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THE MUMFORD–SHAH CONJECTURE IN IMAGE PROCESSING

by **Jean–Michel MOREL**

0. INTRODUCTION

Natural, digital and perceptual images

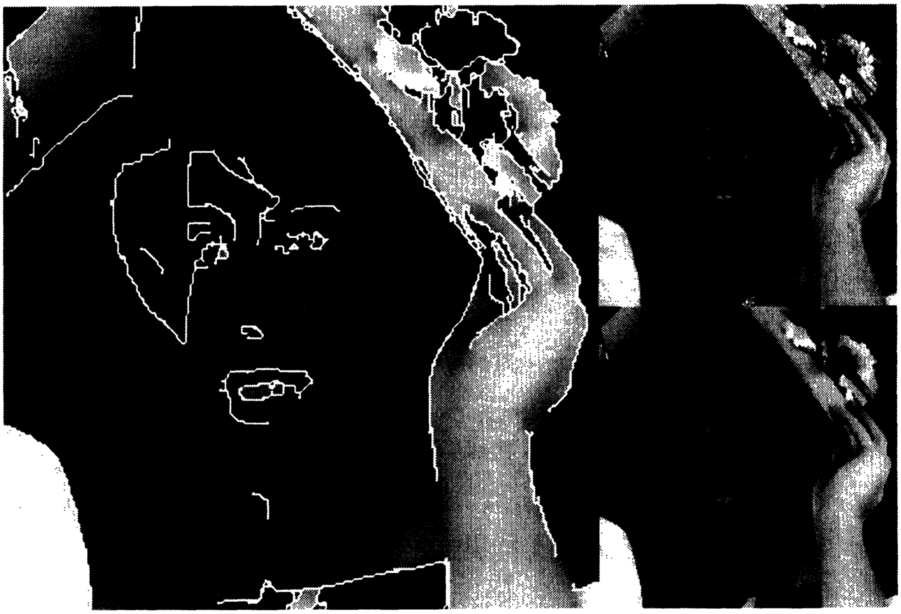
When one looks directly at scenes from the natural or the human world, or at any image (painting, photograph, drawing,...) representing such scenes, it is impossible to avoid seeing in them structures, which in many cases can be identified with real objects. These objects can be somehow concrete, as in photographs where we see trees, roads, windows, people, etc., or abstract perceptual structures, as the ones which appear in abstract paintings and can only be described in geometrical terms. However, we know that the "visual information" arriving at our retina, far from being structured, is a purely local information, for which a good model is given by digital images.

From the mathematical (and engineering) viewpoint, digital images simply are functions $g(x)$, where x is a point of the image domain Ω (the plane, a rectangle, ...) and $g(x)$ is a real number representing the "brightness" or "grey level" of the image at point x. This is the unstructured datum with which engineers have to deal in image analysis, robotics, etc. And it also is somehow the basic datum which arrives at our retina. The question is: How do we pass from the unstructured digital image to the structured perceptual one ? One of the attempts to formalize this question is the so called segmentation problem. Segmenting a digital image means finding (by a numerical algorithm) its *homogeneous regions* and its *edges*, or *boundaries*. Of course, the homogeneous regions are supposed to correspond to meaningful parts of objects in the real world, and the edges to their apparent contours. More than a thousand algorithms have been proposed for segmenting images or detecting "edges". It is of course impossible (and unnecessary) to review them all. We refer to the first part of the book [MoS3] for a classification of these algorithms and their translation from a discrete into a continuous framework (more adapted to the mathematical analysis): It is shown therein that most segmentation algorithms try to minimize, by several very different procedures, one and the same *segmentation energy.* This energy measures how smooth the regions are, how faithful the "analyzed image" to the original image and the obtained "edges" to the image discontinuities are.

If we keep the three more meaningful terms of the functional, we obtain the Mumford-Shah energy. Thus the Mumford-Shah variational model, although initially proposed as one model among other ones, happens to somehow be **the general model** of image segmentation, and all the other ones are variants, or algorithms tending to minimize these variants. The Mumford-Shah model defines the segmentation problem as a joint smoothing/edge detection problem: given an image $g(x)$, one seeks simultaneously a "piecewise smoothed image" $u(x)$ with a set K of abrupt discontinuities, the "edges" of g. Then the "best" segmentation of a given image is obtained by minimizing the functional

$$E(u,K)=\int_{\Omega\setminus K}(|\nabla u(x)|^2+(u-g)^2)dx+\text{length}(K).$$

The first term imposes that u is smooth outside the edges, the second that the piecewise smooth image $u(x)$ indeed approximates $g(x)$ and the third that the discontinuity set K has minimal length (and therefore in particular is as smooth as possible). The model is minimal in the sense that removing one of the above three terms would lead to a trivial solution. Needless to tell it: such a simple functional cannot give a good account of the geometric intricacy of most natural images, nor of our perception of them. What is expected from algorithms minimizing such a functional is a sketchy, cartoon-like version of the image, and these algorithms will give perceptually good results when the processed images somehow match this *a priori model*: contrasted images with objects presenting piecewise smooth surfaces. The success of algorithms minimizing the Mumford-Shah functional can be, however, impressive (see the enclosed figure, due to an algorithm by A. Chambolle [Cham].)



Top right, the original image (GRECO). Bottom right, the reconstructed datum u. Left, a superposition of the reconstructed datum u and the edge set K (in white).

