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SOME RECENT DEVELOPMENTS IN THE THEORY OF PROPERLY EMBEDDED MINIMAL SURFACES IN $\mathbf{R}^3$

by **Harold ROSENBERG**

In the past decade there has been considerable progress in our understanding of minimal surfaces in three dimensional manifolds. In this seminar I would like to discuss a small part of the work that has been done concerning properly embedded minimal surfaces (which I will refer to as m–surfaces) in $\mathbb{R}^3$.

Until 1982, the only examples of such surfaces we knew were periodic minimal surfaces and the catenoid and plane, and they came to us from the last century : the helicoid, Scherk's surfaces, Riemann's surface, Schwarz's surfaces, etc. An m–surface is periodic if it is invariant by a non trivial discrete group of isometries acting freely on $\mathbb{R}^3$. Our surfaces are always assumed connected unless stated otherwise. We denote by $C(M)$ the total curvature of $M : C(M) = \int_M K$, K the gaussian curvature of M.

In 1982, C. Costa wrote down the formulae for a complete minimal surface, modelled on a 3–punctured torus, of $C(M) = -12\pi$, which he believed was embedded [Cost.-1,2]. D. Hoffman and W. Meeks *looked at* the surface on a computer and with the aid of the symmetries they detected, they proved the Costa surface is embedded (James Hoffman did the graphics). Subsequently families of finite total curvature m–surfaces have been constructed [H.-M.-2], figures 1 and 2.

All the examples we know today, of m–surfaces in $\mathbb{R}^3$, are periodic or of finite total curvature. One of the important open problems is to decide if there are other examples.

We will discuss some of the main results concerning m–surfaces of finite total curvature : the theorems of R. Schoen [Sch.-1] and Lopez–Ros [Lo.-Ros] ; each theorem is a characterization of the catenoid among m–surfaces of finite total curvature. Schoen's theorem assumes exactly two ends and the Lopez–Ros theorem assumes genus zero.

We discuss the curvature estimates of stable minimal surfaces, initiated by Heinz for graphs and in general by R. Schoen. We show how the curvature estimates are used to construct stable limits of least area surfaces, and we give applications.

We discuss the annular end theorem and the strong halfspace theorem of Hoffman–Meeks. This latter result says that two properly immersed disjoint minimal surfaces in $\mathbb{R}^3$ are planes ; this is very useful.

We discuss the work of Meeks and myself on the finite total curvature conjecture : an m–surface in $\mathbb{R}^3$ of finite topology and at least two ends is of finite total curvature. A corollary of our work is that such a surface is of finite conformal type.

We discuss the work of Meeks and myself on periodic minimal surfaces. The main result is that finite topology of the quotient surface implies finite total curvature of this quotient surface. If this is so then the (quotient) surface is parametrized by meromorphic data on a compact Riemann surface (a Weierstrass type representation).

This theorem yields topological and geometrical obstructions for the existence of such surfaces. For example, the number of ends of such surfaces is always at least two (except for the plane). If the surface is doubly periodic and orientable (in the quotient) then the number of ends is at least four. These are topological obstructions, we will discuss geometrical obstructions in section VII. For example, if all the ends are not parallel (as in Scherk's doubly periodic surface) then the group G is commensurable. This means there are two independent elements of G of the same length.

We prove the plane and the helicoid are the only simply connected m–surfaces in $\mathbb{R}^3$ with an infinite symmetry group.

We discuss the sum of minimal surfaces and some applications.

There is no known topological obstruction to realizing a complete, orientable, non compact surface as an m–surface in $\mathbb{R}^3$.

Finally we discuss some problems, conjectures, and related results.

I have decided not to discuss the construction of the beautiful examples of Costa, Karcher, Hoffman and Meeks. Their influence on this subject has been enormous, and as H. Karcher says : "What a magnificent picture of a conformal map." I would like to thank David Hoffman and Hermann Karcher for their work and inspiration. Some of the computer graphics were done at the Geometry, Analysis, Graphics Laboratory at the University of Massachusetts at Amherst by Jim Hoffman, Ed Thayer and Fusheng Wei. The remaining computer graphics were done by Hermann Karcher and Konrad Polthier working with SFB256 at Bonn. I thank you all. I received a great deal of help with the material preparation of this manuscript by Hermann Karcher and Katrin Wendland. I thank you both.

The paper is organized as follows.

1. How the classical examples are constructed.
2. The Weierstrass representation and the geometry of the ends of a finite total curvature minimal surface in $\mathbb{R}^3$.
   2.1 Osserman's parametrization of finite total curvature surfaces
   2.2 The geometry of finite total curvature ends
3. The characterizations of the catenoid by R. Schoen and Lopez–Ros.
   3.1 The theorem of R. Schoen
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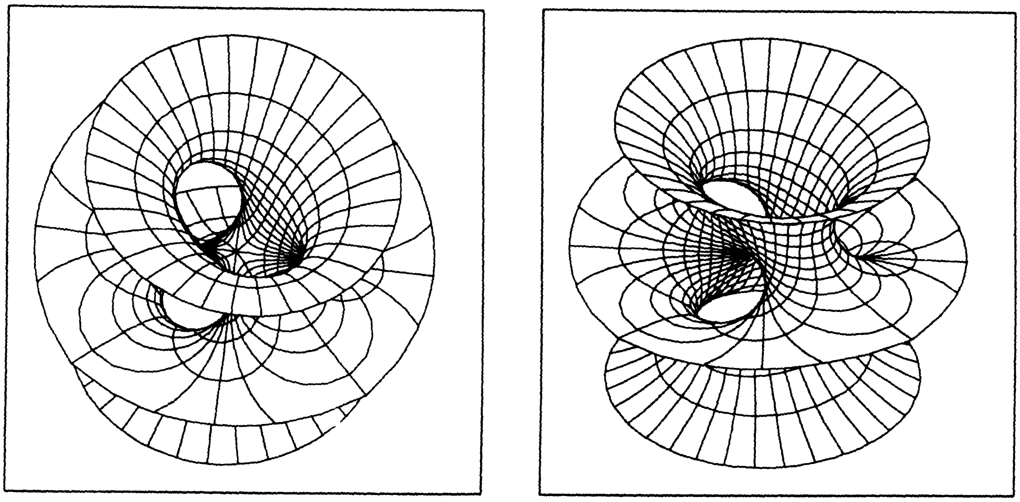
   3.3 and 3.4 The maximum principle at infinity
   3.5 The monotonicity formula
4. Curvature estimates for stable minimal surfaces.
   4.1 The Barbosa-Do Carmo stability criteria
   4.2 An idea of the proof of Heinz's theorem
5. Compactness of least area families and construction of complementary finite total curvature surfaces.



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1. HOW THE CLASSICAL EXAMPLES ARE CONSTRUCTED

Let Γ be a polygonal Jordan curve in $\mathbb{R}^3$ and M_0 a compact minimal surface with $\partial M_0 = \Gamma$. The Schwarz reflection principle [Oss.-1] allows us to extend M_0 across each edge of Γ by rotating M_0 by π about each edge. Continuing the reflections along each edge that develops one obtains a complete minimal surface M, which will have singularities in general and self intersections; i.e. M will be immersed.

If the angle at which two edges of Γ meet is irrational the M will turn infinitely often about this vertex. So if one wants embedded examples then Γ should be chosen to have vertex angles of the form $2\pi/n$, and M_0 should be chosen embedded. For example the polygons of figures 3 and 4 provide properly embedded examples which are invariant by 3–independant translations (triply periodic examples).

Riemann, Schwarz and Weierstrass found minimal surfaces M_0 with $\partial M_0 = \Gamma$ by explicitly solving the Riemann mapping problem and the Weierstrass representation (this is well explained in Darboux [Darb.]). Today we find M_0 by other techniques. Douglas and Rado proved that any rectifiable Jordan curve Γ in $\mathbb{R}^n$ bounds a least area minimal disc M_0, [Doug.],[Rado-1,2]. Subsequently, R. Osserman proved M_0 had no geometric branch points (i.e. a least area disc with boundary Γ is immersed [Oss.-1]. Finally using geometric measure theory, Reifenberg proved there is always an embedded minimal surface M_0 with $\partial M_0 = \Gamma$, [Reif.].

Now if one choses Γ and M_0 well, the complete surface M obtained by the reflections of M_0 in all edges (that develop) will be an embedded triply periodic surface. The quotient of M by a group G generated by 3–independant translations will be a compact minimal surface of finite genus embedded in the flat 3–torus $\mathbb{R}^3/G$. The geometry and topology of these surfaces has been studied by W. Meeks [M.-3], and H. Karcher [K.-3],[K.-4].

Many of the other known examples of infinite total curvature (doubly and singly periodic examples) are constructed by taking Γ to be a non compact polygon and M_0 a complete embedded minimal surface with boundary Γ. Again one does all possible reflections of M_0 across the edges of Γ (and its'iterates) to construct a complete minimal surface M. So how does one find M_0 when Γ is infinite? There is a general theory which attacks this problem (the Jenkins–Serrin theorem [J.-S.] and the conjugate Plateau construction [K.-1]) but rather than discuss this. I will describe how one can obtain some examples directly.

First, let us construct Scherk's (first) surface by solving a compact Plateau problem and taking limits. Consider the polygon $\Gamma(n)$ of figure 5-a. For each integer n, choose $\Gamma(n)$, as in figure 5-a, so that

— $\Gamma(n)$ projects to a square in the horizontal plane, and

— the top edges are at height n, and the bottom edges at height $-n$.

Now let $\Sigma(n)$ be the least area disc with boundary $\Gamma(n)$. It is not hard to prove, that $\Sigma(n)$ is a graph over the square in the horizontal plane to which $\Gamma(n)$ projects. (More generally, Rado has proved that if a Jordan curve Γ projects to a convex planar curve C, then any minimal surface bounded by Γ is a graph over the planar domain bounded by C [M.-2]).

Now $\Sigma(n)$ inherits the symmetries of $\Gamma(n)$ so there is a point p_n of $\Sigma(n)$ at vertical height zero where the tangent plane of $\Sigma(n)$ is horizontal.

Now as $n \to \infty$, the surfaces $\Sigma(n)$ all pass through the same point $p_n = p$. Then the functions defining the graphs $\Sigma(n)$ converge to a function f, defined on the interior of the square, with boundary values $+\infty$ on two opposite sides of the square and $-\infty$ on the two other sides. The graph of f is a minimal surface with boundary the four vertical lines over the vertices of the square, figure 5-b.

Now do Schwarz reflection of the graph of f about the four vertical lines, and about all the vertical lines one obtains. This yields Scherk's minimal surface M. M projects to the infinite array of squares in the horizontal plane, which form the (black squares say) of an infinite checkerboard pattern, figure 5-c.

Now one can form quotients of M by independent horizontal translations to obtain properly embedded, finite topology (and finite total curvature) surfaces in flat manifolds $\mathbb{T}^2 \times \mathbb{R}$, $\mathbb{T}^2$ a flat 2–torus.

The simplest way to do this yields a projective plane punctured in two points. We now describe some of these examples.

Let P be the square to which $\Gamma(n)$ projects and let v_1, v_2 be the vectors determined by the sides of P. Let $G(v_1 + v_2, v_1 - v_2)$ denote the group generated by the translations $v_1 + v_2, v_1 - v_2$. Then $G(v_1 + v_2, v_1 - v_2)$ leaves M invariant and the quotient is topologically a projective plane minus two points, of total curvature -2π, figure 6-a.

A fundamental domain for $G(2v_1, 2v_2)$ is two continguous copies of P (figure 6-b) and the quotient of M by this group is conformally diffeomorphic to a 4–punctured sphere and is of total curvature -4π in $T^2 \times \mathbb{R}$.

One can realize S^2 minus any even number of points this way : let $G = G(2nv_1, v_2)$, a fundamental domain consists of $2n$ copies of P (figure 6-c). To obtain a torus minus four points, let $G = G(2(v_1 + v_2), 2(v_1 - v_2))$. A fundamental domain is four copies of P (figure 6-d). The total curvature is -8π and there are four ends.

One obtains the Klein bottle from the group $G(v_1 - v_2, 2(v_1 + v_2))$. A fundamental domain is given in figure 6-e. There are two ends and the total curvature is -4π. Placing n copies of P diagonally and letting $G = G(v_1 - v_2, n(v_1 + v_2))$ we obtain the connected sum of n projective planes minus two points. By taking appropriate oriented two–sheeted covers of the nonorientable examples just described one obtains every possible orientable surface minus four points.

Notice that in all these examples, the ends are asymptotic to flat cylinders, which happen to be vertical. Also the top ends are not parallel to the bottom ends here. There are examples with all the ends parallel and non vertical [M.-R.1]. H. Karcher has constructed, an easy to visualize example of a torus minus four points in $\mathbb{T} \times \mathbb{R}$, with all the ends parallel; we call this a Karcher saddle, [K.-2].

In Karcher's example one has a rectangle P and a minimal graph over the part of P bounded by L_1, L_2, C_1 and C_2 (figure 7-a). The function is 0 on C_1, C_2 and $+\infty$ on L_1, L_2. Moreover, the graph is vertical along $C_1 \cup C_2$. This implies C_1, C_2 are planar lines of curvature and the graph can be extended by reflection in the plane of P. This new surface has four vertical lines as boundary (figure 7-b).

To obtain a torus minus four points, place two copies of P diagonally and quotient by the group $G(2v_1, 2v_2)$, figure 7-c.

In all of the above examples, the geometry and topology of the quotient surfaces are related by the formula :

$$C(M) = 2\pi\mathcal{X}(M) .$$

This is a special case of the result :

THEOREM 1.1 [M.-R.1]. — *Let $M \subset T \times \mathbb{R}$ be a properly embedded minimal surface of finite topology. Then M has finite total curvature and*

$$C(M) = 2\pi\mathcal{X}(M) .$$

In our construction of Scherk's surface we started with a square P over which we took limits of minimal graphs. If we started with a rhombus P the same construction works; the graphs with boundary the polygons $\Gamma(n)$ would still have their vertical point p_n at height zero. So Scherk's surfaces exist over checkerboard patterns defined by rhombi. However, had we started with a parallelogram P with sides of unequal length, the points p_n will always *drift off* to infinity and the limiting surface will be two disjoint vertical strips (figure 8).

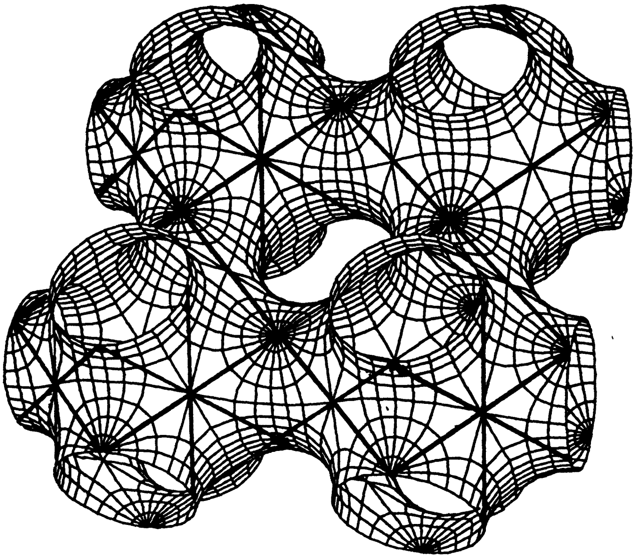
This a special case of our result :

THEOREM 1.2 [M.-R.-1]. — *Let M be a properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ of finite topology. If the ends of M are not parallel then $\mathbb{T} \times \mathbb{R}$ has a commensurable lattice and the ends of M are vertical.*

By commensurable lattice we mean $\mathbb{T} \times \mathbb{R} = \mathbb{R}^3/G$, and G has two linearly independent vectors of equal length.

There is a theorem of Jenkins and Serrin which yields Scherk's surface over a rhombus (hence the complete surface by reflection in the vertical lines over the vertices). We state a special case of their result.

