H) : \text{Gal}(\overline{L}/L) \longrightarrow H \cdot s(\text{Gal}(\overline{K}^+/K^+)) \subset \pi_1(Y^+)$$

lifting  $s$ . As  $H$  decreases the  $s^*(H)$  must converge to  $s$ , and thus the  $Y$ -projections of the  $y^*(H)$  converge to a point  $y \in Y(L^\wedge)$  (considering the induced map on Tate-modules, which determines the points as  $Y$  is not hyperelliptic). Also  $y$  will take values in  $K^+$ . Similarly for fixed  $H$  we consider the  $Y(H)$ -projections of all  $y^*(H')$  (for  $H' \subset H$ ), which have a limit  $y(H)$ . These  $y(H) \in Y(H)(L^\wedge)$  form a compatible projective system of  $K^+$ -points. Considering induced maps on Tate-modules one sees that on  $\text{Gal}(\overline{K}^+/K^+)$  they induce the section  $s$  (as the geometric fundamental group has trivial center), and are non-constant. Especially  $Y$  is equal to the smallest  $K$ -scheme containing the point  $y(H) \in \mathbb{P}^{g-1}(K^+)$ .

By Hodge-Tate theory the map  $\alpha$  induces an injective

$$\alpha^* : H^0(Y, \omega_Y) \longrightarrow H^0(X, \omega_X)$$

which maps the linear form given by evaluation on  $x$  to a multiple for the corresponding form for  $y$ . For any element  $f$  of the symmetric algebra of  $H^0(Y, \omega_Y)$  which vanishes on  $Y$  and thus on  $y \in Y(K^+)$ , its pullback  $\alpha^*(f)$  thus vanishes on  $x$  and also  $X$ . That is  $\alpha^*$  defines a map (first birational, but then regular) of (canonically embedded) curves  $X \rightarrow Y$ .

By the same reasoning we lift to a compatible system of pointed maps from  $X^+(H)$  to  $Y^+(H)$ . On fundamental groups the induced transformation  $\alpha' : \pi_1(X^+) \rightarrow \pi_1(Y^+)$  thus has the property that it sends  $\alpha^{-1}(H)$  into  $H$ , for all characteristic open  $H \subset \pi_1(Y \otimes_K \overline{K})$ . It is thus equal to the composition of  $\alpha$  with an endomorphism of  $\pi_1(Y)$  which respects all such  $H$ , and which induces (Hodge-Tate) the identity on global differentials of  $Y(H)$ . As it also induces the identity on the covering group (over  $\overline{K}^+$ ) of  $Y^+(H) \rightarrow Y$  it is the identity on  $\pi_1(Y \otimes_K \overline{K})$  and thus on all of  $\pi_1(Y)$ .

15. *Remark.*— a) The proof needs only the maximal pro- $p$ -quotient of the geometric fundamental group, that is any open homomorphism of the extensions of  $\text{Gal}(\overline{K}/K)$  of these groups comes from a dominant morphism. Even quotients by suitable commutator-subgroups suffice (but not in the descending central series).

From this one derives again the previous result, by applying it to all coverings like  $X(H)$ .

b) One can treat open hyperbolic curves: Suppose  $U$  is such a curve, with compactification  $X$ . Replace the Jacobian  $J_X$  by the generalised Jacobian which classifies line-bundles on  $X$  with a trivialisation on the divisor at infinity  $X - U$ , and  $\omega_X$  by differentials with simple poles along this divisor. Then for two curves  $U_1, U_2$  any open homomorphism (of extensions) between  $\pi_1(U)$ 's comes first from a map between compactifications  $X$ . As étale coverings of the open curve  $U_2$  (allowed to be ramified along the boundary) pull back to such coverings of  $U_1$ , this maps  $U_1$  to  $U_2$ .

c) For complete curves (but not for affines) one may replace the condition “ $\alpha$  is open” by “ $\alpha$  has non-zero degree”, that is the induced map

$$\alpha^* : H^2(Y \otimes_K \overline{K}, \mathbb{Q}_p) = H^2(\pi_1(Y \otimes_K \overline{K}, \mathbb{Q}_p)) \longrightarrow H^2(X \otimes_K \overline{K}, \mathbb{Q}_p)$$

is an isomorphism. This property is inherited by all finite coverings.

d) The result over  $K$  implies it over any subfield. Namely suppose one has found a  $K$ -rational point of the Hom-scheme  $\mathbf{Hom}(X, Y)$ . If the induced map on Tate-modules is defined over this subfield, then so is the map. Especially this applies to number fields.

e) The theorem extends to finitely generated extensions of  $\mathbb{Q}_p$ .

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