This figure shows a low-energy Mumford and Shah segmentation. Various methods exist to reach such low-energy states. The method used here by Antonin Chambolle [Cham1,2] is a hybrid method: first the two-dimensional part of the energy is minimised (this results in the minimisation of a convex functional). Then the main edges are detected by a standard method (extrema of gradient). This set of edges K_1 being fixed, the two-dimensional energy is once again minimised on $\Omega \setminus K_1$, leading to the detection of a set of finer edges, K_2, etc. On this image, there are 50 such iterations.

The Mumford-Shah conjecture

In this conference, we deal with the mathematical consistency of the Mumford-Shah model. Let us mention that the Mumford-Shah functional derives from a discrete energy proposed by Geman and Geman [GG]. In this discrete framework, coming from a Markov random field model, no questions about the geometry of minimizers can be raised: Existence of minimizers is an obvious compactness theorem in finite dimension. The decision had to be taken to translate it into a continuous framework where the image is a function instead of being a matrix. This is a noticeable progress. In particular, we can then ask whether segmentations exist which minimize the Mumford-Shah energy, and are the boundaries thus obtained smooth ? Mumford and Shah [MumS1] conjectured the existence of minimal segmentations made of a finite set of $C^{1,1}$ embedded curves. In addition, they predict the following local behaviour for the possible endpoints and crossing of the curves.

1) Curves can only meet in a propeller-like configuration, that is, three curves meet at their endpoints and make a 120° angle with each other.

2) "Crack tips", or free endpoints, where K locally looks like a half line and (taking this half line to be the real positive one) u is written in polar coordinates as $u(r,\theta) = (\frac{2}{\pi})^{\frac{1}{2}} r^{\frac{1}{2}} \cos\frac{\theta}{2}$.

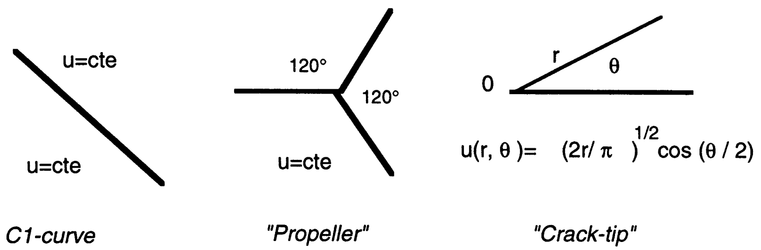


Figure 1 : Three kinds of local behaviour for K

This last possibility is well-known in the theory of fracturation of elastic media ([Kn], [BM]), but remains quite puzzling : so far, we do not know whether the two preceding configurations really are minimizers of the Mumford-Shah energy. (By the way, in the following we shall spell "MS energy" or "MS conjecture" in order to gain some space.) We shall give an account of how Bonnet [Bo] proves that a crack tip is a global minimizer of the Mumford-Shah functional if we restrict ourselves to perturbations preserving the connectedness of K.

Analogues of the MS conjecture have been stated in higher dimension by E. De Giorgi [DeGi0], who aimed at modelling mixed energies arising in the physics of liquid crystals. Unfortunately, there is, to our knowledge and in contrast with dimension 2, no exact statement of how the singularities of the set K should look like. Now, the functional has some interest, particularly in dimension $N = 3$, for (e.g.) medical images, and we shall therefore set in dimension N,

$$E(u, K) = \int_{\Omega \setminus K} (|\nabla u(x)|^2 + (u - g)^2) dx + \mathcal{H}^{N-1}(K),$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure.

So far, the MS conjecture has not been proved and only partial but meaningful enough results are at hand. The problem has proved a difficult problem for the present mathematical technique because of the subtle interaction of the two-dimensional term (in u) and the one-dimensional term $\mathcal{H}^1(K)$. The same difficulty arises when one wishes to define a computer program minimizing the energy and the mathematical analysis somehow clarifies the numerical debate.

Edge sets and rectifiable sets

The first mathematical task is to correctly define the functional $E(u, K)$. Indeed, we cannot *a priori* impose that an edge set K minimizing E is made of a finite set of curves: This precisely is what has to be proved. That situation is classical in mathematical analysis and is dealt with by enlarging the "search space", that is, in our case, by looking for a solution in a wider class of sets with finite length than just finite sets of curves. This is done by defining the "length" of K as its one-dimensional Hausdorff measure, $\mathcal{H}^1(K)$, which is the most natural way of extending the concept of length to non–smooth sets.

We call $(N-1)$*–rectifiable* any set with finite $(N-1)$-Hausdorff measure which is contained in a countable family of $(N-1)$–dimensional C^1 surfaces except for a $\mathcal{H}^{N-1}$-negligible set. Clearly, if the MS conjecture is true, the "edge sets" sought for in image processing must be 1-rectifiable, so that an extended and weakened version of the MS conjecture states that the minimizers of E are rectifiable. From the computational viewpoint, quantified estimates are highly desirable, because if (e.g.) C^1 minimizers are available but are too much ragged in many components, this will in fact contradict the spirit of the conjecture, if not its letter. We shall list what is known about the regularity of minimal segmentations.

1. WHAT HAS BEEN PROVED ABOUT THE MUMFORD–SHAH CONJECTURE

In the statements which follow, the considered constants only depend (unless otherwise stated) on Ω and $||g||_\infty$. All of the stated results will be given for a minimizing set K defined up to a $\mathcal{H}^{N-1}$–negligible set. It is easily proved that the value of E is not altered when we add or remove such a set from K.