THEOREM 1.3 [J.-S.]. — *Let C be a polygonal Jordan curve in the plane with an even number of sides. Let P be the compact planar domain bounded by C and let φ be the data on C which is $+\infty$ and $-\infty$ on adjacent sides of C. A necessary and sufficient condition that φ extend to a (finite valued) minimal graph over P, is the sum of the lengths of the edges of C where φ is $+\infty$, equals the sum of the lengths of the edges of C where φ is $-\infty$.*



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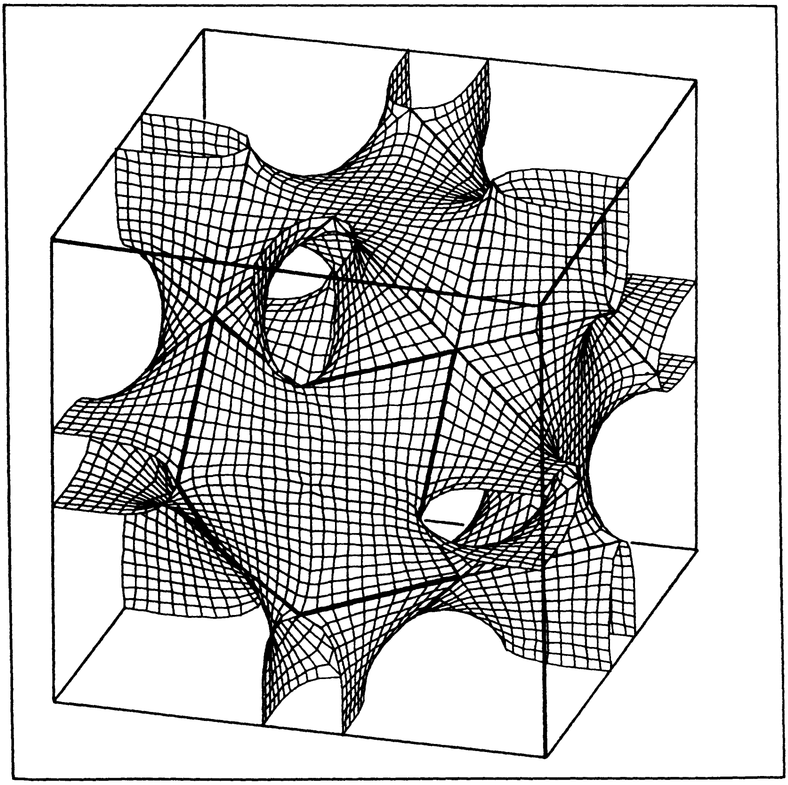

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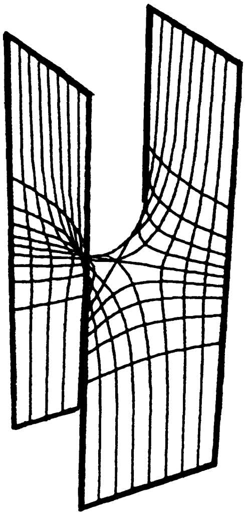

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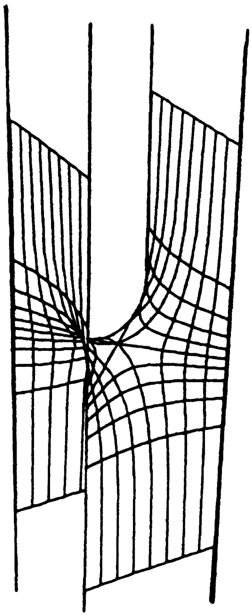

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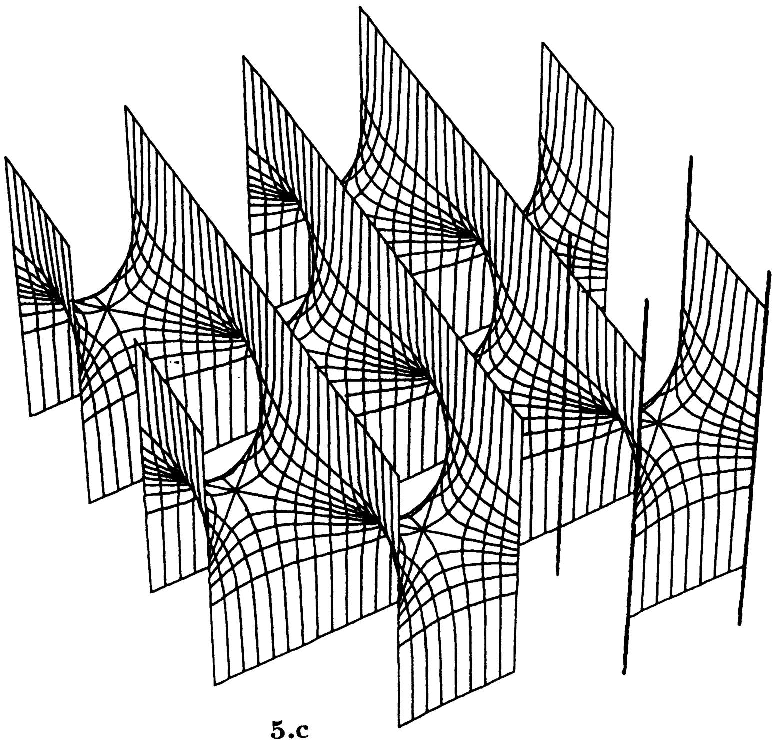

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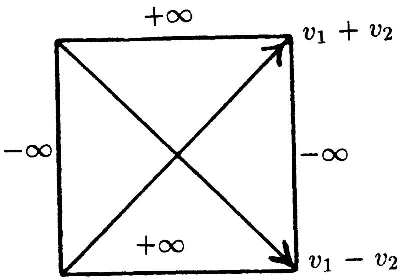
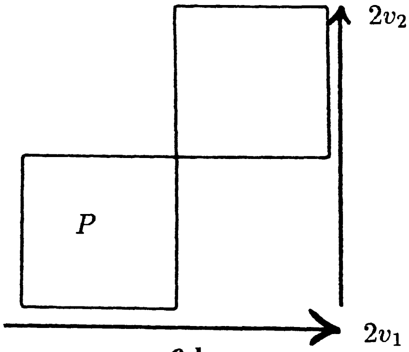


6.a



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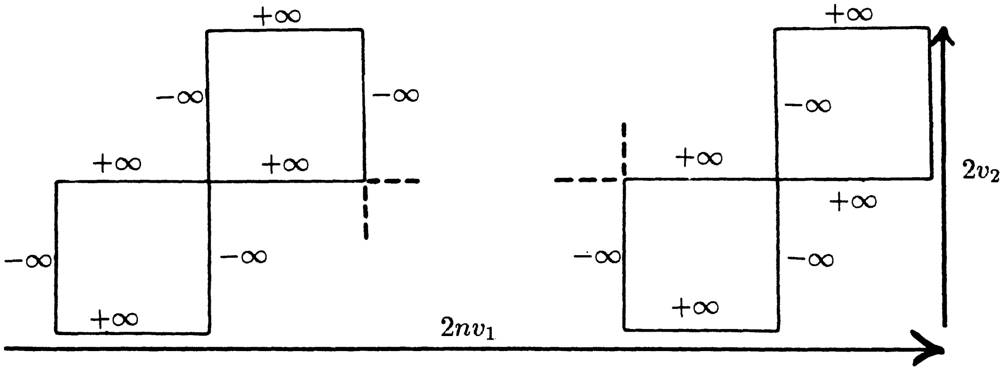

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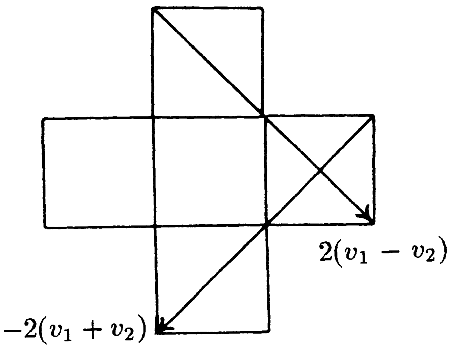

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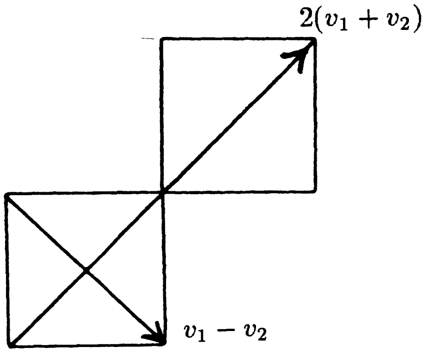

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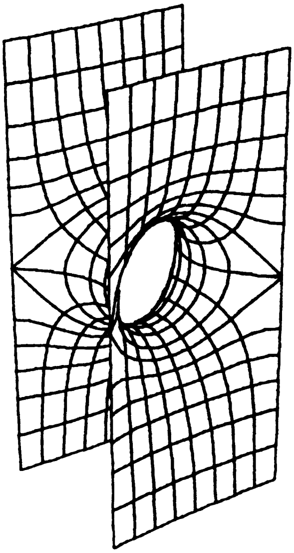

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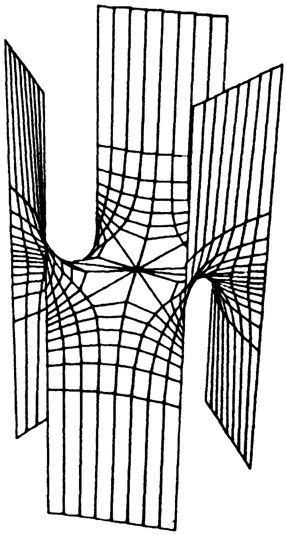

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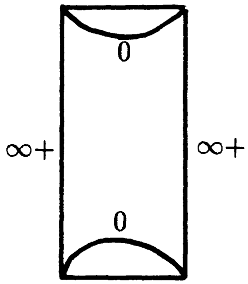
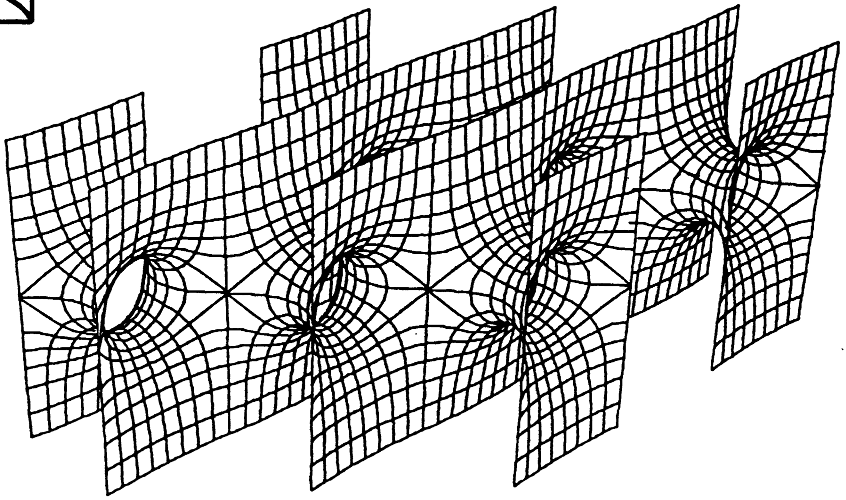


7.a



7.c

When φ does extend to a minimal graph, the graph is bounded by the vertical lines over the vertices of C. For example, the Jenkins-Serrin graph over a regular hexagon is shown in figure 9.

Clearly the theorem of Jenkins–Serrin implies Scherk's surface exists over a parallelogram precisely when it is a rhombus.

There are generalizations of Jenkins–Serrin theorem to non compact domains which have proved useful to construct complete surfaces [R.-S.E.], [K.-1].

2. THE WEIERSTRASS REPRESENTATION AND THE GEOMETRY OF THE ENDS OF A FINITE TOTAL CURVATURE MINIMAL SURFACE IN $\mathbb{R}^3$

Consider a Riemann surface M and a conformal map $\phi : M \to \mathbb{C}^3$ satisfying $\sum_{i=1}^{3} \phi_i^2 = 0$, $\phi = (\phi_1, \phi_2, \phi_3)$. Then $X(z) = Re \int_{z_0}^{z} \phi$, is a minimal surface in $\mathbb{R}^3$. It is not hard to see (and can be found in [Oss.-1], for example) that every surface in $\mathbb{R}^3$, whose mean curvature vanishes, is locally of this form. In order for the surface in $\mathbb{R}^3$ to the modelled on M, one needs the integral of ϕ to be independent of the path on M between z_0 and z; this is called the period condition : for every cycle γ on M,

$$Re \int_{\gamma} \phi(z)dz = 0 .$$

Also, in order for M to be immersed in $\mathbb{R}^3$, one requires $\sum_{i=1}^{3} |\phi_i(z)| \neq 0$ for $z \in M$.

We summarize this in the definition : a minimal surface, in $\mathbb{R}^3$, modelled on the Riemann surface M, is a conformal map $\phi : M \to \mathbb{C}^3$ satisfying :

- $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ on M and $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2$ never vanishes,
- the period condition : $Re \int_{\gamma} \phi(z)dz = 0$ for all cycles γ on M.

In case the period condition is not satisfied, one considers the minimal surface modelled on the universal conformal covering space of M (i.e. $\mathbb{C}$ or the open unit disc). It is common, even when X is not an embedding, to speak of $X(M)$ as the minimal surface modelled on M.

The three coordinate functions of ϕ and the one equation : $\phi_1^2+\phi_2^2+\phi_3^2 = 1$, can be reduced to two conditions. One classical way to do this is called the Weierstrass representation. Assuming $\phi_1 - i\phi_2$ is not identically zero (this corresponds to M a plane), let

$$g = \frac{\phi_3}{\phi_1 - i\phi_2} \quad \text{and} \quad \omega(z) = (\phi_1 - i\phi_2)dz .$$

Then g is a meromorphic function on M, ω a holomorphic one form, and M is obtained from (g, ω) by :

$$(W) \qquad X(z) = Re \int_{z_0}^{z} \left(\frac{\omega}{2}(1-g^2),\ \frac{i\omega}{2}(1+g^2), g\omega\right) .$$

This is called the Weierstrass representation of M. Notice, the poles of g are the zeros of ω and a pole of order k of g corresponds to a zero of order $2k$ of ω.

It is easy to see that a pair (g, ω) as above, i.e. g is meromorphic on M, ω holomorphic on M, satisfying the zero–pole condition, defines a minimal surface by using the formula (W). Naturally one needs the period condition for the surface to be modelled on M.

The meromorphic map g has an important geometrical meaning : it is the Gauss map of M ; more precisely, it the composition of the usual Gauss map of $X(M)$ with stereographic projection of the unit sphere (centered at the origin) to the equatorial plane, from the north pole.

The geometric invariants of M are expressed in terms of (g, ω). The induced metric on M is given by :

$$ds = \frac{|\omega|}{2}(1+|g|^2) ,$$

and the curvature of M :

$$K = -\left[\frac{4|g'|}{|\omega|(1+|g|^2)^2}\right]^2 .$$

2.1. Osserman's parametrization of finite total curvature surfaces

A Riemann surface M is said to be of finite conformal type if there is a compact Riemann surface $\overline{M}$ such that M is conformally equivalent to $\overline{M}$ punctured in a finite number of points.

The importance of finite total curvature for a complete minimal surface in $\mathbb{R}^3$ is made clear by the following theorem of Osserman.

THEOREM 2.2 [Oss.-1]. — *Let M be a complete immersed minimal surface in $\mathbb{R}^3$ and $|C(M)| = \int_M |K| dA < \infty$. Then M is of finite conformal type and M can be parametrized by meromorphic data on a compact Riemann surface. More precisely, if $\overline{M}$ denotes the conformal compactification of M (so M is conformally $\overline{M} - \{p_1, \ldots, p_n\}$) then the Weierstrass representation (g, ω) of M extends meromorphically to $\overline{M}$.*

Thus, in some sense, the theory of finite total curvature minimal surfaces in $\mathbb{R}^3$ is a problem in Riemann surface theory. But, we are very far from an understanding of this subject. How does one *see* M in terms of (g, ω)? When is M embedded? Which M exist?

It is interesting to understand what Osserman's theorem has to do with minimal surfaces. In fact, an important part of this theorem is independant of minimality. A complete Riemannian two manifold of finite total curvature is of finite conformal type. This is a theorem of Huber [Hub.]; a modern proof can be found in [Wh.]. The hard part of Huber's theorem is the conformal type since the Cohn–Vossen inequality ($C(M) \leq 2\pi\mathcal{X}(M)$ for complete 2–manifolds of non positive curvature) implies the topological type is finite when $C(M)$ is finite. Assuming this, it is not hard to extend (g, ω) to the punctures. An end A of M is conformally a punctured disc : $A = \mathbb{D}^* = \{0 < |z| \leq 1\}$. The Gauss map g extends meromorphically to the origin since the total curvature of A is the area of the Gaussian image of A, counted with multiplicity. If the puncture were an essential singularity then g would take on almost every value infinitely often and the spherical area would be infinite. Now rotate M so g is finite at the puncture. Since the metric on M is $\frac{|\omega|}{2}(1+|g|^2)$, and the metric is complete, one has $\int_\gamma |\omega| = \infty$, for every divergent path γ on A; i.e. γ converges to the origin viewed in $\mathbb{D}^*$.

Then one proves ω has a pole at the origin by a function theory argument (cf; [Oss.-1]).

2.2. The geometry of finite total curvature ends

Now it is not difficult to analyse an end A of finite total curvature. Parametrize A conformally by $\mathbb{D}^*$ and let (g, ω) be the Weierstrass representation of A. After a rotation of A in $\mathbb{R}^3$ we can suppose $g(0) = 0$, so $g(z) = z^k$ after a conformal reparametrization of a subend of A. The metric complete at 0 tells us that ω must have a pole at $0 : \omega(z) = \left(\frac{C_\ell}{z^\ell} + \mathcal{O}(|z|^{-\ell-1})\right)dz$ in a neighbourhood of 0.

A direct calculation, using the Weierstrass representation (W), yields :

$$E) \qquad 2[x_1(z) - ix_2(z)] = \int_{z_0}^{z} \omega - \overline{\int_{z_0}^{z} g^2\omega} .$$

We know $g^2\omega$ has a milder pole than ω at 0 so $x_1 - ix_2$ has a pole of order ℓ at 0. Consider the image of the circle $re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $r > 0$, r small, by the map $x_1 - ix_2$. The image curve turns $\ell - 1$ times about the x_3–axis. Since the curve must close (i.e. A is an annulus) we have $\ell > 1$; in fact, the coefficient of $\frac{1}{z}$ in ω must be 0, since x_1 and x_2 are single valued on A.