- *Rectifiability in arbitrary dimension* N (Ambrosio [Amb1, 2, 3,4]) : u is a BV function and K is the (rectifiable) discontinuity set of u.
- *Uniform Lower density bound,* $N = 2$, ("Elimination property", Dal Maso,M., Solimini [DalMMS1, 2], generalized in arbitrary dimension in [CaLe, CaLe1]) : There exists a constant c such that for any disk $D(x,r)$ centered on K,

$$cr \leq \mathcal{H}^1(K \cap D(x,r)) \leq c^{-1}r.$$

- *Closedness.* K is a closed set (true in arbitrary dimension N). De Giorgi, Carriero, Leaci [DeGiCL]. This property can be viewed as a consequence of the lower density bound.
- *Concentration property,* $N = 2$ (Dal Maso, M., Solimini [DalMMS1, 2]). There exists for any $\varepsilon > 0$ a constant $\alpha(\varepsilon)$ such that every disk $D(x,R)$ centered on K contains a subdisk $D(y,r)$ with

$$r \geq \alpha R \quad \text{and} \quad \mathcal{H}^1(K \cap D(x,r)) \geq 2(1-\varepsilon)r.$$

- *Uniform projection property,* $N = 2$ (Dibos, Koepfler [DibK, Dib, Lég]). There exists a constant c_1 such that for every square $S = S(x,r)$ centered on K, denoting by p_1 and p_2 the orthogonal projections onto the sides of S,

$$\mathcal{H}^1(p_1(K \cap S)) + \mathcal{H}^1(p_2(K \cap S)) \geq c_1 r.$$

- *Uniform Rectifiability Property,* $N = 2$ (David-Semmes [DaSe1]). For every $\varepsilon > 0$ there exists $\alpha > 0$ such that inside each disk $D(x,R)$ centered on K there is a curve γ satisfying

$$\mathcal{H}^1(\gamma) \geq \alpha R \quad \text{and} \quad \mathcal{H}^1(\gamma \setminus K) \leq \varepsilon \mathcal{H}^1(\gamma).$$

  This property is extended to arbitrary dimensions in [DaSe2].

- *Quantified $C^{1,\alpha}$ arcs, $N = 2$,* (David [Da]) There exists a constant c such that any disk $D(x,r)$ centered on K contains a subdisk $D(x,\rho)$ with $\rho \geq cr$ and $D(x,\rho) \cap K$ is a $C^{1,\alpha}$ curve.
- *Proof of the MS conjecture when the number of connected components of K is constrained* (Bonnet, [Bo]) : All isolated connected components of K satisfy the MS conjecture. In addition, K is $C^{1,1}$ except at an exceptional set with zero Hausdorff length.
- *Regularity almost everywhere in arbitrary dimension* (Ambrosio-Pallara [AP], Ambrosio-Fusco-Pallara [AFP]) : Minimizers are $C^{1,\alpha}$, except at an exceptional set with zero $\mathcal{H}^{N-1}$ measure.

Notice that each one of the mentionned properties (except the two last ones) implies the preceding ones, so that (e.g.) the uniform rectifiability property implies the uniform projection, the uniform concentration and the uniform density bounds. The uniform projection property implies the rectifiability by a founding result of Besicovitch (see [MoS3]).

Two words about our strategy of exposition. Of course, we won't be able to do justice in 15 pages to some 400 pages of thick and concise proofs with very little overlap. So we selected a series of fast arguments able to convey the conviction that the above statements are true. We shall start with the Ambrosio-De Giorgi approach, which takes advantage of the well-known rectifiability of the discontinuity set of BV functions to manage a short–cut to the rectifiability of K. Then we shall concentrate on the two-dimensional case and give the most salient arguments for all of the announced results in dimension 2. For the uniform properties, we have used the line of presentation of [MoS3]. We shall directly give the main arguments leading to the David-Semmes uniform rectifiability property, since it implies all the preceding ones. We extract arguments towards the $C^{1,1}$ regularity from Bonnet [Bo] rather than David [Da]. Indeed, the Bonnet techniques yield a little less but are easier to explain in a few sentences. Regarding the last mentionned regularity results by Ambrosio, Fusco and Pallara in arbitrary dimension, we dedicate them much less space than they really deserve, since they bring into the discussion valuable new arguments (as e.g. the remarkable "Tilt Lemma" in [AP], adapted from Brakke [Br]). Now, as a consolation for the reader, many of the techniques therein are a combination of quantified estimates in the same line as what we shall prove in dimension 2 and blow-up techniques similar to what we extract from Bonnet [Bo]. So the next few pages will provide anyway some training for readers of [AP-AFP].

Some reduction of the problem

Since we start with the mathematical arguments, it will be convenient to give a slightly simplified framework for working with the MS functional. First, it is always assumed that the image is bounded, say $||g||_\infty \leq 1$. By an obvious troncature argument, this also implies that if u is a minimizer of the MS functional, then $||u||_\infty \leq 1$. When we talk about u (resp K) as a minimizer of the MS functional $E(u, K)$, this of course means that u is obtained as the first item of a pair (u, K) minimizing the functional (resp. K is the second item of a minimizing pair).

Next, we shall forget about Ω, or, to be at ease, assume most of the time that $\Omega = I\!R^N$. This will avoid us the tedious work of adapting interior estimates to the boundary but raises some incertitude about the functional $E(u, K)$, which may well be infinite when we integrate over the full space ! This is fixed by defining *global minimizers* of E. We say that (u, K) globally minimizes E if no change which alters (u, K) inside a ball B and leaves it unchanged outside can decrease the MS energy restricted to B.

As a last and more surprising simplification, we shall assume that $g = 0$. Indeed, we have already fixed that u and g are uniformly bounded. Since our only concern is the regularity of K, we are dealing with very local phenomena. If we look at fine scales, that is, consider the energy E (e.g. in dimension 2) inside a disk $D = D(x, r)$ with $r \to 0$, we easily see that the term $\int_D (u - g)^2 \leq 4\pi r^2$ is the "parent pauvre" of the functional. Indeed, whenever x is centered at a point of interest, that is, on K, the term $\mathcal{H}^1(D(x, r) \cap K)$ scales (generically) as $2r$. As for the first term, $\int_D |Du|^2$, we can invoke classical estimates near the boundary to say that it can scale like r^2 only when K is quite flat in a neighborhood of x. In fact, it is actually proved in [AF] that if it scales faster that r, then K is smooth at x. Thus, what we state is *a posteriori* true. So the reader is invited to believe that in all of the work on the MS conjecture, including the estimates to come, the $\int (u - g)^2$ term has never the leading part.

2. BV SPACES AND THE MUMFORD–SHAH FUNCTIONAL

We call space of functions with bounded variation, $BV(\Omega)$, the set of integrable functions whose distributional gradient Du is a bounded measure, i.e. $|Du|(\Omega) < \infty$. Let us recall some elements about the structure of BV functions. If u is BV, we consider the set $K(u)$ of points at which u is essentially discontinuous. Then $K(u)$

is rectifiable, that is, contained in a countable family of C^1 $(N-1)$-dimensional manifolds, except for a set with zero $\mathcal{H}^{N-1}$ measure. As a consequence, a normal ν exists almost everywhere. In addition, at most points of $K(u)$, the function u has essential limits on both sides of K which we denote by u^+ and u^-. Then the gradient of u can be split into three parts

$$Du = \nabla u + (u^+ - u^-)\nu d\mathcal{H}^{N-1} + C(u),$$

where : Du is the distributional gradient, ∇u is the part of Du which is absolutely continuous with respect to the Lebesgue measure (and therefore is integrable). $C(u)$ is the so called Cantorian part of Du. In order to let this structure be of use for the Mumford-Shah conjecture, it is enough to get rid of the Cantorian part : Indeed, it is obvious that if K is rectifiable and has finite $\mathcal{H}^{N-1}$ Hausdorff measure, and if $u \in H^1(\Omega \setminus K)$, then u is in BV. Conversely, De Giorgi [DeGiAm, DeGi0] has defined what he calls "Special functions with Bounded Variation" (SBV-functions), functions of BV for which $C(u) = 0$. SBV is proposed as the right "search space" for finding minimizers of the MS functional. This proposition is justified by the following theorem.

THEOREM 1 ([Amb1, Thm 2.1]).— *Let $u_n \in SBV(\Omega)$ be a sequence of functions such that $-C \leq u_n \leq C$ and*

$$\int_\Omega |\nabla u_n|^2 + \mathcal{H}^{n-1}(K(u_n)) \leq C < \infty.$$

Then, there exists a subsequence (still denoted by u_n) converging almost everywhere to a function $u \in SBV(\Omega)$. In addition, ∇u_n converges weakly in $L^2(\Omega)$ to ∇u and $E(u) \leq \liminf_{n\to\infty} E(u_n)$.

Sketch of Proof.— Let us start with the case of dimension $N = 1$. In that case, $K(u_n)$ simply is a set of points and $\mathcal{H}^0(K(u_n))$ its cardinality. Since the assumption asserts that this cardinality is uniformly bounded, we can extract a subsequence such that $K(u_n)$ converges in the Hausdorff distance to a finite set of points K. It is then easily checked by compactness in H^1 that u_n converges strongly in $L^2(\mathbb{R}\setminus K)$ to some fonction u and ∇u_n converges weakly in $L^2(\mathbb{R}\setminus K)$ to ∇u. Thus the discontinuity set of u is contained in K and by Fatou's lemma we get the last assertion of the theorem.

A "sclicing theorem" for SBV functions ([Amb2, Prop. 3]) permits to reduce the N-dimensional case to the one-dimensional. Let us denote by $e_1, ..., e_i, .., e_N$ the canonical euclidian basis of $\mathbb{R}^N$ and by π_i the hyperplane orthogonal to e_i. For every $x \in \pi_i$, we consider the restriction $u_x(t) = u(x + te_i)$ to the line parallel to e_i and passing by x. Let us assume for simplicity that $\Omega_N = [0,1]^N$. Then it is not difficult to prove that a function $u \in BV(\Omega_N)$ satisfies $u \in SBV(\Omega_N)$ if and only if its restriction $u_x \in SBV([0,1]$ for almost every $x \in \Omega_{N-1}$. In addition, the $N-1$-dimensional area of a rectifiable set K is easily recovered from the cardinalities of its slices by using the reconstruction formula

$$\int_K < \nu, e_i > d\mathcal{H}^{N-1} = \int_{\pi_i} \mathcal{H}^0(\{t, x + te_i \in K\}) d\mathcal{L}_{n-1},$$

where $\mathcal{L}_{N-1}$ is the $N-1$-dimensional Lebesgue measure and ν the normal to K. From this and Fatou's lemma, one deduces that $\mathcal{H}^{N-1}(K(u)) \leq \liminf_{n\to\infty} \mathcal{H}^{N-1}(K(u_n))$ and the same for the Sobolev norm of u_n, so that $E(u)$ appears to be lower discontinuous.

3. PROOF OF THE UNIFORM RECTIFIABILITY

We shall now give sketches of the main arguments which can be used for giving shape to the conjecture. We restrict ourselves to the case of dimension 2. Let us begin with a formula which relates the optimality of a minimizing segmentation (u, K) to the shape of K. In what follows, we denote the first term of the MS functional by $I_{\Omega\setminus K}(u) = \int_{\Omega\setminus K} |Du|^2$.