If the end A is embedded, then the image curve : $(x_1 - ix_2)(re^{i\theta})$, $0 \leq \theta \leq 2\pi$ turns once around the x_3–axis, hence $\ell = 2$ and

$$\omega = \left(\frac{c}{z^2} + h(z)\right)dz, \quad h \text{ holomorphic}$$

in a neighbourhood of 0.

Now $x_3 = Re \int g\omega$, and $g(z) = z^k$ near 0, so $|x_3|$ is bounded means $k \geq 2$; these are the planar ends. Notice that if $k = 1$, c is real since x_3 is well defined on the end.

Integrating ϕ_3 we obtain the development of x_3 (the constants are the integration constants) :

$$x_3(z) = \begin{cases} c\ell n|z| + c_0 + \mathcal{O}(|z|^2), & \text{if } k = 1, \\ d_0 + \mathcal{O}(|z|), & \text{if } k \neq 1. \end{cases}$$

From the equation E we obtain :

$$|z| = \frac{|c|}{2|x|} + \mathcal{O}(|x|^{-2}) .$$

Substitute this in $x_3(z)$ to obtain :

$$x_3 = a\ \ell n|x| + b + \frac{c_1x_1 + c_2x_2}{|x|^2} + \mathcal{O}(|x|^{-2}) ,$$

where the coefficients are real constants ($a = -c$).

Hence an embedded finite total curvature end is asymptotic to a planar or a catenoid end.

3. THE CHARACTERIZATIONS OF THE CATENOID BY R. SCHOEN AND LOPEZ–ROS

In the class of finite total curvature minimal surfaces in $\mathbb{R}^3$ we have two fundamental theorems; each a characterization of the catenoid.

THEOREM 3.1 [Sch.-1]. — *Let M be a complete immersed finite total curvature minimal surface in $\mathbb{R}^3$ with two ends, each embedded. Then M is a catenoid.*

Remark : We will see that M is embedded; this follows immediately from the monotonicity formula.

THEOREM 3.2 ([Lo.-Ros] and [P.-Ros]). — *Let M be an m-surface in $\mathbb{R}^3$ of finite total curvature and genus zero. Then M is a plane or a catenoid.*

We make a short digression. The theorem of R. Schoen is rather suprising. Why can't one add a handle to a catenoid (figure 10)?

Related to this, we have the unsolved conjecture of W. Meeks : let C_1 and C_2 be convex curves in parallel planes and let M be a compact connected minimal surface with $\partial M = C_1 \cup C_2$. Then M has genus zero.

Convexity is certainly necessary here as the following example shows. Let M_1, M_2 be two pieces of catenoids placed as in figure 11-a.

The boundaries of M_1, M_2 are in parallel planes. Now join the top and bottom boundary circles by narrow bridges (figure 11-b). By the Bridge principle [Cour.], [Smale], there is a minimal surface M, which is close to M_1 and M_2 (near M_1, M_2) and fills the bridges.

Now we shall begin the proofs of theorems 3.1, 3.2. A useful tool is the maximum principle at infinity. The usual maximum principle implies that the distance between two disjoint properly immersed minimal surfaces in $\mathbb{R}^3$ cannot be realized at points $p_1 \in \text{int}(M_1)$, $p_2 \in \text{int}(M_2)$, unless M_1 and M_2 are parallel planes. Now what happens when M_1 and M_2 are asymptotic at infinity?

THEOREM 3.3 (Maximum Principle at Infinity [L.-R.],[M.-R.-2]). — *Let M_1 and M_2 be disjoint properly immersed minimal surfaces with compact boundary in a complete flat three manifold. If $\partial M_1 = \partial M_2 = \phi$ then M_1 and M_2 are flat. Otherwise*

$$\text{dist}(M_1, M_2) = \min\{\text{dist}(M_1, \partial M_2),\ \text{dist}(M_2, \partial M_1)\}\ .$$

In fact, the case $\partial M_1 = \partial M_2 = \phi$ is the strong halfspace theorem of Hoffman–Meeks. This case does not arise in the proofs of the theorems of Schoen and Lopez–Ros and we discuss the strong halfspace theorem in VI. What we need (and prove) here is the special case of the maximum principle at infinity, first proved by Langevin and Rosenberg.

THEOREM 3.4 [L.-R.]. — *Let M_1 and M_2 be disjoint finite total curvature embedded minimal surfaces in $\mathbb{R}^3$ with compact boundaries. Then* $\text{dist}(M_1, M_2) > 0$.

The proof of this theorem uses the notion of flux on a minimal surface M.

Let α be an oriented cycle on M and denote the complex structure operator of M by J (J is rotation by $\pi/2$ in each tangent space of M). The flux of α is

$$Flux(\alpha) = \int_\alpha J(\alpha')\ ,$$

where α' is the unit tangent vector to α : $J(\alpha')$ is a conormal field to M along α; a unit normal to α, tangent to M.

Since the coordinate functions are harmonic on M, a direct application of the divergence theorem shows that $Flux(\alpha)$ only depends on the homology class of α; so flux should be thought of as an $\mathbb{R}^3$–valued function on the homology of M.

Now, we established in II that an embedded finite total curvature end of a minimal surface in $\mathbb{R}^3$, has a limiting normal vector (which we suppose vertical here) and the end can be written as a graph $u(x)$, for $|x|$ large :

$$u(x) = a\ \ell n|x| + b + \frac{c_1x_1 + c_2x_2}{|x|^2} + \mathcal{O}(|x|^{-2}) \ .$$

We say the end is a catenoid type end if $a \neq 0$ (a is the logarithmic growth rate of the end) and the end is planar if $a = 0$. In the first case the end is geometrically asymptotic to a catenoid and to a horizontal plane when $a = 0$.

The development of u can be differentiated term by term, so the outward pointing conormal vector to the end, along the curve $C_R = \{(x, y, u(x, y)/x^2 + y^2 = R^2\}$ is easily calculated to be

$$\nu = \frac{1}{R}(x, y, a) + \mathcal{O}(|R|^{-2}) \ .$$

Hence $Flux(C_R) = (0, 0, 2\pi a) + \mathcal{O}(|R|^{-1})$.

Since the flux only depends on the homology class of C_R, we have $Flux(C_R) = (0, 0, 2\pi a)$. In particular, the $Flux$ vector only depends on the logarithmic growth rate of the end and is parallel to the limiting normal vector.

Now we can prove theorem 3.4. Assume there are two embedded disjoint minimal surfaces M_1, M_2 of finite total curvature, compact boundaries, and $\text{dist}(M_1, M_2) = 0$. We will see this leads to a contradiction.

Since finite total curvature of M_1, M_2, implies finite topological type, each M_i has a finite number of ends, each end topologically an annulus. Since $\text{dist}(M_1, M_2) = 0$ and M_1, M_2 have compact boundaries, there must be an end E_1 of M_1 and E_2 of M_2 such that $\text{dist}(E_1, E_2) = 0$. After a rotation in $\mathbb{R}^3$, we can assume E_1 and E_2 are asymptotic to the same horizontal catenoid end of growth rate a (if $a = 0$ its a horizontal plane).

Since $E_1 \cap E_2 = \phi$, we can assume E_1 lies above E_2 (we can take E_1 and E_2 to be graphs). After a small vertical downward translation E_1' of E_1, $\partial E_1'$ still lies above E_2, but outside of a large ball, E_1' lies below E_2. Hence $E_1' \cap E_2$ is a compact nonempty one dimensional analytic subset of both E_1' and E_2.

We now show that $E_1' \cap E_2$ is a simple closed curve γ, homotopically non trivial on E_1' and E_2, and E_1' is transverse to E_2 along γ. Since E_1' is a graph, the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ of $E_1' \cap E_2$ is a compact nonempty one–dimensional analytic variety of $\mathbb{R}^2$. Let D be a disc in $\mathbb{R}^2$ so that E_1' is a graph over $\mathbb{R}^2 - D$. If $\pi(E_1' \cap E_2)$ is not a connected, homotopically non trivial simple closed curve in $\mathbb{R}^2 - D$, then $\mathbb{R}^2 - \pi(E_1' \cap E_2)$ contains a compact component disjoint from D. This is impossible since the lifts of this component to E_2 and E_1' correspond to different solutions to the minimal surface equation with the same boundary values (impossible by the usual maximum principle). Hence E_1' intersects E_2 transversely in a single curve γ that is homotopically non trivial on E_1' and E_2. Let $\widetilde{E}_1$ and $\widetilde{E}_2$ denote the ends of E_1', E_2 respectively, with boundary γ.

The surfaces $\widetilde{E}_1$ and $\widetilde{E}_2$ represent distinct solutions to the minimal surface equation (they are graphs) over the unbounded region Δ of $\mathbb{R}^2$ with boundary $\pi(\gamma)$, and they have the same boundary values along $\pi(\gamma)$. Since $\widetilde{E}_1$ and $\widetilde{E}_2$ are asymptotic to translates of a fixed vertical catenoid, they have the same logarithmic growth rate.

Let X_1 and X_2 denote the gradient of the third coordinate functions of $\widetilde{E}_1$ and $\widetilde{E}_2$, respectively. If ν_1, ν_2 denote the conormal (upward pointing) vectors to $\widetilde{E}_1$ and $\widetilde{E}_2$ along γ, then

$$Flux(\widetilde{E}_1) = (0, 0, \int_\gamma X_1 \cdot \nu_1) = (0, 0, \int_\gamma X_2 \cdot \nu_2) .$$

However, along γ, $\widetilde{E}_1$ lies below $\widetilde{E}_2$, so $X_1 \cdot \nu_1 < X_2 \cdot \nu_2$ at each point of γ, and this contradicts $\int_\gamma X_1 \cdot \nu_1 = \int_\gamma X_2 \cdot \nu_2$.

A basic result in minimal surface theory is the monotonicity formula; a proof may be found in [G.-T.].

THEOREM 3.5 (Monotonicity Formula). — *Let M be a properly immersed minimal surface in $\mathbb{R}^3$, $x \in M$, and $D_R(x) =$ the euclidean ball of $\mathbb{R}^3$, of radius R, centered at x. Let k be the number of sheets of M passing through x and $\Sigma(R) = M \cap D_R(x)$. Then $\frac{|\Sigma(R)|}{\pi R^2 k}$ is a monotone increasing function of R, which tends to one as $R \to 0$. Here $|\Sigma(R)|$ denotes the area of $\Sigma(R)$. In fact, each sheet of $\Sigma(R)$, passing through x, has area growing at least as fast as πR^2, the area of the flat disc through x of radius R.*

COROLLARY. — *Let M be a complete finite total curvature surface in $\mathbb{R}^3$ with exactly two embedded ends. Then M is embedded.*

Proof of Corollary : Each end E of M can be written as the graph of a function u over $\mathbb{R}^2$ – a compact disc. A simple calculation shows the area growth of E is Euclidean, i.e. $\frac{|E \cap D_R|}{\pi R^2} \to 1$ as $R \to \infty$. Since M has exactly two ends, each embedded, we conclude $f(R) = \frac{|M \cap D_R|}{2\pi R^2} \to 1$ as $R \to \infty$. If M had a point of self intersection, the monotonicity formula implies $f(R) = 1$ for all R hence M is the union of two flat planes.

Proof of Theorem 3.1 : Let E_1 and E_2 be the ends of M and let γ_1 and γ_2 be smooth Jordan curves on E_1 and E_2 respectively, each homotopically non trivial on its end. Let Σ be the compact submanifold of M bounded by $\gamma_1 \cup \gamma_2$. Since γ_1 is homologous to γ_2 in Σ, the flux of γ_1 equals the flux of γ_2. Hence, if the flux of γ_1 is different from 0, then the limiting normal vectors to E_1 and E_2 are parallel and if a_1 and a_2 are the logarithmic growth rates of E_1 and E_2 respectively, then $a_1 = -a_2$. Also the flux formula implies that if a_1 is zero then so is a_2. Thus both ends are simultaneously catenoid type ends or planar type ends. Moreover, since we know M is embedded, the ends are always parallel, i.e. their limiting normal vectors are parallel.

After a rotation of M we can assume the ends are horizontal.

We observe first that neither end of M is planar. For if E_1 were planar then we could find a horizontal plane P disjoint from M. Then move P towards M by parallel translation. There would be a first point of contact with the horizontal plane (at a finite point of M, or at infinity) and this contradicts the maximum principle at infinity, or the usual maximum principle.

So we can assume E_1 is a catenoid type end (above E_2) with growth rate $a_1 \neq 0$ and $a_1 = -a_2$.

We will now prove there is a horizontal plane P which is a plane of symmetry of M. This allows us to say that the catenoids to which E_1 and E_2 are asymptotic, have the same vertical axis. One finds P by applying the Alexandrov reflection technique and the maximum principle at infinity.

Let $P(t)$ be the horizontal plane $x_3 = t$. For each t, let M_t^+ be the part of M, on and above $P(t)$ and M_t^- the part of M on and below $P(t)$. Let M_t^* be the symmetry of M_t^+ by $P(t)$:

$$M_t^* = \{(x, 2t - x_3)/(x, x_3) \in M_t^+\} .$$

A surface S has locally bounded slope if the tangent plane to every interior point of S never contains a vertical line. Finally we say a subset A is above a subset B, written $A \geq B$, if for every $x \in \mathbb{R}^2$ for which $p^{-1}(x) \cap A \neq \phi$ and $p^{-1}(x) \cap B \neq \phi$, we have all points of $p^{-1}(x) \cap A$ lying above all points of $p^{-1}(x) \cap B$. Here $p : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection to the horizontal.

Now for t large, M_t^+ is a graph of locally bounded slope over (a part of) $P(t)$; M_t^+ is part of E_1 for t large. M_t^* is then a catenoid type end of growth rate $-a_1$; the same as E_2. Thus for t sufficiently large, M_t^* is above M_t^-.

Now consider decreasing t, to s say, and the surface M_s^*. We claim that if for each τ, $s \leq \tau \leq t$, M is never vertical along $P(\tau)$, then M_s^* is above M_s^-. Otherwise, there would be a first interior point of contact of some M_τ^* with M_τ^- and the usual maximum principle yields $M_\tau^* = M_\tau^-$ so M is vertical along $P(\tau) \cap M$. Here we have used the maximum principle at infinity to say the end of M_τ^* is a strictly positive distance from E_2.

Since it is not possible that M_s^* is above M_s^- for all s, there is a largest τ such that M is vertical at some point p of $M \cap P(\tau)$. M_τ^* is above M_τ^- hence M_τ^* and M_τ^- have a common boundary, they are tangent at p, and one is locally on one side of the other at p. Thus $M_\tau^* = M_\tau^-$ by the boundary maximum principle and M is invariant by reflection in $P(\tau)$.

Now consider the development of E_2 as a graph :

$$u_2(x) = a\ \ell n|x| + b + \frac{c_1 x_1}{|x|^2} + \frac{c_2 x_2}{|x|^2} + \mathcal{O}(|x|^{-2}) .$$

We wish to translate M so that the axis of the catenoid passes through the origin. Let $x_1 = y_1 + \alpha_1$, $x_2 = y_2 + \alpha_2$. Then u_2 in terms of y is given by :

$$u_2(y) = a\ \ell n|y| + b + \frac{\widetilde{c}_1 y_1}{|y|^2} + \frac{\widetilde{c}_2 y_2}{|y|^2} + \mathcal{O}(|y|^{-2}) ,$$

where $\widetilde{c}_i = c_i + a\alpha_i$, $i = 1, 2$. Thus letting $\alpha_i = -\frac{c_i}{a}$, $i = 1, 2$, and calling y by x again, we have :

$$u_2(x) = a\ \ell n|x| + b + \mathcal{O}(|x|^{-2}) .$$

Assuming the horizontal plane $x_3 = 0$ is the plane of symmetry of M (which we can suppose after a vertical translation of M), we have $u_2(x) = -u_1(x)$ and the development of $u_1(x)$ is :

$$u_1(x) = -a\ \ell n|x| - b + \mathcal{O}(|x|^{-2}) .$$

We will now prove M is a surface of revolution about the x_3–axis, thus a catenoid. Since the expression of u_2 is invariant by rotation about the origin in the $x = (x_1, x_2)$ plane, it suffices to prove the plane $x_1 = 0$ is a plane of symmetry of M.

Denote by $P(t)$ the planes $x_1 = t$ (now we shall think of the x_1 axis as vertical) and for K a (large) constant let $M \cap \{|x_3| = K\} = B_1 \cup B_2 = B$, where $B_2 = M \cap \{x_3 = K\}$, $B_1 = M \cap \{x_3 = -K\}$.

Here is the idea of what we shall do next. Fix $t > 0$. For K large, B_1 and B_2 are close (C^1–close) to circles in the planes $|x_3| = K$, centered at the x_3 axis. So B_t^+ is a graph of bounded slope over $P(0)$ and $B_t^* \geq B_t^-$, cf. figure 12.