ENERGY JUMP LEMMA [Kn].— *Let K be a rectifiable subset of Ω. Let u minimize $I_{\Omega\setminus K}$ and v be the minimum of the MS energy associated with an empty segmentation, $I_\Omega(v) = \int_\Omega |\nabla v|^2$. Denote by $\frac{\partial v}{\partial n}$ the derivative of v in the direction normal to K. Then*

$$I_\Omega(u) - I_\Omega(v) = \int_K (u^+ - u^-)\frac{\partial v}{\partial n} + \mathcal{H}^1(K).$$

As a consequence, if (u, K) is an optimal segmentation, then

$$\mathcal{H}^1(K) \leq |\int_K (u^+ - u^-)\frac{\partial v}{\partial n}|.$$

Sketch of proof.— This theorem is an obvious consequence of the Green formula, together with the consideration that $\frac{\partial u}{\partial n}$ is zero on K and $\partial\Omega$ and v is continuous so that $v^+ - v^- = 0$ on K. The use of Green formula when the boundary of a domain is rectifiable is justified in [DalMMS2]. The last announced inequality is obtained because v is no successful competitor to u.

All regularity results for the MS conjecture start with more or less sophisticated excision arguments. Let us begin with the most basic one.

EXCISION LEMMA.— *If $(u, K(u))$ minimizes E, then for every disk $D = D(x,r)$,*

$$\int_D |Du|^2 + \mathcal{H}^1(K) \leq 2\pi R.$$

Proof. If this inequality is false, we can build a competitor $(v, (K \setminus D) \cup \partial D)$, with $v = u$ outside D, $v = 0$ inside D. In other terms, we add to K a circle, which permits to put to zero the MS energy inside the disk, and we obtain a lower energy, contradicting the minimality of (u, K).

We shall now see how to obtain information on the geometric behaviour of K with the simple use of both preceding lemmas. The next lemma states roughly that each part of K must be large enough (or close enough to another part) in order to survive. When $\Omega = I\!R^2$, this also implies that K is unbounded.

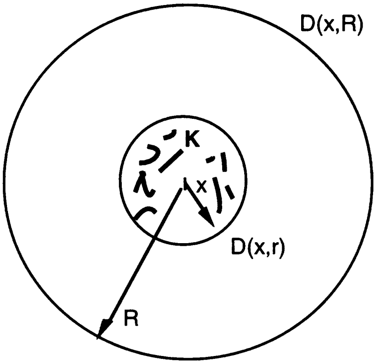


Figure 2 : Elimination Lemma

ELIMINATION LEMMA [DalMMS1,2].— *There is a constant $M > 0$ such that for any disk $D(x,R)$ contained in Ω, if $R \geq Mr$ and $(D(x,R) \setminus D(x,r)) \cap K = \emptyset$, then $K \cap D(x,r) = \emptyset$.*

In this lemma and in the remainder of this conference, we denote by C any constant which only depends on the image domain Ω. The Elimination Lemma reads in the theory of fracturation of homogeneous elastic media as "a fracture cannot be too small". If the model is correct, then the fracturation process must be sudden, at least at its beginning ([Kn]).

Sketch of proof.— Let $r \le \rho \le R$ be a radius to be adequately fixed. By the Energy Jump Lemma, we know that $\mathcal{H}^1(K) \le |\int_K (u^+ - u^-)\frac{\partial v}{\partial n}|$, where v is a continuous function such that $v = u$ outside $D(x,\rho)$ and v minimizes $I(v)$ inside $D(x,\rho)$. Let us estimate both $(u^+ - u^-)$ and $|\frac{\partial v}{\partial n}|$ on K. By Maximum Principle, we have

$$|u^+ - u^-| \le \max_{\partial D(x,r')} u - \min_{\partial D(x,r')} u$$

for every disk $\partial D(x,r')$ containing K. Now, using the EXCISION LEMMA, we have $\int_{D(x,2r)} |Du|^2 \le 4\pi r$ and we can therefore select some $r \le r' \le 2r$ such that $\int_{\partial D(x,r')} |Du|^2 \le C$. By integrating along this circle and using Hölder inequality, we obtain that for $y \in K$,

$$|u^+(y) - u^-(y)| \le \max_{\partial D(x,r')} u - \min_{\partial D(x,r')} u \le Cr^{\frac{1}{2}}.$$

Let us now proceed to estimate the other term of the ENERGY JUMP LEMMA, that is, $|\frac{\partial v}{\partial n}(y)|$ for $y \in K$. By the same argument which yields r', we can find ρ such that $\frac{R}{2} \le \rho \le R$ and $\int_{\partial D(x,\rho)} |Du|^2 \le C$. Since $v = u$ on $\partial D(x,\rho)$, the same estimate is valid for the tangential derivative of v on $\partial D(x,\rho)$. By standard estimates on the Poisson kernel, we then have $|Dv(y)| \le C(\rho - |x|)^{-\frac{1}{2}}$ for any point y inside $D(x,\rho)$. Thus, for $y \in K \cap D(x,r)$, using $R \ge Mr$, we get

$$|Dv(y)| \le C(Mr)^{-\frac{1}{2}}.$$

Summarizing the obtained inequalities, we see that for $y \in K$,

$$|(u^+ - u^-)\frac{\partial v}{\partial n}| \le CM^{-\frac{1}{2}}.$$

Integrating this inequality over $K \cap D(x,r)$ and using the ENERGY JUMP LEMMA, we finally obtain

$$\mathcal{H}^1(K \cap D(x,r)) \le CM^{-\frac{1}{2}}\mathcal{H}^1(K \cap D(x,r)),$$

which implies $\mathcal{H}^1(D(x,r)) = 0$ if M is large enough.

It is worth noticing that the preceding argument can be sharpened so as to yield more and more uniform density estimates on the geometry of K. The uniform projection property, the concentration property and the uniform rectifiability property can be obtained in the same line (see [MoS3] for a unified presentation). Let us give a sketch of proof for the uniform rectifiability property, which anyway implies all of the other ones. We begin with a lemma which we obtain by sharpening the arguments of the previous proof.

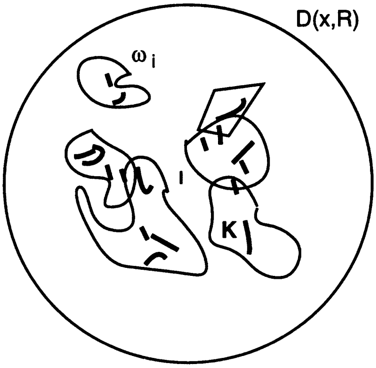


Figure 3 : Small oscillation covering

SMALL OSCILLATION COVERING LEMMA .— *There is a function $C(\nu) > 0$ such that whenever we can find a covering of $K \cap D(x,R)$ by simply connected open sets ω_i, $i \in I$ such that $\partial\omega_i$ does not meet K, the ω_i are not redundant in the sense that*

$$\Sigma_i \mathcal{H}^1(K \cap \omega_i) \leq C\mathcal{H}^1(K \cap D(x,R))$$

and the oscillation of u on $\partial\omega_i$ is uniformly controlled,

$$\int_{\partial\omega_i} |Du| \leq \nu,$$

then $\max_i(\mathcal{H}^1(\partial\omega_i)) \leq C(\nu)R$ *implies* $\mathcal{H}^1(K \cap D(x,\frac{R}{2})) = \emptyset$.

Sketch of proof.— Of course, we must think of ν as rather large and $C(\nu)$, which measures how "thin" the covering is, as a small enough constant. We shall take a simplifying assumption, which is inessential but without which the proof which follows would be significantly longer : We assume that all of the ω_i's are at a distance to $\partial D(R)$ larger than cR for some small fixed constant c. In reality, "most" of the points in a disk are far from its boundary so that this simplifying assumption can be removed, or rather replaced, by a mean value argument on R. Under this slightly stronger assumption, we shall actually prove more, that is $\mathcal{H}^1(K \cap D(x,R)) = 0$ when $C(\nu)$ is small enough. The ENERGY JUMP LEMMA and a straightforward adaptation of the bounds on $u^+ - u^-$ and $\frac{\partial v}{\partial n}$ computed in the proof of the ELIMINATION LEMMA yield

$$\mathcal{H}^1(K \cap D(R)) \le C\Sigma_i(\frac{\nu \mathcal{H}^1(\partial\omega_i))^{\frac{1}{2}}}{R^{\frac{1}{2}}}\mathcal{H}^1(K \cap \omega_i))$$

$$\le C\nu^{\frac{1}{2}}(\frac{\max_i \mathcal{H}^1(\partial\omega_i)}{R})^{\frac{1}{2}}\mathcal{H}^1(K \cap D(x,R)) \le C\nu^{\frac{1}{2}}C(\nu)^{\frac{1}{2}}\mathcal{H}^1(K \cap D(x,R)).$$

Thus, if $C(\nu)$ is small enough, we see that $\mathcal{H}^1(K \cap D(x,R)) = \emptyset$.

Of course, the statement of the SMALL OSCILLATION COVERING LEMMA must be reversed, and it tells us that if $K \cap D(x, \frac{R}{2}) \neq \emptyset$, then no small oscillation covering can be built. This essentially means that, in a quantifiable way, K is not too much spread out. Let us take an example.

DEFINITION (QUANTIFIED NONCONNECTEDNESS).— *We say that K satifies the quantified nonconnectedness property in a disk $D(x,R)$ in correspondence of two constants $\alpha > 0$ and $\varepsilon > 0$ if for every disk $D(y,r) \subset D(x,R)$ with $r \ge \alpha R$ and for every rectifiable curve γ connecting the circles $\partial D(y, \frac{r}{2})$ and $\partial D(y,r)$, one has $\mathcal{H}^1(\gamma \setminus K) \ge \varepsilon r$.*

We wish now to explain why the quantified nonconnectedness implies the existence of a small oscillation covering. This is the main argument in [DaSe1], based on an elegant use of the coarea formula.

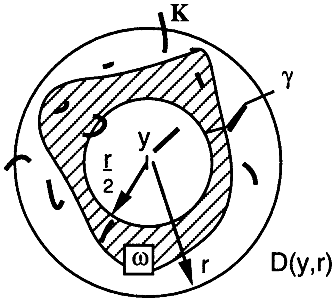


Figure 4 : Quantified nonconnectedness property

LEMMA [DaSe1].— *Assume that K satisfies the quantified nonconnectedness property in a disk $D(x, R)$. Then for every $y \in K$ and $r \geq \alpha R$, there exists a simply connected open set ω such that $D(y, \frac{r}{2}) \subset \omega \subset D(y, r)$, whose boundary $\partial\omega$ does not meet K and satisfies*

$$\mathcal{H}^1(\partial\omega) \leq \frac{C}{\varepsilon} r, \quad \int_{\partial\omega} |Du| \leq \frac{C}{\varepsilon} r^{\frac{1}{2}}.$$