Let $\Sigma = \Sigma(K)$ be the compact part of M bounded by B. By doing the Alexandrov reflection technique with the horizontal planes $P(s)$, coming down to $P(t)$, from above $\Sigma(K)$, one proves that Σ_t^+ is a graph of bounded slope over $P(0)$ and $\Sigma_t^* \geq \Sigma_t^-$. By construction, this will continue to hold for any larger value of K. So $M_t^* \geq M_t^-$. Since this is true for any $t > 0$, we have $M_0^* \geq M_0^-$. Now do the same argument from below, i.e. start with $-t$ and come up, from below with horizontal planes to conclude $M_0^- \leq M_0^+$. hence M is invariant by symmetry in $P(0)$.

We now make the above discussion precise. First we prove B_t^+ is a graph of bounded slope and $B_t^* \geq B_t^-$. Clearly it suffices to do this for B_2 (B_1 is the same argument). We have

$$u_2(x) = a\ \ell n|x| + b + \mathcal{O}(|x|^{-2})\ ,$$

hence

$$\frac{\partial u_2}{\partial x_1} = \frac{ax_1}{|x|^2} + \mathcal{O}(|x|^{-3})\ .$$

Consequently for $x_1 \geq t$ and $|x|$ sufficiently large (depending on t) we have $\frac{\partial u_2}{\partial x_1} > 0$. The normal vector η to B_2, in the plane $x_3 = K$ is

$$\eta = \left(\frac{ax_1}{|x|^2} + \mathcal{O}(|x|^{-3}),\quad \frac{ax_2}{|x|^2} + \mathcal{O}(|x|^{-3}), 0\right)\ .$$

The first coordinate of η is non zero for $|x|$ large, and $x_1 \geq t$, so B_t^+ is a graph of bounded slope.

Now on B_2 we have $u_2 = K$, so

$$\ell n|x| + \mathcal{O}(|x|^{-2}) = \frac{1}{a}(K - b)$$

hence $|x|e^{\mathcal{O}(|x|^{-2})} = R$ for a large R. Since $e^{\mathcal{O}(|x|^{-2})} = 1 + \mathcal{O}(|x|^{-2})$, it follows that $|x| = R + \mathcal{O}(|x|^{-1})$, so B_2 is close to a circle for $|x|$ large. Let C denote the circle of radius R in $x_3 = K$, centered at the origin. Clearly

$$\mathrm{dist}(C_t^*, C_{t/2}^-) \geq \varepsilon(t)\ ,$$

where $\varepsilon(t) > 0$, depends only on t. Hence if K is sufficiently large, $B_{2,t}^* \geq B_{2,t/2}^-$. Since $B_{2,t/2}^+$ is a graph over $P(0)$, it follows that $B_{2,t}^* \geq B_{2,t}^-$.

To complete the proof of R. Schoen's theorem, it remains to show $\Sigma_t^* \geq \Sigma_t^-$, where $\Sigma = \Sigma(K)$ is the compact submanifold of M bounded by B.

Let T be the maximum value of x_1 on Σ; it is realised on B since x_1 is harmonic.

Define $J = \{s \in [t, T] / \Sigma_s^+$ is a graph of locally bounded slope over $P(s)$ and $\Sigma_s^* \geq \Sigma_s^-\}$. We prove the theorem by showing J is a non empty open and closed subset of $[t, T]$, so $t \in J$.

Let $p \in B$ be a point where $x_3 = T$. Then $T \in J$ so J is non empty. Since a compact minimal surface is in the convex hull of its boundary (the maximum principle applied to the coordinate functions) Σ is in the slab between the planes $x_3 = K$ and $x_3 = -K$. By the boundary maximum principle, Σ is never tangent to the planes $|x_3| = K$; i.e; Σ is transverse to $|x_3| = K$ along B. So for s near T, $s < T$, Σ_s^+ is a graph over $P(s)$ of locally bounded slope. Notice also, that if $s \in J$ and $s < s_1 \leq T$, then $s_1 \in J$.

First we show J is closed. Suppose $(s,T) \subset J$. If Σ_s^+ is not a graph then there is s_1, $s < s_1 \leq T$ and $x \in P(o)$ such that (x,s) and (x,s_1) are on the same vertical. We choose s_1 so there are no other points of Σ on the vertical between (x,s) and (x,s_1). We know that (x,s) is an interior point of Σ since B_t^+ is a graph and Σ touches $|x_3| = K$ only along B.

Now Σ is a graph in a neighborhood of (x,s_1) and not vertical in the neighborhood. So the vertical lines to this neighborhood, descend to fill a neighborhood of (x,s). It follows that Σ is below $P(s)$ in a neighborhood of (x,s), otherwise there would be an $(\overline{x},\overline{s})$ on Σ, near (x,s), with $\overline{s} > s$. But then $(\overline{x},\overline{s})$ would have another point of Σ, above it, on the vertical, so $\Sigma_{\overline{s}}^+$ would not be a graph. Clearly Σ below $P(s)$ at (x,s) implies $\Sigma = P(s)$ by the maximum principle, which is a contradiction. Thus J is closed.

Next we show J is open. Let $(x,s) \in \Sigma$ and let assume $[s,T] \subset J$.

Let $D \subset \Sigma$ be a disc containing (x,s). Notice that Σ is not vertical at (x,s), for if this were so, consider the discs D_s^* and D_s^-. Then D_s^* is locally on one side of D_s^- in a neighborhood of (x,s), so $D_s^* = D_s^-$ by the boundary maximum principle and $P(s)$ is a plane of symmetry of Σ. This is impossible since B is not symmetric in $P(s)$.

Thus Σ is a graph in a neighbourhood U of $P(s)$ and not vertical in U. It remains to show $\Sigma_\tau^* \geq \Sigma_\tau^-$ for τ near s. Since $\Sigma \cap U$ is a graph, we have $\Sigma_\tau^* - V \geq \Sigma_\tau^-$ for V a neighborhood of $P(s)$, $V \subset U$, and τ near s. Also $\Sigma_\tau^+ - V$ is compact and its image under reflection in $P(s)$ is disjoint from Σ_s^-, so by continuity, for τ near s, we have $\Sigma_\tau^* - V \geq \Sigma_\tau^-$. This means $\Sigma_\tau^* \geq \Sigma_\tau^-$ for τ near s.

The last argument using Alexandrov reflection actually proves more : under certain circumstances, a minimal surface inherits the symmetries of its boundary. More precisely, R. Schoen proved :

THEOREM 3.6. — *Suppose $\Omega \subset \mathbb{R}^n$ is a compact domain whose boundary is mean convex. Let $B^{n-1} \subset \mathbb{R}^{n+1}$ be a compact embedded manifold (not necessarily connected) satisfying : $B \subset (\partial\Omega) \times \mathbb{R}$, B_0^+ is a graph of locally bounded slope over $P(0) = \mathbb{R}^n \times (0)$ and $B_0^+ \geq B_0^-$. Let M be an embedded minimal hypersurface with $\partial M = B$ and all interior points in $\Omega \times \mathbb{R}$. Then M_0^+ is a graph over $P(0)$ of locally bounded slope and $M_0^+ \geq M_0^-$.*

In fact, R. Schoen proved this theorem only assuming M immersed, but one has to work a little more than we did to obtain this generality. The hypothesis, the interior points of M are in $\Omega \times \mathbb{R}$ is not serious, since if an interior point p of M is in $(\partial\Omega) \times \mathbb{R}$ then, by the maximum principle, the connected component of p in M is entirely contained in $(\partial\Omega) \times \mathbb{R}$, so one can disregard these components of M.

It is easy to construct examples of boundaries B whose symmetries do not pass to a minimal M with $\partial M = B$. For example consider two copies of a dumbell curve in parallel planes as in figure 13.

The (asymmetric) minimal M can be obtained by applying the bridge principle to a catenoid bounded by the two circles on the right, and the two discs bounded by the two circles on the left.

We now begin the proof of the Lopez–Ros theorem : the plane and the catenoid are the only properly embedded minimal surfaces with finite total curvature and genus zero.

Here is the idea. Suppose M is a embedded finite total curvature surface in $\mathbb{R}^3$. We know M has a finite number of ends which we can assume horizontal (i.e. their limiting normals are vertical) after a rotation of M in $\mathbb{R}^3$. Each end is asymptotic to a horizontal plane or to a catenoid.

Lopez and Ros deform M, among minimal surfaces by deforming the Weierstrass data. If (g, ω) is the Weierstrass data of M they consider the data $\left(\lambda g, \frac{\omega}{\lambda}\right)$, where λ is a positive real number. One checks that this data defines an immersion X_λ of M. In fact this holds whenever the flux of M is vertical. In our case this is so, since M has genus zero so all the flux is at the ends. The planar ends have zero flux and the catenoid type ends have vertical flux since their limiting normals are vertical.

Next one observes that each catenoid end of M is a (vertical) catenoid end of X_λ with the same logarithmic growth rate; it can move up or down during the deformation. The planar ends stay planar ends at the same height.

By the maximum principle at infinity, the distance between the ends of X_λ is strictly positive, as λ goes from 1 to infinity. Therefore each X_λ is an embedded surface.

If M has a point p where the normal vector is vertical, then there is a neighborhood D of p and a $\lambda > 1$ such that $X_\lambda(D)$ is not embedded (assuming M is not a plane). One sees this by proving X_λ converges to an Enneper surface (which is not embedded) near p, as $\lambda \to \infty$. Thus M has no points p where the normal is vertical.

If M has a planar end A, then a similar analysis proves $X_\lambda(D)$ is not embedded, for λ large and D a subend of A (assuming M not flat).

Thus all the ends of M are catenoid type and the Gauss map has no zeros or poles on M. The conformal compactification of M is the sphere S and the foliation by the level curves of x_3 is non singular on M and has a singularity of positive index at each puncture. Hence there are exactly two ends in M.

We could now refer to R. Schoen's theorem to conclude M is a catenoid but it is easy to prove this directly. Since the Gauss map has all it's zeros and poles at the ends, g has degree one. After a conformal reparametrisation of M one can assume $g(z) = z$. A little residue theory then proves $\omega(z) = c\frac{dz}{z^2}$, $c \in \mathbb{R}$, hence M is a catenoid [Oss.-1].

Now we enter into the details of this argument.

For each cycle γ on M, one can calculate the flux of γ by :

$$\int_\gamma (\phi_1, \phi_2, \phi_3) = i \text{ Flux}(\gamma) .$$

From this formula, it follows easily that the following three conditions are equivalent to M having vertical flux :

1) the forms ϕ_1 and ϕ_2 are exact,

2) the forms ω and $g^2\omega$ are exact,

3) for each $\lambda > 0$, the immersion X_λ is well defined on M.

The formula for the metric and curvature show that each $X_\lambda(M) = M_\lambda$ is a complete minimal surface of the same total curvature as M.

Now suppose $p \in M$ is a point where the normal to M is vertical, say $(0,0,-1)$. Parametrize a neighborhood of p conformally by $\{0 \leq |z| < \varepsilon\}$, with

$$g(z) = z^k, \quad \omega = (a + zh(z))dz \ ,$$

where $a \in \mathbb{C}^*$ and h is holomorphic in $D(t) = \{|z| < \varepsilon\}$. Introduce the conformal coordinate $\xi = \lambda^{1/k} z$ on $D(\lambda^{1/k}\varepsilon)$. Then X_λ is parametrized by

$$g_\lambda(\xi) = \xi^k, \quad \omega_\lambda = \frac{1}{\lambda^{1+1/k}}\Big(a + \frac{\xi}{\lambda^{1/k}} h\Big(\frac{\xi}{\lambda^{1/k}}\Big)\Big) d\xi \ .$$

Now dilate X_λ by $\lambda^{1+1/k}$ to obtain $\widetilde{X}_\lambda$. As $\lambda \to \infty$, $\widetilde{X}_\lambda$ converges, on compact subsets of $\mathbb{C}$, to the minimal surface $X_\infty : \mathbb{C} \to \mathbb{R}^3$, with Weierstrass data :

$$g_\infty(\xi) = \xi^k, \quad \omega_\infty = a d\xi \ .$$

This is a complete surface with a non embedded end; there are transversal self intersections. Hence X_λ, for λ large, has self intersections.

If the normal at p is $(0,0,1)$, then turn M upside down.

Now suppose A is a planar end of M (and M is not a plane), and let the limiting normal vector to A be $(0,0,-1)$. Parametrize a subend of A by the Weierstrass data in $D(\varepsilon)$:

$$g(z) = z^k, \quad \omega = \Big(\frac{a}{z^2} + h(z)\Big) dz \ ,$$

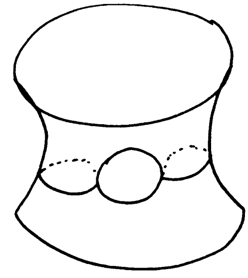
where $a \in \mathbb{C}^*$ and h holomorphic in $D(\varepsilon)$.

We have a parametrization of the end of M_λ in $D(\lambda^{1/k}\varepsilon)$, $\xi = \lambda^{1/k} z$,

$$g_\lambda(\xi) = \xi^k, \quad \omega_\lambda = \frac{1}{\lambda^{1-1/k}}\Big(\frac{a}{\xi^2} + \frac{1}{\lambda^{2/k}} h\Big(\frac{\xi}{\lambda^{1/k}}\Big)\Big) d\xi \ , \ k > 1 \ .$$

After a homothety by $\lambda^{1-1/k}$, we obtain a new minimal surface $\widehat{M_\lambda}$. When $\lambda \to \infty$, $\widehat{M_\lambda}$ converges uniformly on compact subsets of $\mathbb{C}^*$ to $X_\infty : \mathbb{C}^* \to \mathbb{R}^3$ defined by

$$g_\infty(\xi) = \xi^k, \quad \omega_\infty = \frac{a}{\xi^2} d\xi, \quad \xi \in \mathbb{C}^* \ .$$



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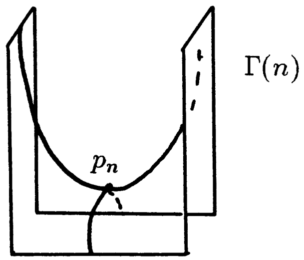


Figure 8

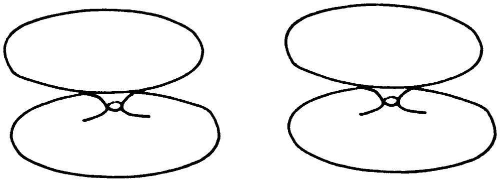


Figure 11.a

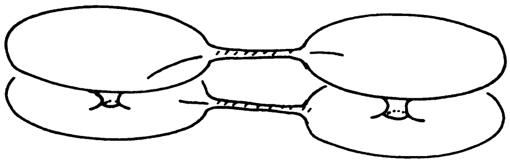
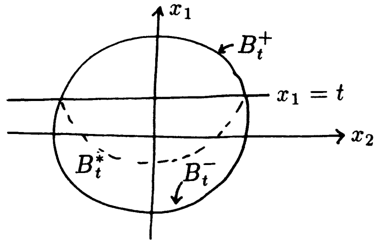


Figure 11.b



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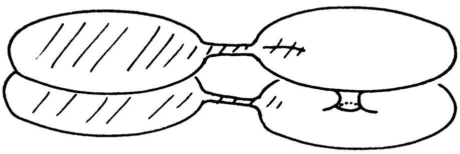


Figure 13

If $k = 1$, this is a catenoid, and if $k > 1$, the surface has a non embedded end at infinity. So for λ large, M_λ is not embedded, since A is a planar end.

It remains to prove each X_λ is an embedding. Let $J = \{\lambda / X_\lambda$ is injective$\}$. If $\lambda_0 \in J$ then the distance between two fixed ends of $X_{\lambda_0}(M)$ is strictly positive, by the maximum principle at infinity. Clearly this distance is a continuous function of λ (it may be infinite). Hence for λ near λ_0, X_λ is also an embedding and J is open.

Suppose $\lambda_n \in J$, $\lambda_n \to \lambda$ as $n \to \infty$. If X_λ is not injective then there are points $x, y \in M$, $x \neq y$, with $X_\lambda(x) = X_\lambda(y)$. The intersection of M_λ at $X_\lambda(x)$ and $X_\lambda(y)$ cannot be one dimensional so a neighborhood of x and a neighborhood of y have the same image by X_λ, by the usual maximum principle (we used $\lambda_n \in J$ here). Hence $X_\lambda : M \longmapsto M_\lambda$ is a finite covering of the (embedded) minimal surface M_λ. Again, by the maximum principle at infinity, there is an ε–tubular neighborhood U of M_λ that is embedded. The ends of M_{λ_n} vary continuously so for n large, $M_{\lambda_n} \subset U$. But then the orthogonal projection of M_{λ_n} to M_λ is a diffeomorphism so X_λ is also a diffeomorphism. This contradiction shows J is closed and completes the proof of the Lopez-Ros theorem.

4. CURVATURE ESTIMATES FOR STABLE MINIMAL SURFACES

In 1952, E. Heinz proved that if M is a minimal graph over the disc D_R of radius R ($D_R = \{x^2 + y^2 \leq R^2\}$) and if K_0 is the Gaussian curvature of M at the origin, then [Heinz] :

$$|K_0| \leq \frac{4\pi^3}{3R^2} .$$

This result was generalized by E. Hopf, Finn and Osserman, [E.Hopf-1],[F.-Oss.], to parametric minimal surfaces whose Gauss map misses an open set.