Sketch of proof.— We define a geodesic distance $\delta(x)$ in the closed disk $D = D(y, r)$ as the infimum of the values of $\mathcal{H}^1(\gamma \setminus K)$ extended to all the rectifiable curves contained in D and connecting ∂D to x. Obviously, δ is a Lipschitz function, is zero on ∂D and satisfies by assumption $\delta(x) \geq \varepsilon r$ if $x \in D(y, \frac{r}{2})$. By the coarea formula, considering the slicing of the disk made by the isolevel sets $\delta^{-1}(t)$, we can assert that for every integrable function $f \geq 0$ on the disk,

$$\int_0^{\varepsilon r} \int_{\delta^{-1}(t)} f d\mathcal{H}^1 dt \leq \int_D f.$$

Applying this to $f = |Du|$ and noting that by Hölder inequality and the EXCISION LEMMA, $\int_{D(y,r)} |Du| \leq C r^{\frac{3}{2}}$, we get

$$\int_0^{\varepsilon r} \int_{\delta^{-1}(t)} |Du| d\mathcal{H}^1 dt \leq C r^{\frac{3}{2}}.$$

So by a mean value argument, we see that for most of the $t \in [0, \varepsilon r]$ we have, for a large enough constant C,

$$\int_{\delta^{-1}(t)} |Du| d\mathcal{H}^1 dt \leq C r^{\frac{3}{2}}.$$

In the same way, since $\mathcal{H}^1(K \cap D) \leq Cr$, we can select, by the coarea formula again, t among the preceding ones such that

$$\mathcal{H}^1(\delta^{-1}(t)) \leq \frac{C}{\varepsilon} r.$$

Proof of the quantified rectifiability property.— We have now all ingredients to prove the David-Semmes result. We first notice that if K does not satisfy the quantified rectifiability property, then we can find $\varepsilon > 0$ such that for every $\alpha > 0$, there is a disk $D(x, R)$ centered on K which satisfies the Quantified Nonconnectedness Property. Then the previous David-Semmes Lemma asserts that we can surround every x in $K \cap D(R)$ by a set $\omega(x)$ whose perimeter is proportional to $\frac{r}{\varepsilon}$ and therefore to $\frac{\alpha R}{\varepsilon}$. In addition, on the boundary of $\omega(x)$ the oscillation of u is controlled by a constant $\nu = \frac{C}{\varepsilon}$. Using a Besicovich covering lemma, it is easy to extract from the covering $\{\omega(x), x \in K \cap D(R)\}$ a nonredundant covering $(\omega_i), i \in I$ in the sense of the Small Oscillation Covering Lemma. Since α can be arbitrarily small for fixed ε, we see that the perimeters of the ω_i, of order $\frac{\alpha R}{\varepsilon}$, can be chosen to be arbitrarily small with respect to R. So we conclude by the SMALL OSCILLATION COVERING LEMMA that $D(x, \frac{R}{2}) \cap K$ is empty, which contradicts the assumption that it is centered on K.

4. CLASSIFICATION OF CONNECTED GLOBAL MINIMIZERS

We now give the elements of the Bonnet proof that all isolated connected components of K indeed answer the Mumford-Shah conjecture. From now on, we call *global minimizers* of the MS functional pairs (u, K) whose energy cannot be decreased by any modification inside a disk *which preserves the connected components of* $\mathbb{R}^2 \setminus K$ *outside the disk.* This is a less restrictive notion of global minimizer than the one introduced at the beginning of this conference. The central new estimate in [Bo] is the following monotonicity formula, inspired from [ACF].

MONOTONICITY FORMULA ([Bo]).— *Let* (u, K) *be a global minimizer of* E *and set* $\varphi(r) = \int_{D(x,r)\setminus K} |Du|^2$. *If* K *is connected, then* $r \to \frac{\varphi(r)}{r}$ *is a nondecreasing function of* r. *If* $\frac{\varphi(r)}{r}$ *is constant, then in some polar coordinates* (r, θ) *we have*

$$u(r, \theta) = C\sqrt{r} cos\frac{\theta}{2} \quad for \quad \theta \in [0, 2\pi]$$

and K is the half axis $\{\theta = 0\}$. In addition, we have either $C = 0$ or $|C| = \sqrt{\frac{2}{\pi}}$, which corresponds to $\frac{\varphi(r)}{r} \equiv 0$ or $\frac{\varphi(r)}{r} \equiv 1$.

The proof, which we omit, combines the ENERGY JUMP LEMMA and a tricky combination of elementary inequalities. The value of the constant C is given in [Kn] and is computed by doing a first variation in E when we propagate the "crack tip" in a straight line.

CLASSIFICATION THEOREM ([Bo]).— *If (u, K) is a global minimizer (for perturbations preserving connectedness) such that K is connected, then $\frac{\varphi(r)}{r}$ is either identically equal to 0 or to 1 and (u, K) is one of the following:*

(i) *K is empty and u is constant.*

(ii) *K is a straight line defining two half-planes and u is constant on each half-plane.*

(iii) *K is the union of three half lines with angle $\frac{2\pi}{3}$ and u is constant on each sector.*

(iv) *(Crack tip.) In a polar set of coordinates, $u(r, \theta) = Cr^{\frac{1}{2}} \cos\frac{\theta}{2}$ and K is the half axis $\theta = 0$. In addition, $|C| = \sqrt{\frac{2}{\pi}}$.*

Sketch of proof.— Let us first define a blow-up technique. We set

$$u_\varepsilon(x) = \frac{u(\varepsilon x) - c_\varepsilon(x)}{\sqrt{\varepsilon}},$$

$$K_\varepsilon = \{x, \varepsilon x \in K\}.$$

The real function $c_\varepsilon(x)$ is piecewise constant on the complementary of K. It is chosen in order to keep u_ε bounded as $\varepsilon \to \infty$. Then it is easily proved that a subsequence of $(u_\varepsilon, K_\varepsilon)$ converges to a global minimizer u_0, K_0 of E. More precisely, $K_\varepsilon \to K_0$ in the Hausdorff metric, $u_\varepsilon \to u_0$ strongly in $H^1_{loc}(\mathbb{R}^2 \setminus K_0)$. This is easily obtained as a consequence of the Concentration Property stated in the introduction, which implies that the Hausdorff length is lower semicontinuous when restricted to minimizers of E. The main argument is to show that $\frac{\varphi(r)}{r}$ is constant for a properly chosen origin, which is obtained by studying the limits $\lim_{r\to\infty \text{ or } 0} \frac{\varphi(r)}{r}$. We define a "blow-down" by

$$u_l(x) = \frac{u(lx)}{\sqrt{l}}, \quad K_l = \{x,\ lx \in K\},$$

where $l \to +\infty$. We know that a blow-up subsequence converges to a global minimizer (u_0, K_0) and, by the same argument, a blow-in sequence converges to a global minimizer (u_∞, K_∞). It is immediately deduced that, setting $\varphi_0(r) = \int_{D(r)} |Du_0|^2$ and $\varphi_\infty(r) = \int_{D(r)} |Du_\infty|^2$,

$$\frac{\varphi_0(r)}{r} = lim_{r\to 0}\frac{\varphi(r)}{r}, \quad \frac{\varphi_\infty(r)}{r} = lim_{r\to\infty}\frac{\varphi(r)}{r}.$$

By the MONOTONICITY FORMULA LEMMA, we know then that $\lim_{r\to 0} \frac{\varphi(r)}{r}$ and $\lim_{r\to\infty} \frac{\varphi(r)}{r}$ take either the value 0 or the value 1. We only have two cases to consider

Case 1 : We have $\lim_{r\to\infty} \frac{\varphi(r)}{r} = 0$. By the monotonicity formula we then have $\varphi(r) \equiv 0$. Thus u is a constant in any connected component of $I\!R^2 \setminus K$ and K appears to be a simple minimizer of $\mathcal{H}^1(K)$ under the connectedness contraint. It is well-known, and anyway easy to show, (see [MumS2] for all details) that then K satisfies one of the situations (i), (ii) or (iii) of the theorem.

Case 2 : We have $\lim_{r\to\infty} \frac{\varphi(r)}{r} = 1$. Then if also $\lim_{r\to 0} \frac{\varphi(r)}{r} = 1$, we conclude, again by the MONOTONICITY FORMULA LEMMA, that we are in the case (iv) of the theorem. Then it only remains to rule out the case where $\lim_{r\to\infty} \frac{\varphi(r)}{r} = 1$ and $\lim_{r\to 0} \frac{\varphi(r)}{r} = 0$. This is done by proving by contradicition that there exists a point x_0 such that if we perform a blow-down with x_0 as origin, then $\lim_{r\to 0} \frac{\varphi(r)}{r} = 1$. We shall not detail this argument, which uses "cut and paste" arguments on K and of course the connectedness.

In order to complete this review, we would have liked to explain how further regularity on the edge set is obtained. Let us just point out that the process leading to regularity proofs is heuristically contained in the formula given in [MumS2],

$$curv(K) = |Du^+|^2 - |Du^-|^2,$$

where it is assumed that K locally is a C^2 embedded curve, $curv(K)$ denotes the curvature of K, u^+ and u^- the limits of u on both sides of K. The formula is easily obtained by taking the first variation of E when we perturb K sidewise. From this formula, it is easily deduced that if u is smooth enough on the complementary of K, then so is K and conversely. The problem is : how to start the bootstrap argument ? Weak versions of the formula must be found to this effect. A first possibility, developped in David [Da] is to first find a quantified "close-the-holes" argument which shows that at almost points, K is an embedded curve. Once this is obtained, the bootstrap starts by proving that K must if fact be a chord arc curve. Then estimates on Du are obtained by conformal mapping on both sides of K. In the works of Ambrosio-Pallara, Ambrosio-Fusco-Pallara and Bonnet, the "close the holes" argument is obtained by a blow-up argument.

CONCLUSION

Although much evidence in favour of the MS conjecture has been gathered, it is not clear how far we are from a conclusive argument. Let us give a small list of still puzzling problems. The first one, raised in [MumS2], also of interest for fracturation theory, is the question of whether a crack tip is a global minimizer (without connectedness constraint). In some extent, the Bonnet arguments seem to indicate that this problem is of the same level of difficulty as the conjecture itself.

Mumford and Shah [MumS2] conjecture that some crack tips might be C^1, but not $C^{1,1}$ at their endpoint. They give an example apparently supporting this view. If this happens to be true, we shall deduce that the analogy between the segmentation problem and fracturation theory only is very superficial : because this will imply that the MS energy is not adequate for the modelling of quasi static crack propagation ([Kn]).

Let us finally note that the higher dimension functional is wanting precise conjectures about the shape of K, in particular for the shape of possible "cracks".

Me must apologize not to have reviewed works on variants of the MS functional. Of much interest for applications in image processing is the case where we enforce u to be piecewise constant on each connected components : see [MumS2], [MoS2,3] for a proof of the MS conjecture in this particular case and [MaTa] for existence and regularity results in higher dimensions. [BZ] proposes variants of the MS functional which improve its performance in image analysis and an interesting work on its mathematical properties is [CaLT].

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