The most general theorem was obtained by R. Schoen [Sch.-2] : there is a universal constant $C > 0$ such that if M is a stable minimal (immersed and complete) surface in a flat three manifold then

$$|K(p)| \leq \frac{C}{d(p)^2} ,$$

where $p \in M$, and $d(p)$ is the intrinsic distance of p to ∂M. Stable means that every compact domain D of M minimizes area up to second order, among normal variations of D leaving the boundary fixed; we will make this precise shortly.

This theorem of Schoen is a very important tool for the study of minimal surfaces in three–manifolds. Notice that this implies the only complete immersed stable minimal surfaces with no boundary in flat 3–manifolds are totally geodesic. So, for example, in $\mathbb{R}^3$ they are planes. This result was also proved by Do Carmo and Peng [Do C.-P.].

Why is Schoen's result a generalization of Heinz's theorem, i.e. why is a minimal graph stable? In general a foliation by minimal hypersurfaces implies each leaf is stable. For the unit vector field n to the foliation is divergence free in the ambient space. Let D be a compact domain in a leaf and $\widetilde{D}$ a chain with $\partial\widetilde{D} = \partial D$; so that $D \cup \widetilde{D}$ is a cycle, homologous to zero. The divergence theorem implies the flux of n across D equals the flux of n across $\widetilde{D}$; i.e.

$$area(D) = \int_D < n, n >= \int_{\widetilde{D}} < n, n_{\widetilde{D}} > \leq area(\widetilde{D}) ,$$

where $n_{\widetilde{D}}$ is the unit normal vector field to $\widetilde{D}$. Hence D is area minimizing in its homology class.

Now the vertical translation of a graph foliates a solid cylinder and the above argument shows that for every $R' < R$, the part of M over $D_{R'}$ minimizes area up to second order. Letting $R' \longrightarrow R$ we see this is also true for M.

4.1. The Barbosa–Do Carmo stability criteria

There is an important criteria for stability of a domain on a minimal surface in $\mathbb{R}^3$ due to Barbosa and Do Carmo [B.-Do C.] which implies that graphs are stable. Their theorem says that an immersed minimal surface in

$\mathbb{R}^3$ is stable if the area of the spherical image (by the Gauss map) is less than 2π.

I would like to make a few comments on their theorem. Let D be a compact domain on the minimal surface M, n a unit normal vector field to M and f a piecewise smooth function on D which vanishes on ∂D. The vector field $Y = fn$ on D, induces a normal variation of D and the second derivative of area of this variation is :

$$-\int_D f(\Delta f - 2Kf)$$

where Δ is the intrinsic Laplacian of M. The operator $L = \Delta - 2K$ is the stability (or Jacobi) operator of M. M stable means the above integral is strictly positive for all compact domains D and non constant f on D, vanishing on ∂D. Hence if one can find a non constant f, $f = 0$ on ∂D, in the kernel of L (such f are called Jacobi fields), D is not stable.

Now suppose D is a domain on which the Gauss map g is a branched covering onto $g(D)$. Then Schwarz proved that if the first eigenvalue λ_1 of the spherical Laplacian Δ_s on $g(D)$ is less than two, D is not stable. Here is the proof. Let u be a function on $g(D)$, u positive in interior $g(D)$, zero on $\partial g(D)$ and $\Delta_s u + \lambda_1 u = 0$. Define $f = u \circ g$. Since $g(\partial D) = \partial(gD)$, f vanishes on ∂D and is positive in interior D. The second variation defined by f is :

$$-\int_D f\Delta_s f + 2f^2 = (\lambda_1 - 2)\int_D f^2 < 0 .$$

Hence D is not stable (the above integrals are taken on the complement of the branch points of g).

Now here is the idea of the proof of the Barbosa–Do Carmo stability criteria. If D is not stable then one can find a domain $\widetilde{D} \subset D$ and a function u on $\widetilde{D}$, $u > 0$ on int $\widetilde{D}$, $u = 0$ on $\partial\widetilde{D}$ and $\Delta u - 2Ku = 0$.

One then averages u via the Gauss map to obtain a function f on $g(\widetilde{D})$ satisfying

$$\int_{g(\widetilde{D})} |grad\ f|^2 \leq 2\int_{g(\widetilde{D})} f^2 .$$

This inequality implies $\lambda_1(g(\widetilde{D})) \leq 2$.

However, among all spherical domains having a fixed area, the spherical cap minimizes the first eigenvalue of the Laplacian. But a spherical cap in an open hemisphere has $\lambda_1 > 2$ (the coordinate functions of $\mathbb{R}^3$ satisfy $\Delta_s + 2 = 0$ and they are positive on an open hemisphere, zero on its boundary) so this is a contradiction.

4.2. An idea of the proof of Heinz's theorem

Let us suppose M is the graph of a function f whose gradient vanishes at the origin. This gradient hypothesis makes the proof simpler. The idea is to compare M to a Scherk graph.

We can assume f defines a minimal graph on D_R, $f(0,0) = 0$, and $|\nabla f(0,0)| = 0$. Rotate the graph of f so the x–axis is a principal direction, curving upwards.

Let N be a Scherk graph defined over a square of side length 2, centered at the origin, with boundary values $+\infty$ on the vertical sides of the square and $-\infty$ on the horizontal sides. Assume also N passes through the origin; clearly N is horizontal at the origin.

Let $\widehat{K}_0$ be the Gauss curvature of N at the origin. A homothety of N by $C > 0$, from the origin, transforms N to a minimal graph N_C defined over a square $\diamond(C)$ containing D_C. Since curvature is multiplied by $\frac{1}{C^2}$ under this homothety, the curvature K_C of N_C at the origin, satisfies

$$|K_C| \leq \frac{|\widetilde{K}_0|}{C^2} .$$

Notice that N_C is horizontal at the origin and one of the principal curvatures of N_C is along the x–axis and points upward.

Choose $C > 0$ so that the principal curvature of N_C, along the x–axis at the origin, equals the corresponding principal curvature of M at the origin. Then $K_0 = K_C$ at this point.

Now if $R \leq C$ then $D_R \subset D_C$ and

$$|K_0| = |K_C| \leq \frac{|\widetilde{K}_0|}{C^2} \leq \frac{|\widetilde{K}_0|}{R^2} .$$

If $\diamond(C) \subset D_R$ then consider $M \cap N_C$. Both surfaces are tangent at $\mathcal{O}$ so they are equal or $M \cap N_C$ is a one dimensional analytic curve, singular at $\mathcal{O}$, and with at least six branches passing through the singularity. Since N_C is asymptotic to infinity on the boundary of the square $\diamond(C)$, except at the four vertices, there must be a compact component of $M \cap N_C$ strictly contained in the vertical region over the interior of $\diamond(C)$ (at most one branch of $M \cap N_C$ can go to a fixed vertex of $\diamond(C)$ since M is a graph). Then there is a Jordan curve α in $\diamond(C)$ along which M and N_C agree. Since they are both graphs over the interior of α and one has unicity of such minimal graphs by the usual maximum principle, we have $M = N_C$; a contradiction. Thus $\diamond(C)$ is not contained in D_R, hence $C > R/\sqrt{2}$ and

$$|K_0| = |K_C| \leq \frac{|\widetilde{K}_0|}{C^2} < \frac{2|\widetilde{K}_0|}{R^2} ,$$

and a Heinz type estimate is established.

5. COMPACTNESS OF LEAST AREA FAMILIES AND CONSTRUCTION OF COMPLEMENTARY FINITE TOTAL CURVATURE SURFACES

A technique used often to study a complete minimal surface M in a flat 3–manifold N is to construct finite total curvature minimal surfaces Σ, with $\partial\Sigma$ compact and non empty, Σ non compact, such that $\partial\Sigma \subset M$ and $\text{int}(\Sigma) \cap M = \phi$. Such surfaces Σ trap M in small regions of N which makes the geometry of M understandable. We will see several examples of this technique.

First I would like to explain how Σ can be obtained. Let Ω be a complete region of N, whose boundary is a good barrier for solving the least–area Plateau problem (this theory was developed by Meeks and Yau [M.-Y.]). This means $\partial\Omega = C$ is a 2–dimensional variety, smooth except along an analytic one dimensional variety, such that

– C is mean convex at the smooth points, i.e., the mean curvature vector at such points, points into Ω (the zero vector points into Ω), and

– at a non smooth point of C, the angle between the smooth faces of C, at the point, is less than or equal to π (measured in Ω).

Then Meeks and Yau proved that any smooth embedded 1–cycle Γ in $\partial\Omega$, that is null homologous in Ω is the boundary of a compact least area surface Σ_Γ in Ω, and Σ_Γ is smooth and embedded. The idea is to solve the Plateau problem in N by taking a limit of embedded surfaces with boundary Γ whose areas converge to the infimum of all possible areas. Then one checks that such a minimizing sequence can be constructed to stay in Ω. The mean convexity (and angle condition) implies that surfaces leaving Ω will increase area when crossing $\partial\Omega$. Then one works (considerably) to extract a subsequence that converges to a smooth embedded surface.

One can also use geometric measure theory to obtain Σ_Γ [Simon]. Again, we are assuming $C = \partial\Omega$ is a good barrier and $\Gamma \subset \partial\Omega$ a smooth one cycle (i.e. a collection of disjoint smooth Jordan curves). If Γ bounds an oriented 2–chain in Ω then Γ bounds a smooth embedded orientable surface Σ_Γ in Ω which minimizes area among all orientable 2–chains in Ω with boundary Γ. If Γ is a Z_2–boundary in Ω then Γ bounds a smooth embedded least area surface in the same relative Z_2–homology class. If Γ bounds an orientable (immersed) surface of genus n in Ω, then Σ_Γ can be chosen of genus at most n and of least area in its homotopy class.

Now we will discuss how the least area compact minimal surfaces Σ_Γ can converge to finite total curvature, non compact, minimal surfaces Σ.

We assume M orientable, A an end of M, $A \subset \partial\Omega$, and Γ a smooth Jordan curve on A, not homologous to zero in Ω. Let $A_1 \subset A_2 \subset \cdots$ be an increasing sequence of compact submanifolds of A, which exhausts A, and $\partial A_i = \Gamma \cup \Gamma_i$. By our previous discussion of how one can solve the Plateau problem in Ω using geometric measure theory, we know there exists a least area smooth embedded surface Σ_i in Ω such that $\partial\Sigma_i = \Gamma \cup \Gamma_i$ and Σ_i is Z_2–homologous to A_i rel ∂A_i. Since A_i is orientable and $\Sigma_i \cup A_i$ is Z_2–homologous to zero, Σ_i is also orientable. Since Γ is not homologous to zero in Ω, Σ_i can be chosen connected.

Now we will show a subsequence of the Σ_i converges to a stable embedded minimal surface Σ with $\partial\Sigma = \Gamma$.

Observe that there are uniform local area bounds for the family Σ_i. For if $B \subset \Omega$ is a ball of radius r, ∂B transverse to Σ_i, then $\partial B \cap \Sigma_i$ is a 1–cycle on ∂B that bounds $(\bmod 2)$ a 2–chain on ∂B of area at most $2\pi r^2$. Since Σ_i

minimizes area bounded by ∂A_i (in the Z_2–homology class), we conclude $B \cap \Sigma_i$ has area at most $2\pi r^2$. Similarly if B is a ball centered at a point of $\partial\Sigma_i$, then the area of $\Sigma_i \cap B$ is at most the area of ∂B.

Now let $B(r) \subset \Omega$. By the curvature estimates of R. Schoen, after choosing a possibly smaller r, each component of $\Sigma_i \cap B(r)$ that intersects $B(r/2)$ can be expressed as a graph, of small gradient, over a plane P_i in $B(r)$, passing through the center of the ball, and P_i does not depend on the component. By the uniform area estimates, $\Sigma_i \cap B(r/2)$ contains a bounded number of components independent of i and hence there a a bounded number of associated graphs. Suppose for the moment that for every i, $\Sigma_i \cap B(r/2)$ contains one component. Choose a subsequence of the P_i to converge to a plane P through the center of the ball. Then the standard compactness theorem for minimal graphs implies a subsequence of the graphs $\Sigma_i \cap B(r/2)$ converge to a minimal graph over $P \cap B(r/2)$. When $\Sigma_i \cap B(r/2)$ has more than one component, we do the above argument to each component and obtain a (uniformly bounded) finite number of graphs over $P \cap B(r/2)$, to which the subsequence of $\Sigma_i \cap B(r/2)$ converges.

Now Ω has a countable basis of balls B_n where for every n and subsequence Σ_{i_λ} of Σ_i, the $\Sigma_{i_\lambda} \cap B_n$ have a convergent subsequence in B_n. Suppose the subsequence $\Sigma_{i_\lambda} \cap B_1$ converges in B_1. Then the associated sequence of graphs in $B_2 \cap \Sigma_{i_\lambda}$ has a subsequence converging in $B_2 \cup B_1$. Continue in this manner from B_i to B_{i+1} and take a diagonal subsequence. This yields a subsequence of Σ_i that converges to a smooth minimal surface Σ, with $\partial\Sigma = \Gamma$. It is not hard to see that Σ is embedded and stable (since it's a limit of least area embedded surfaces). Also the boundary regularity theorem of Hardt and Simons implies Σ is smooth along Γ [H.-S.]. Finally, the theorem of Doris Fisher Colbrie yields that Σ has finite total curvature [F.C.].

In particular, this technique yields :

LEMMA 5.1. — *Let M be a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end. Then there is an end of a catenoid or of a plane in the complement of M.*

Proof : Let Γ be a smooth Jordan curve on M that separates M into two non compact components, one of which we denote by A. M separates $\mathbb{R}^3$ into two connected components and Γ cannot be homologous to zero in both components; let Ω be a component such that Γ is not homologous to zero in Ω.

By our previous discussion, there is a finite total curvature embedded minimal surface Σ_Γ in Ω with $\partial\Sigma_\Gamma = \Gamma$. More precisely, $C = \partial\Omega = M$ is a minimal surface hence a good barrier for solving the Plateau problem. Let $A_i \subset A_{i+1}$ be an exhaustion of A with $\partial A_i = \Gamma \cup \Gamma_i$. Let Σ_i be an embedded minimal surface in Ω, Z_2–homologous to A_i, with $\partial\Sigma_i = \partial A_i$. As before, a subsequence of Σ_i converges to Σ_Γ.

Now it may be that $\Sigma_\Gamma \subset M$ (if it touches M at one interior point, then since it's on one side of M at this point, it is contained in M). In this case, at least one end of M is of finite total curvature, so asymptotic to a planar end or catenoid end B. By the maximum principle at infinity, the distance between the ends of M is strictly positive. So B can be translated into Ω to be disjoint from M. Similarly, if $\text{int}\Sigma_\Gamma \subset \text{int}\Omega$, then the ends of Σ_Γ are a strictly positive distance from M so the conclusion of the lemma is clear.

There is a slight refinement of this lemma which is useful.

LEMMA 5.2. — *Let B be a ball in $\mathbb{R}^3$ and A_1, A_2 properly embedded minimal surfaces, non compact with ∂A_1, ∂A_2 smooth Jordan curves such that $B \cap (A_1 \cup A_2) = \partial A_1 \cup \partial A_2$ and $A_1 \cap A_2 = \phi$. Let Δ be the annulus on ∂B, bounded by $\partial A_1 \cup \partial A_2$ and let Ω be the connected component of $\mathbb{R}^3 - (A_1 \cup A_2 \cup \Delta)$ disjoint from B. Then there is an end of a plane or a catenoid in the interior of Ω. Moreover, ∂A_1 is the boundary of a smooth embedded surface Σ in Ω and outside of a larger ball $\widetilde{B}$ containing B, Σ is a finite total curvature minimal surface that separates ends of A_1 and A_2, i.e. any path from A_1 to A_2 in $\mathbb{R}^3 - \widetilde{B}$, meets Σ.*

Proof : Let $\Gamma = \partial A_1$ and consider Ω, with $\partial\Omega = A_1 \cup \Delta \cup A_2$. If $\partial\Omega$ were a good barrier for solving the Plateau problem then the construction of $\Sigma = \Sigma_\Gamma$ proceeds exactly as in the previous lemma. However ∂B is not mean convex with respect to Ω. One changes the Riemannian metric of $\mathbb{R}^3$

in a neighborhood of Δ in Ω so that $\partial\Omega$ is a good barrier in the new metric (cf. [M.-Y.] for the details). Then one proceeds as before. (This lemma remains true even if A_1 and A_2 are properly immersed; [M.-R.-2].)

6. THE ANNULAR END THEOREM AND THE STRONG HALFSPACE THEOREM OF HOFFMAN–MEEKS

We can now give an idea of the proof of the following important result.

THEOREM 6.1 [H.-M.-3]. — *Let M be a properly embedded minimal surface in $\mathbb{R}^3$, then M can have at most two annular ends of infinite total curvature.*

Sketch of Proof: Let A_1, A_2 and A_3 be distinct annular ends of M. It is not hard to find a ball B such that $B \cap (A_1 \cup A_2 \cup A_3) = \partial A_1 \cup \partial A_2 \cup \partial A_3$. Using the previous lemma, one traps one of the ends, A_1 say, between standard ends E_1, E_2 (each is a catenoid or planar end).

Now one proves that A_1 has finite total curvature. This is the difficult part of the proof. One proves the tangent plane to A_1 is never vertical outside of some compact set (then the Gaussian image of this subend is in a hemisphere hence has area less than 2π, so by the stability theorem of Barbosa–Do Carmo, A_1 is stable so of finite total curvature). T_0 prove the tangent plane of A_1 is eventually never vertical one constructs foliations of the region between E_1, E_2 by minimal annuli whose boundaries are on $E_1 \cup E_2$. Then one studies the contact of A_1 with the foliation. The only contact points are of saddle type (by the usual maximum principle) and the topology of A_1 being simple one is able to show A_1 is eventually tranverse to the foliation which implies there are no vertical tangent planes. hence at most two annular ends of M can have infinite total curvature.

Now what about the two remaining annular ends, can they have infinite total curvature? This is unknown and it is one of the most important problems in this subject today. Meeks and I have proved :

THEOREM 6.2 [M.-R.-3]. — *Let M be a properly embedded minimal* *surface in* $\mathbb{R}^3$*, with more than one end. If* A *is an annular end of* M *then (after a rotation of* M *in* $\mathbb{R}^3$*), either* A *is asymptotic to a horizontal plane (hence has finite total curvature) or* x_3/A *is a proper harmonic function. In particular, every such* A *is conformally the punctured disc* $D^* = \{z \in \mathbb{C}/0 < |z| \leq 1\}$.

COROLLARY 6.3. — *If* M *is a properly embedded minimal surface of finite topology and more than one end, then* M *has finite conformal type.*

COROLLARY 6.4. — *If* M *is a properly embedded minimal annulus then after a rotation of* M*,* M *intersects every horizontal plane in a simple closed curve.*

COROLLARY 6.5. — *An* m*–surface in* $\mathbb{R}^3$ *with a helicoidal type end has exactly one end.*

The strong halfspace theorem

There are complete immersed non planar minimal surfaces in a halfspace of $\mathbb{R}^3$. Jorge and Xavier constructed such examples in a slab [J.-Xav.]. It is not known if such examples exist in a ball.

However, if the immersion is proper, Hoffman and Meeks proved this is not possible. They prove more :

THEOREM 6.6. — (the strong halfspace theorem [H.-M.-4]) *If* M_1 *and* M_2 *are disjoint properly immersed minimal surfaces in* $\mathbb{R}^3$ *then they are parallel planes.*

Proof : Assume first that M_2 is a plane (the (x, y) plane say) and M_1 is in the upper halfspace. After a vertical translation we can assume $\text{dist}(M_1, M_2) = 0$.

Let D_t be the disc of radius t in M_2 centered at the origin. Since M_1 is properly immersed, there is a $t > 0$ such that $\text{dist}(D_1, M_1) > t$. Choose $t < 1/4$. Let γ be the vertical upward translation of ∂D_1, a distance t. By our choice of t and D_1, $\gamma \cup \partial D_1$ is the boundary of a stable catenoid C_1. For each $t > 1$, $\gamma \cup \partial D_t$ is the boundary of a stable catenoid C_t and C_{t_1} is above C_{t_2} when $1 \leq t_2 \leq t_1$. As $t \to \infty$, the C_t converge to the horizontal plane at height t, less the disc E in this plane bounded by γ; figure 14.

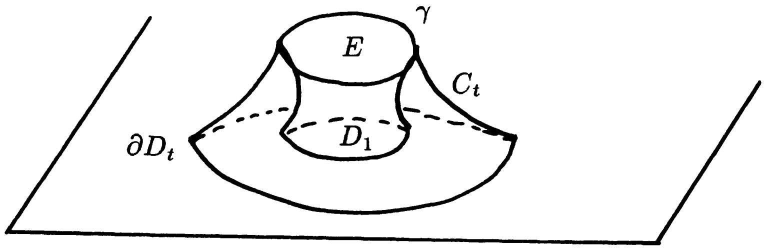


Figure 14

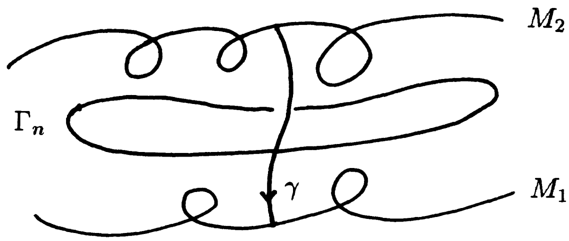


Figure 15

Clearly $E \cup C_t \cup D_t$ bounds a compact topological ball and the limit of these balls as $t \to \infty$ is the slab between M_2 and the horizontal plane at height t.

Now M_2 is properly immersed in $\mathbb{R}^3$ and $\text{dist}(M_2, M_1) = 0$ so there is a smallest t such that $C_t \cap M_2 \neq \phi$. But then C_t is on one side of M_2 at this point of first contact so $C_t = M_2$ by the maximum principle. This proves the strong halfspace theorem in the special case that M_2 is a plane.

Now suppose M_1 and M_2 are disjoint and properly immersed. We will find a plane between M_1 and M_2 so by what we have just proved M_1 and M_2 are planes too.

Let Ω be the connected region of $\mathbb{R}^3$ whose boundary is contained in

$M_1 \cup M_2$ and the boundary contains points of both M_1 and M_2. Notice that $\partial\Omega$ is a good barrier for solving the Plateau problem.

Let γ be an arc in Ω joining a point of M_1 to a point of M_2 and let Γ_n be Jordan curves in Ω such that the linking number of Γ_n and γ is one and Γ_n is in the complement of the ball of radius n centered at a fixed point of γ; figure 15.

Let Σ_n be a least area smooth immersed minimal surface in Ω with $\partial\Sigma_n = \Gamma_n$. As in V, a subsequence of the Σ_n converge to a complete stable minimal surface $\Sigma \subset \Omega$. Σ is non empty since each Σ_n intersects γ by our linking number restriction.

By R. Schoens theorem Σ is a plane. Clearly if Σ ever touched M_1 or M_2 then they would be planes too. This completes the proof of the strong halfspace theorem.

7. DOUBLY PERIODIC MINIMAL SURFACES

We call a minimal surface in $\mathbb{R}^3$ periodic if it is connected and invariant by a non trivial discrete group G of isometries that acts freely on $\mathbb{R}^3$. In fact we study the quotient minimal surface in $\mathbb{R}^3/G$. In fact, all connected properly embedded minimal surfaces M in $\mathbb{R}^3/G$ arise this way, since, by the strong halfspace theorem the lift of M to $\mathbb{R}^3$ is a connected minimal surface invariant by G (assuming M not planar). Notice that this implies that $\pi_1(M) \to \pi_1(R^3/G)$ is surjective under our hypothesis on M.

Our main result relates the topology of M to its total curvature $C(M)$.

THEOREM 7.1 (the finite total curvature theorem, [M.-R.-4]). — *Let M be a properly embedded minimal surface in a non simply connected complete flat 3-manifold. Then M has finite topology if and only if $C(M)$ is finite. When $C(M)$ is finite, we have the formula*

$$C(M) = 2\pi(\mathcal{X}(M) - W(M)) ,$$

where $W(M)$ is the total winding number of the ends of M (we define this later). When $N = T^2 \times \mathbb{R}$, $W(M) = 0$.

Notice that one needs to assume N not simply connected; the helicoid in $\mathbb{R}^3$ has infinite total curvature and finite topology.

I would like to discuss the proof of this theorem (at least for doubly periodic surfaces) and give some applications. A complete flat 3–manifold is finitely covered by T^3, $T^2 \times \mathbb{R}$ or $\mathbb{R}^3/S_\theta$, S_θ a screw motion around the x_3–axis, followed by rotation by θ about this axis. So our theorem concerns $T^2 \times \mathbb{R}$ and $\mathbb{R}^3/S_\theta$ (doubly and singly periodic surfaces).

Now let G be generated by two independant translations so that $\mathbb{R}^3/G = T \times \mathbb{R}$, T a flat 2–torus. Let $x_3 : T \times \mathbb{R} \to \mathbb{R}$ denote the *third* coordinate function, $T_t = T \times (t)$ the level set of x_3 at height t. We let $D^* = \{0 < |z| \leq 1\}$ be the punctured disc in $\mathbb{C}$.

LEMMA 7.2. — *Let A be an annulus diffeomorphic to D^* and $X : A \to T \times \mathbb{R}$ a proper minimal immersion of A. Then A contains a proper subannulus A' which can be conformally parametrized by D^*. In this parametrization $x_3/A'(z) = c\ \ell n|z|$ where c is constant.*

Proof : Let $X_3 = x_3 \circ X : A \to \mathbb{R}$; X_3 is a proper map. Since A has one end, X_3 is bounded from above or below but not both, so assume X_3 is bounded below. After translating $X(A)$ vertically downward, we can make the boundary of the annulus have negative x_3–coordinate and X_3 has 0 as a regular value. Hence $\Delta = X_3^{-1}(-\infty, 0]$ is a compact smooth submanifold of A. Δ contains exactly one component containing ∂A and the other components have x_3–coordinate zero. The maximum principle for the harmonic function X_3 implies Δ is connected and by elementary topology Δ is an annulus, and $A' = X_3^{-1}[0, \infty)$ is a proper subannulus of A.

The function X_3/A' is a proper nonnegative harmonic function with zero boundary values. It is an easy exercise in elementary complex analysis to prove that A' can be conformally parametrized by D^* and $X_3 = c\ \ell n|z|$ for some constant c.

Now let M be a properly immersed minimal surface in $T \times \mathbb{R}$, of finite topology. By the above lemma, each annular end of M is conformally D^* so M has finite conformal type. We want to know M is of finite total curvature when embedded so we may as well assume M is orientable (by passing to a two sheeted covering). Then the Gauss map $g : M \to S^2$ can be defined and is conformal; two liftings of a point of M to $\mathbb{R}^3$ differ by a translation that leaves the oriented unit normal vector field to the lifted surface, invariant.

Our technique to prove M has finite total curvature is to prove the punctures of the annular ends of M are removable singularities of the Gauss map g. Since the total ourvature is the area of the spherical image of M by g, this suffices. In general one shows the puncture is not an essential singularity by trapping an end A in a region of space which controls the values of g. If g misses to many values near the puncture then the singularity is removable. Now we can do this.

THEOREM 7.3. — *Let $X : A \to T \times \mathbb{R}$ be a proper minimal embedding of D^*. Then A has finite total curvature.*

Proof : By the previous lemma 7.2, we can suppose $A = D^*$ and $X_3(z) = c\ \ell n|z|$; we shall identify A with $X(A)$. We take $C < 0$ so that $X_3 \geq 0$ on A. Let $C_t = A \cap T_t$; each C_t is a simple closed curve. The proof divides into two cases : C_0 a generator of $\pi_1(T_0)$ or not. We shall consider the first case here and we refer the reader to [M.-R.-1] for the second case.

C_0 generates a cyclic subgroup G in $\pi_1(T \times \mathbb{R})$. Let $p : \widetilde{T \times \mathbb{R}} \to T \times \mathbb{R}$ be the Riemannian covering space such that $p_* \pi_1(\widetilde{T \times \mathbb{R}}) = G$. $\widetilde{T \times \mathbb{R}}$ is isometric to $(S^1 \times \mathbb{R}) \times \mathbb{R}$ and the generator of $\pi_1(T \times \mathbb{R})/G$ acts naturally on $H = p^{-1}(x_3^{-1}[0, \infty))$ as a translation. For notational convenience, let A also denote a lifting of A to H. Since ∂A is compact and $S^1 \times \mathbb{R} = \partial H$ is non compact, we can choose a closed geodesic α in ∂H such that $\alpha \cap \partial A = \phi$. Choose a covering transformation σ such that α is contained in the interior of the compact annulus Δ with boundary ∂A and $\sigma(\partial A)$; figure 16.

Let $\Omega \subset \widetilde{T \times \mathbb{R}}$ be the component of $H - (A \cup \sigma A)$ whose boundary contains $A \cup \sigma A$, and Ω_t the points of Ω at height at most t. Notice that Ω_t is a good barrier for the Plateau problem; its boundary consists of four minimal surfaces meeting at angles less than or equal to π.

Let α_t be a Jordan curve in the interior of the smooth annulus of $\partial \Omega_t$ at height t, such that α_t is homotopic to α. Let Σ_t be a least area embedded smooth surface with $\partial\Sigma_t = \alpha \cup \alpha_t$, $\Sigma_t \subset \Omega_t$, figure 16. First observe that Σ_t is orientable : this will follow by showing Σ_t separates Ω_t. If not then there is a simple closed curve δ in Ω_t which intersects Σ_t tranversely in one point. But $\pi_1(\Omega_t)$ is generated by $\pi_1(\partial A)$ hence δ is homotopic to a multiple of ∂A and ∂A has zero intersection number with Σ_t. Since the Z_2–intersection numbers are well defined in homotopy classes, this is impossible and Σ_t is orientable.

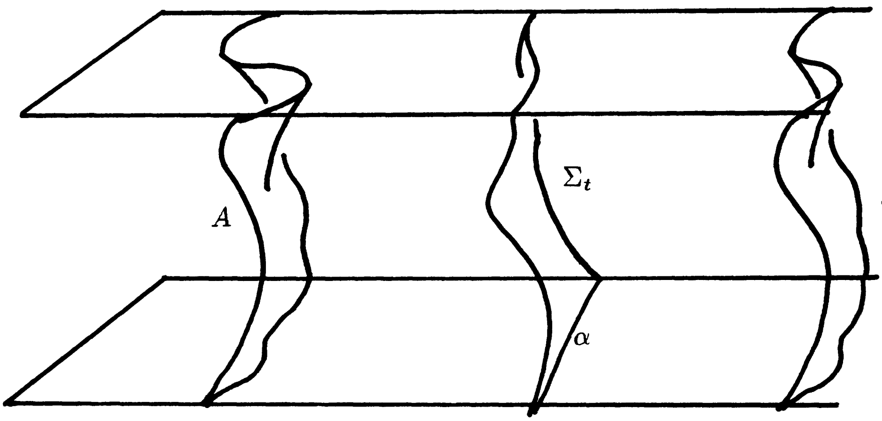


Figure 16

By 5, a subsequence of Σ_t converge to a smooth embedded stable surface Σ, $\partial\Sigma = \alpha$, $\Sigma \subset W$. By the usual maximum principle int$\Sigma \subset$ int W.

We now prove Σ is part of a plane. Since α is the quotient of a straight line in $\mathbb{R}^3$, we can extend Σ by Schwarz reflection R_α to a properly embedded minimal surface $\widetilde{\Sigma} \subset \widetilde{T \times \mathbb{R}}$. Note that since $\sigma(\Omega) \cap \Omega = \sigma(A)$ and $\Sigma \cap \partial\Omega = \alpha$, $\sigma\Sigma \cap \Sigma = \phi$. Let $R_{\sigma\alpha}$ be Schwarz reflection about $\sigma\alpha$ (i.e. rotation by π about $\sigma\alpha$). We claim that $\widetilde{\Sigma}$ and $\Sigma' = \sigma\Sigma \cup R_{\sigma\alpha}(\sigma\Sigma)$ are two properly embedded disjoint minimal surfaces. Note that $R_{\sigma\alpha} \circ \sigma = R_{\widetilde{\alpha}}$, and $R_{\widetilde{\alpha}} \cdot R_{\sigma\alpha} \cdot \sigma = id\cdot$, where $\widetilde{\alpha}$ is the geodesic on ∂H halfway between α and $\sigma\alpha$. Hence if $\widetilde{\Sigma} \cap \Sigma' \neq \phi$, then $R_\alpha\Sigma \cap R_{\sigma\alpha}(\sigma\Sigma) \neq \phi$. Composing the last inequality with $R_{\widetilde{\alpha}}$ yields $R_{\widetilde{\alpha}}R_\alpha(\Sigma) \cap \Sigma \neq \phi$. But $R_{\widetilde{\alpha}}R_\alpha = \sigma$ so

$\sigma\, \Sigma \cap \Sigma \neq \phi$, a contradiction.

Now that $\widetilde{\Sigma}$ and Σ' are disjoint the strong halfspace theorem says their lifts to $\mathbb{R}^3$ are planes, hence Σ and $\sigma\Sigma$ are parallel flat annuli in H.

Let $P(\theta)$ be a flat annulus which contains α and which makes an angle θ with the horizontal plane ∂H. Choose θ sufficiently small so that $P(\theta)$ intersects the region bounded by Σ and $\sigma\Sigma$ in a compact set and $P(\theta)$ intersects A transversally in a smooth curve. This is possible since A intersects ∂H transversally in a single curve.

Consider the foliation of $\widetilde{T \times \mathbb{R}}$ by planes parallel to $P(\theta)$ (flat annuli in fact). Notice that each leaf intersects $P(\theta)$ in a compact set. This foliation is defined by the level sets of a linear function whose restriction to A is a proper harmonic function. Hence this harmonic function has no critical points on A above $P(\theta)$. In particular, the normal to $P(\theta)$ is never attained as a normal vector to the part of A above $P(\theta)$.

Since θ can vary in an interval, the Gauss map on A misses a curve of values, hence the puncture is not an essential singularity and A has finite total curvature.

Now we can analyse the geometry of the ends of an m–surface of $T \times \mathbb{R}$ of finite topological type; we will see they converge geometrically to flat annuli. Before proving this we analyse immersed finite total curvature surfaces in $T \times \mathbb{R}$.

THEOREM 7.4. — *Let M be a properly immersed minimal surface in $T \times \mathbb{R}$, of finite total curvature. Let $A_1, \ldots, A_\ell$ be the ends of M with vertical limiting normal vectors and let n_i be the branching order of the Gauss map at the end A_i. Then $C(M) = 2\pi(\mathcal{X}(M) - \sum_{\lambda=1}^{\ell} n_\lambda)$. In particular, if M has no horizontal ends, then $C(M) = 2\pi\mathcal{X}(M)$.*

Proof: When M is nonorientable and we pass to the oriented two sheeted cover of M, then all the terms in above formula multiply by two. This is obvious for $C(M)$ and $\mathcal{X}(M)$; each end of M lifts to two ends in the cover. Hence we can assume M is orientable.

Let $M_t = M \cap (T \times [-t,t])$. By Gauss–Bonnet, we have :

$$C(M) = \lim_{t\to\infty} \int_{M_t} K = \lim_{t\to\infty} (2\pi \mathcal{X}(M_t) - \int_{\partial M_t} \kappa_g) .$$

For large values of t, $\mathcal{X}(M_t) = \mathcal{X}(M)$ so we must calculate $\int_{\partial M_t} \kappa_g$.

First consider a component C_t of ∂M_t that is on an end E having a nonvertical limiting normal vector v. We shall prove $\int_{C_t} \kappa_g \to 0$ as $t \to \infty$. Let $\vec{a}$ be a horizontal unit vector orthogonal to v; since v is not vertical, there are exactly two such vectors. Choose the orientation of $\vec{a}$ so that C_t' converges to $\vec{a}$ as $t \to \infty$ (C_t' is oriented by M_t and $'$ denotes derivative with respect to arc length).

Let $d\vec{a}$ be the closed one form defined by orthogonal projection on $\vec{a}$ (the line parallel to $\vec{a}$). We have $\int_{C_{t_1}} d\vec{a} = \int_{C_{t_2}} d\vec{a}$ sinc $C_{t_1} - C_{t_2}$ bounds on E. As $t \to \infty$, $C_t' \to \vec{a}$ hence $\int_{C_t} ds$ converges to $\int_{C_t} d\vec{a}$. In particular, the lengths of the C_t are uniformly bounded.

Let X be the conormal vector field along C_t, i.e. X is tangent to M, $< X, C_t' >= 0$, $|X| = 1$ and X points into M_t. Let $\vec{a}^\perp$ be the unit normal vector to $\vec{a}$, tangent to T_t and whose direction is C_t'', when $C_t'' \neq 0$.

We have

$$\kappa_g(C_t) =< C_t'', X >= |C_t''| \cos(\measuredangle\, (C_t'', X))$$

$$= \kappa \cos(\measuredangle\, (\vec{a}^\perp, X) + \varepsilon) ,$$

where κ is the curvature of C_t, viewed as a planar horizontal curve and $\varepsilon \to 0$ as $t \to \infty$.

Now compute κ by thinking of C_t as a planar section of M. Let P be the plane at $C_t(s)$ generated by the normal n to M at $C_t(s)$ and $C_t'(s)$. Let $\kappa_n(s)$ be the normal curvature, i.e. the curvature of the curve $P \cap M$ at $C_t(s)$.

We have $\kappa_n(s) = \kappa \, \cos\psi$ where ψ is the angle between $C_t''(s)$ and n. Since the limiting normal is not vertical, $\cos\psi$ is bounded away from zero. Hence if $\kappa_n \to 0$ as $t \to \infty$ then so does κ and κ_g.

Since M is minimal, the principal curvatures κ_1, κ_2 of M are equal in modulus. The normal curvatures are between κ_1 and κ_2 so it suffices to prove

$$K = \kappa_1 \kappa_2 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty .$$

In a conformal parametrization of the end by D^* the induced metric is $ds = \lambda |dz|$ where $\lambda = \frac{|f(z)|}{2}(1 + |g(z)|^2)$, f a non vanishing holomorphic function on D^*, g the Gauss map. The curvature $K = \frac{-\Delta \ell n \lambda}{2\lambda}$. Since $\ell n|f|$ is harmonic in D^* and g extends holomorphically to 0, we have $\Delta \ell n \lambda = \Delta \ell n(1 + |g|^2)$ is bounded in a neighborhood of 0. The metric ds is complete at 0 so $\lambda \to \infty$ as $|z| \to 0$, and this proves $K \to 0$, hence $\int_{C_t} \kappa_g \, ds \to 0$ as $t \to \infty$.

Now consider an end A of M with a vertical limiting normal vector v. By lemma 7.2, we can conformally parametrize A by D^* so that $x_3 = C \; \ell n|z|$. The Gauss map has a zero or pole at 0 of order n, the branching order of A.

We can assume $g(z) = z^n +$ higher order terms near 0. As z goes once around the circle $|z| = r$ counterclockwise, the normal vector to A along C_t goes n times around the vertical vector v, always turning in the same sense when r is small. Hence the normal vector to the curve C_t in T_t, turns monotonically counterclockwise,n times around the origin.

Let κ be the planar curvature of C_t (in T_t). By the last paragraph, $\kappa > 0$ and $\int_{C_t} \kappa ds = 2\pi n$. Now $\kappa_g = \kappa \; \cos\phi$, ϕ the angle between the conormal to C_t in M and the horizontal. We have $\phi \to 0$ as $t \to \infty$ hence $\int_{C_t} \kappa_g ds \to 2\pi n$. This completes the proof of theorem 7.4.

Remark : We deduce from the above argument, that if the end A is embedded, it can not have a vertical limiting normal vector v. For if v is vertical, the curves C_t have positive curvature κ and $\int_{C_t} \kappa ds = 2\pi n$. Clearly this means C_t is a convex curve, null homotopic on T_t. Hence A lifts to a finite total curvature embedded end in $\mathbb{R}^3$ which must be a catenoid (asymptotically). Clearly this can not be embedded in $T \times \mathbb{R}$.

THEOREM 7.5. — *Let A be a properly embedded minimal annular end in $T \times \mathbb{R}$. Then A is asymptotic to a flat cylinder. Moreover two distinct annular ends of an m–surface in $T \times \mathbb{R}$, converge to distinct flat cylinders.*

Proof: We know we can assume A is parametrized by D^* and $X_3 = c\ell n|z|$

with $c < 0$. By the analysis in the proof of theorem 7.4, we may assume that for every small $\varepsilon > 0$, there exists a $T > 0$ such that for $t > T$, each curve $C_t = A \cap T_t$ is contained in the interior of an ε–tubular neighborhood B_t of a geodesic; $B_t \subset T_t$. Fix $\varepsilon > 0$, and assume, after possibly translating A downward, that C_t has this property for $t \geq 0$. Let α_t and β_t be the boundary curves of B_t. For small t, it is clear there exists a unique flat annulus $F_t \subset T \times \mathbb{R} - A$ with boundary $\alpha_0 \cup \alpha_t$. We shall check that such an F_t exists for all t and varies continuously with t. Clearly the set of t for which F_t exists is open, since F_t and $A \cap (T \times [0,t])$ are compact. Also, $\partial F_t \cap \partial A = \phi$ so the maximum principle implies the limit of such F_t is also an example.

In the same manner, we define flat annuli E_t with boundary $\beta_0 \cup \beta_t$, disjoint from A. A subsequence of E_t converges to a flat annulus E, with $\partial E = \alpha_0$, $E \cap A = \phi$ (by the maximum principle). Similarly F_t has a subsequence converging to a flat annulus F, $\partial F = \beta_0$, $F \cap A = \phi$ and $E \cap F = \phi$. Hence E and F are parallel at a distance ε and A is between E and F. Now do the same argument at a height such that the C_t are within $\varepsilon/2$ of a geodesic on T_t. Letting $\varepsilon \to 0$ we get the desired limit flat annulus.

By the maximum principle at infinity, it follows that distinct ends converge to distinct flat annuli.

7.6. Global topological and geometrical properties

Recall that $T \times \mathbb{R} = \mathbb{R}^3/G$ has a commensurable lattice if G contains two linearly independent vectors of equal length.

THEOREM 7.6. — *Let M be an m–surface of $T \times \mathbb{R}$ of finite topological type (M not flat). Then :*

1. *If M is orientable, then M separates $T \times \mathbb{R}$. In this case, the number of top ends, as well as the number of bottom ends of M is even. In particular M has at least four ends.*

2. *If M is nonorientable, then the number of top ends, as well as the number of bottom ends, is odd. In particular, whether M is orientable or not, the number of ends is even.*

3. *The top ends of M are parallel to the bottom ends of M if and only if the subgroup of* $H_1(T \times \mathbb{R})$ *generated by the loops on the ends of M is cyclic. If the ends are parallel then the number of top ends equals the number of bottom ends. In particular, by* 1, *if M is orientable and has parallel ends, then the number of ends is a multiple of four.*

4. *If the ends of M are not parallel, then they are vertical and* $T \times \mathbb{R}$ *has a commensurable lattice.*

Proof : Assume M orientable and let $\widehat{M}$ be the connected lifting of M to $\mathbb{R}^3$. Let G be the translation group defining M. if $\sigma \in G$ then $\sigma\widehat{M} = \widehat{M}$ by the strong halfspace theorem. $\widehat{M}$ separates $\mathbb{R}^3$ into two connected components A and B and σ conserves orientation so $\sigma A = A$. Hence M bounds A/G in $T \times \mathbb{R}$ and M separates $T \times \mathbb{R}$.

For t large, we know that $M \cap T_t$ consists of a finite number of pairwise disjoint simple closed curves $C_1, \ldots, C_n$ and each C_i is approximately a geodesic. Here n is the number of top ends. Similarly, $M \cap T_t = D_1 \cup \cdots D_m$ for $t < 0$, $|t|$ large, each D_j an almost geodesic and m equals the number of bottom ends. Since M separates $T \times \mathbb{R}$ into two components, A and B say, each C_i has two sides on T_t, one in A, the other in B. Hence both n and m are even. This proves 1. We leave 2 to the reader, or refer to [M.-R.-1].

Let P be a flat annulus parallel to the limiting top ends and Q a flat annulus representing the limit of the bottom ends. Let $\vec{a}$ and $\vec{b}$ denote the limiting directions of C_i and D_j respectively, $|\vec{a}| = |\vec{b}| = 1$. Let X denote the conormal vector field to ∂M_t; X is tangent to M, $|X| = 1$, $X \perp \partial M_t$, and X points upward. Let $\vec{v}$ be the upward unit vector field tangent to P and normal to $\vec{a}$. Similarly let $\vec{w}$ be the unit field tangent to Q, normal to $\vec{b}$ and pointing upward.

The flux of $\vec{v}$ across a curve C_j is $\int_{C_\lambda} < \vec{v}, X > ds$. As $t \to \infty$, X converges to $\vec{v}$, C_j converges to a geodesic A_j. Hence the flux of $\vec{v}$ across ∂M_t for t large, is

$$\sum_{j=1}^{n} \int_{|A_j|} < \vec{v}, \vec{v} > ds = n|A_1| \ .$$

Similarly, the flux of $\vec{v}$ across D_j is $\int_{D_j} < \vec{v}, \vec{w} > |B_j|$, where B_j is the limiting geodesic of D_j.

Since $\vec{v}$ is the gradient of a coordinate function, which is harmonic on M, the flux of $\vec{v}$ across $\partial(M \cap T \times [-t,t])$ is zero. hence $n|A_1| =< \vec{v}, \vec{w} > m|B_1|$. In particular $n|A_1| \leq m|B_1|$ and equality holds if and only if $\vec{v} = \vec{w}$. Now turn M upside down to conclude $m|B_1| \leq< \vec{v}, \vec{w} > n|A_1|$. Hence $\vec{v} = \vec{w}$ and $n|A_1| = m|B_1|$.

If $\vec{v}$ is not vertical then there is a unique horizontal direction $\vec{a}$ normal to $\vec{v}$. Hence $\vec{a} = \vec{b}$ when $\vec{v}$ is not vertical and the top ends are parallel to the bottom ends.

If the ends are all parallel, then the subgroup of $H_1(T \times \mathbb{R})$ generated by the ends is the cyclic subgroup generated by A_1. If the ends generate a cyclic subgroup with generator A then $\vec{a} = \vec{b}$ and $\vec{v} = \vec{w}$ so the ends are parallel.

If the ends are not parallel then they are vertical and $n|A_1| = m|B_1|$. Since $\vec{a}$ and $\vec{b}$ are independent, the vectors $n|A_1|\vec{a}$ and $m|B_1|\vec{b}$ are independent and of equal length. So the lattice is commensurable. This completes the proof of theorem 7.6.

Now it is not hard to give necessary conditions for a given doubly periodic minimal surface to have nonparallel ends which forces the ambient space to have a commensurable lattice. We leave the proof to the reader or refer to [M.-R.-1].

THEOREM 7.7. — *Let $M \subset T \times \mathbb{R}$ be a non flat m–surface of finite topology. Then the ends are not parallel if* 1, 2 *or* 3 *holds :*

1. *M is orientable and the number of ends is not a multiple of four*
2. *M is a planar domain*
3. *$\mathcal{X}(M)$ is odd.*

7.8. The sum of finite total curvature minimal surfaces (minimal herissons).

Let M_1, M_2 be finite total curvature complete non planar minimal surfaces in $\mathbb{R}^3$ with Gauss maps g_1, g_2. Let $p_1, \ldots, p_n$, $q_1, \ldots, q_m$ be the punctures of M_1, M_2 respectively and $\overline{M_1}$, $\overline{M_2}$ the compactified Riemann surfaces. If one fixes a unit vector $z \in S^2$, one can add (in $\mathbb{R}^3$) all points in $\mathbb{R}^3$ having z as normal. As z varies in S^2 this yields a complete (branched) minimal surface or a point. More precisely, Rosenberg and Toubiana have proved :

THEOREM 7.8 [R.-T.-2]. — *The set*

$$M_1 + M_2 = \Big\{ \sum_{x \in g_1^{-1}(z)} x + \sum_{y \in g_2^{-1}(z)} y / z \in S^2 - W \Big\}$$

is a complete minimal surface in $\mathbb{R}^3$ *(or a point) of total curvature* -4π. *Here* W *is some subset of* $g_1\{p_1, \dots, p_n\} \cup g_2\{q_1, \dots, q_m\}$.

The normal vector to $M_1 + M_2$ at the point $\sum_{x \in g_1^{-1}(z)} x + \sum_{y \in g_2^{-1}(z)} y$ is z.

Thus $M_1 + M_2$ is naturally parametrized by $S^2 - W$; denote this parametrization by $\widehat{g}$. If $M_1 + M_2$ is not a point, then $\widehat{g}$ is a conformal injection, which explains the total curvature -4π of $M_1 + M_2$.

The (possible) branch points of $M_1 + M_2$ are geometric branch points, however the Weierstrass data of $M_1 + M_2$ is meromorphic at these branch points; the $\widehat{\omega}$ of $M_1 + M_2$ vanishes at the branch points. Notice these points are quite distinct from the branch points of the Gauss map; in general, $\widehat{g}$ is injective where $\widehat{\omega}$ vanishes.

The sum operation is very useful for detecting symmetries in a surface M. For example Rosenberg and Toubiana have proved :

THEOREM 7.9 [R.-T.-2]. — *Let* M *be a complete finite total curvature minimal surface in* $\mathbb{R}^3$. *if all the ends of* M *are asymptotic to planes (planar ends) then* $M + M$ *is a point.*

The idea of the proof is simple. At a planar end of M, the points having a fixed normal direction (near the limiting normal) are distributed in space so as to have the same barycenter (like the roots of unity). So a planar end puncture becomes a regular point in $M + M$.

Since $T \times \mathbb{R}$ is an abelian group under addition and the Gauss map is invariant under translation, the sum $M_1 + M_2$ is also defined in $T \times \mathbb{R}$ and has total curvature -4π or 0.

Meeks and Rosenberg have proved.

THEOREM 7.10 [M.-R.-1]. — *Let M be a finite total curvature complete immersed minimal surface in $T \times \mathbb{R}$. Then*

1. *If the ends of M converge to parallel flat annuli, then $M + M$ is a point.*

2. *If M is embedded and the ends of M are not parallel, then $M+M$ is a Scherk surface.*

Finally, we apply this theorem to obtain :

THEOREM 7.11. — *Suppose M is an m–surface of finite topology in $T \times \mathbb{R}$ and $T \times \mathbb{R}$ has an incommensurable lattice. Then*

1. *$M + M$ is a point.*

2. *If M has genus one and four parallel ends (e.g. a Karcher saddle), then after a translation of M (so that a zero of Gaussian curvature occurs at the origin), the order two points in the group $(\mathbb{R}^2/G) \times \mathbb{R}$ are the zeros of the Gaussian curvature of M. In this case M separates $T \times \mathbb{R}$ into two isometric components.*

8. SINGLY PERIODIC MINIMAL SURFACES

We have a wealth of beautiful examples of singly periodic m–surfaces. The helicoid is the easiest to grasp : take a horizontal line ℓ, passing through the x_3–axis; then rotate ℓ with constant velocity while rising vertically with constant velocity. This surface $\widetilde{M}$ is invariant under screw motions S_θ and for a fixed θ, $M = \widetilde{M}/S_\theta$ is conformally a two–punctured sphere of finite total curvature -2θ. In $\mathbb{R}^3/S_\theta$, M has two annular ends, each a helicoid end. Notice that M no longer has a well defined Gauss map. The Gauss map g of $\widetilde{M}$ induces a multivalued meromorphic map on M where distinct determinations of its values differ by λ^m where $\lambda = e^{i\theta}$: if p and q are points of $\mathbb{R}^3$ with $S_\theta^m(p) = q$, $p, q \in \widetilde{M}$, then the normal vectors to $\widetilde{M}$ at p and q differ by rotation about the x_3–axis by $m\theta$, hence their stereographic projections to the horizontal complex plane differ by multiplication by λ^m.

From the point of view of the Weierstrass representation of $\widetilde{M}$ in $\mathbb{R}^3$, $\widetilde{M}$ is the conjugate surface of the catenoid. On $\mathbb{C}^* = \mathbb{C} - (0)$, the data : $g(z) = z$, $\omega_\tau(z) = e^{i\tau}\frac{dz}{z^2}$, defines a complete minimal surface $\widetilde{M}_\tau$ for each real τ; the catenoid is $\tau = 0$ and the helicoid $\tau = \pi/2$. The surfaces $\widetilde{M}_\tau$ are not embedded for $0 < \tau < \pi/2$, however each $\widetilde{M}_\tau$ has two annular ends and they are embedded. Each end is a *helicoid–catenoid* type end. The intersection of M_τ with a large vertical cylinder of radius R, centered at the x_3-axis, consists of two helices (like a barber pole). As $R \to \infty$, the helices rise on the cylinder like $\ell n\ R$; so they look like helicoids and catenoids (actually one helice rises and the other descends). We will see later that when $\widetilde{M}$ is an embedded singly periodic surface, there are no annular ends of this type. The number of ends will be even and half of them would rise as $R \to \infty$, while the other half would descend. So $\widetilde{M}$ couldn't be embedded. Hoffman and Wei have shown that one can add a handle to the helicoid in a periodic manner [H.-Wei.], figures 17-a,b.

Another singly periodic example is the conjugate surface of Scherks doubly periodic m–surface. In terms of Weierstrass data in $T \times \mathbb{R}$, $g(z) = z$ and $\omega(z) = \frac{dz}{z^4-1}$, parametrizes Scherk's 1'st surface by the sphere punctured at the four roots of unity. The reader can easily check that x_1 and x_2 are multi–valued, and x_3 single valued. The data for the conjugate surface (Scherks' second surface) is $g(z) = z$, $\omega = \frac{i\,dz}{z^4-1}$, also modelled on S^2 minus the fourth roots of unity. Now x_3 has a period and x_1, x_2 are single valued; the surface is invariant by a vertical translation T. There are four annular ends that are the quotient, by T, of vertical ends, asymptotic to planes. Ends of this type are called Scherk type ends, figure 18-a.

Karcher has shown that one can construct singly periodic m–surfaces of this nature with $2n$ Scherk type ends, for any $n \geq 2$. Moreover, he is able to deform these surfaces to singly periodic m–surfaces, invariant by screw motions S_θ, so that the Scherk type ends become helicoid type ends [K.-2], figure 18-b . He does this with the generalized Weierstrass representation we develop in this chapter.

We discuss one more example, the Riemann example. This surface is invariant by a translation T (not a vertical translation) and the horizontal sections x_3 = constant are circles and lines. The ends in $\mathbb{R}^3$ are asymptotic to horizontal planes (located at the heights whose sections are lines), figure 19. The simplest orientable quotient of this is a two punctured torus of total curvature -8π. The Riemann examples form a one parameter family and the conjugate surface of a Riemann example is also a Riemann example (this is a good exercise). We refer the reader to [H.-M.-5] for an excellent discussion of these surfaces. Callahan, Hoffman and Meeks have generalized the Riemann examples [C–H–M], figure 20-a. Also Hoffman and Wei have shown that one can add a handle to Riemann's surface (one handle between every other pair of planar ends) to obtain a singly periodic m–surface, a surface of genus one with three punctures [H.-Wei], figure 20-b.

8.1. The finite total curvature theorem

Our theorem 7.1 states that an M surface in $\mathbb{R}^3/S_\theta$ is of finite topology if and only if it is of finite total curvature. I will briefly outline the structure of the proof, and refer the courageous reader to [M.-R.-4] for the details.

Let $M \subset \mathbb{R}^3/S_\theta$ be an m–surface of finite topology. The problem is to show the ends (topologically annular) are of finite total curvature. This is done by trapping an end of M between *standard ends*; i.e. two ends of finite total curvature whose geometry one understands. Then, using foliations by stable minimal annuli of the region between the standard ends (that trapped the end of M) one proves the end of M is stable hence of finite total curvature.

The first part of the proof requires an understanding of the finite total curvature annular ends. In II, we explained the geometry of embedded finite total curvature ends in $\mathbb{R}^3$ using the Weierstrass representation and the fact that the Weierstrass data (g, ω) extends meromorphically to the puncture. An annular end A in $\mathbb{R}^3/S_\theta = N$ has a multi–valued Gauss map so the first thing we need to know is the existence of a limit tangent plane of A at infinity. Assuming A has finite total curvature Huber's theorem tells us A can be conformally parametrized by D^*. Now we have the Picard–type theorem (whose proof uses elementary complex analysis).

THEOREM 8.2 [M.-R.-4]. — *Let g be a multi–valued meromorphic map* *on* D^* : $g = \widehat{g}(\exp^{-1})$, *with* $\widehat{g}(z + 2\pi i) = \lambda\widehat{g}(z)$ *for* $z \in \{x + iy/x \leq 0\}$, *and* $|\lambda| = 1$. *If the area of the image of* g *(i.e. the restriction of* g *to* D^* *slit along a radial line), counted with multiplicity, is finite, then* g *extends continuously to the origin.*

8.3 The generalized Weierstrass representation

Now we use this result to obtain a Weierstrass representation for A (meromorphic on D) as follows. We can assume $\lambda \neq 1$ since this is the usual Weierstrass representation. Then the limiting value of g is 0 or ∞ since it is fixed by multiplication by λ; so assume $g(0) = 0$. Let $\theta = 2\pi a$ with $0 < a < 1$. Clearly the map $z^{1-a}g(z)$ is bounded in a neighborhood of 0, so $g(z) = z^{a-1}h(z)$ with h holomorphic in a neighborhood of 0. Hence $\frac{dg}{g}$ is a well defined meromorphic one form on D^* that extends meromorphically to 0. One obtains the multi–valued g from this form by $g = \exp(\int \frac{dg}{g})$.

Notice that the third coordinate function x_3 is defined, up to a constant on N so dx_3 is well defined on N. Let $\eta = dx_3 + i(*dx_3)$. It is easy to see that η is meromorphic on D^* and extends meromorphically to 0.

We then can take as Weierstrass data on A the pair of one forms $(\frac{dg}{g}, \eta)$, which extend meromorphically to the puncture. In general, we have :

THEOREM 8.4 [M.-R.-4]. — *let* M *be a complete finite total curvature minimal surface in* $\mathbb{R}^3/S_\theta$. *There exists a conformal compactification* $\overline{M}$ *of* M, *and meromorphic one forms* $(\frac{dg}{g}, \eta)$ *on* $\overline{M}$, *such that* M *is parametrized by*

$$X(z) = Re \int (g + \frac{1}{g},\ ig - \frac{i}{g},\ 2)\eta$$

where $g = \exp(\int \frac{dg}{g})$.

8.5. The geometry of finite total curvature ends

Now using this parametrization we describe the asymptotic geometry of embedded annular ends.

THEOREM 8.6 [M.-R.-4]. — *A properly embedded minimal annulus in*

$\mathbb{R}^3/S_\theta$, of finite total curvature, is asymptotic to a plane, a flat vertical annulus (a Scherk type end) or to a helicoid–catenoid type end (with horizontal limit tangent plane). If the end A is part of an m–surface of finite total curvature then A can not be a helicoid–catenoid type end. If $\theta \neq 0$ and A is asymptotic to a plane, then the plane is horizontal. If θ is irrational, then A is not a Scherk type end.

8.6. The winding number of an end

Using this theorem we can calculate the flux and total curvature of m–surfaces M of finite total curvature. For the latter one proceeds as follows. Let $\gamma \subset \mathbb{R}^3/S_\theta$ be the quotient of the x_3–axis and let T_R be a tubular neighborhood of radius R of γ. For R large, $M_R = M \cap T_R$ is bounded by k Jordan curves $C_1, \ldots, C_k$ on ∂T_R, pairwise disjoint, and each C_i converges to a vertical line (Scherk type end) or to a horizontal circle (planar end) or to a helice on ∂T. Now one calculates the total curvature of M_R using Gauss–Bonnet and let $R \to \infty$. The boundary term is what we call the winding number of the end.

More generally, let A be a properly immersed annular end in $\mathbb{R}^3/S_\theta$. We know a subend of A is disjoint from γ so we assume $A \cap \gamma = \phi$. Then ∂A is homotopic to a cycle on ∂T_R of the form $n\alpha + m\beta$ where α is a horizontal circle on ∂T_R and β is the quotient of the right handed helicoidal arc that joins a point p to $S_\theta(p)$ and projects to an embedded cycle on ∂T_R. The winding number of A is defined to be $\frac{1}{2\pi}|2\pi n + m\theta|$. It's easy to see that this doesn't depend on R for R large, and in the case of standard ends it is the limit of the total geodesic curvature of the $C_1, \ldots, C_k$.

When M is a complete minimal surface of finite total curvature in $\mathbb{R}^3/S_\theta$, the winding number of M is defined to be the sum of the winding numbers of it's ends. When M is embedded, this is k times the winding number of one end, k the number of ends.

Now the formula of 7.1 should be clear to the reader :

$$C(M) = 2\pi(\mathcal{X}(M) - W(M)) \ .$$

When the ends are Scherk type ends this is $C(M) = 2\pi\mathcal{X}(M)$. When they are k–planar ends, it is $C(M) = 2\pi(\mathcal{X}(M) - k)$.

Applications of the finite total curvature theorem

We have seen in VII, that a non planar orientable m–surface in a flat 3–manifold separates the space (this followed easily from the strong half–space theorem). This fact, together with 8.1 and our knowledge of the geometry of finite total curvature ends 8.6, yields a topological obstruction for the existence of certain m–surfaces :

THEOREM 8.7 [M.-R.-4]. — *Let M be an orientable non planar m–surface, of finite topology, in a non simply connected flat 3–manifold. Then the number of ends of M is even.*

Erik Toubiana has proved that an m–surface in $\mathbb{R}^3/T$, T a translation, that has finite, non zero, total curvature and the topology of a two punctured sphere (i.e. an annulus) is a helicoid [T]. Using 8.1 we generalize this result to $\mathbb{R}^3/S_\theta$:

THEOREM 8.8 [M.-R.-4]. — *Let M be an m–surface in $\mathbb{R}^3/S_\theta$, topologically an annulus, and not flat. Then M is a helicoid.*

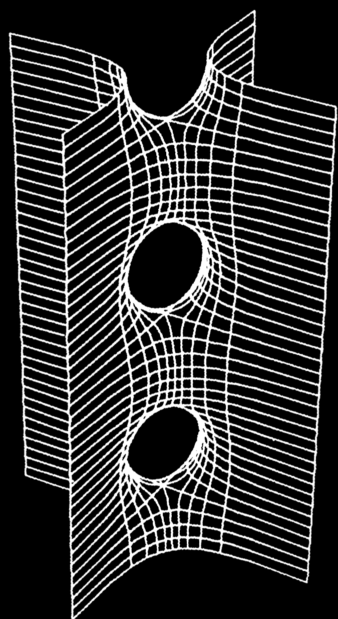
Perez and Ros have generalized the Toubiana theorem to genus zero :

THEOREM 8.9 [P.-Ros]. — *The helicoid is the only genus zero m–surface in $\mathbb{R}^3/T$ with a finite number of helicoidal type ends.*

Their technique of proof uses the Lopez–Ros deformation described in III, and theorem 8.1. They also prove (with this technique) that there are no genus one m–surfaces in $\mathbb{R}^3/S_\theta$, $\theta \neq 0$, with a finite number of planar ends. In other words : one can not screw the Riemann example. Notice that the Karcher deformations of Scherk's singly periodic surface shows that one can screw Scherk's surface [K.-2], figure 18-b.

Theorem 8.8 yields a unicity theorem for the helicoid in $\mathbb{R}^3$:

THEOREM 8.10 [M.-R.-4]. — *The plane and the helicoid are the only simply connected m–surfaces in $\mathbb{R}^3$ with an infinite symmetry group.*



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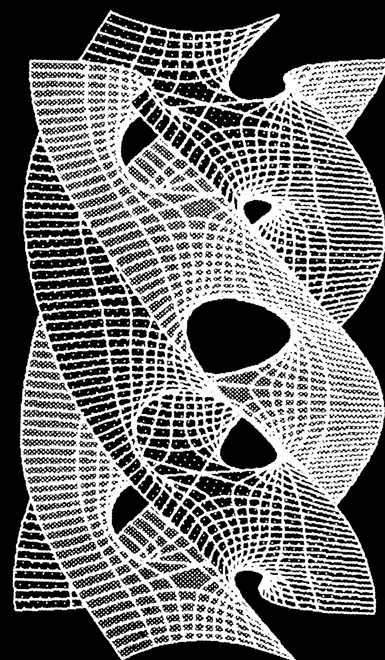

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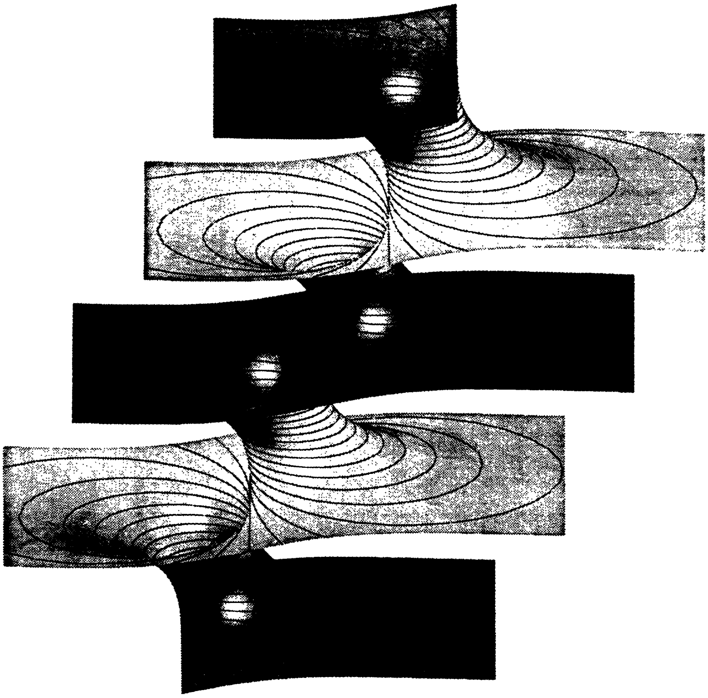
17.a

17.b



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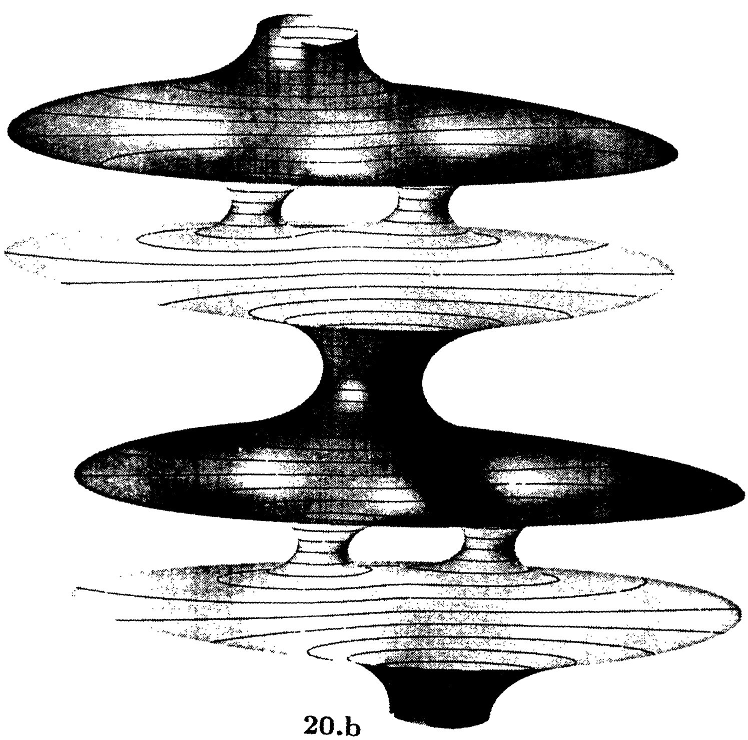

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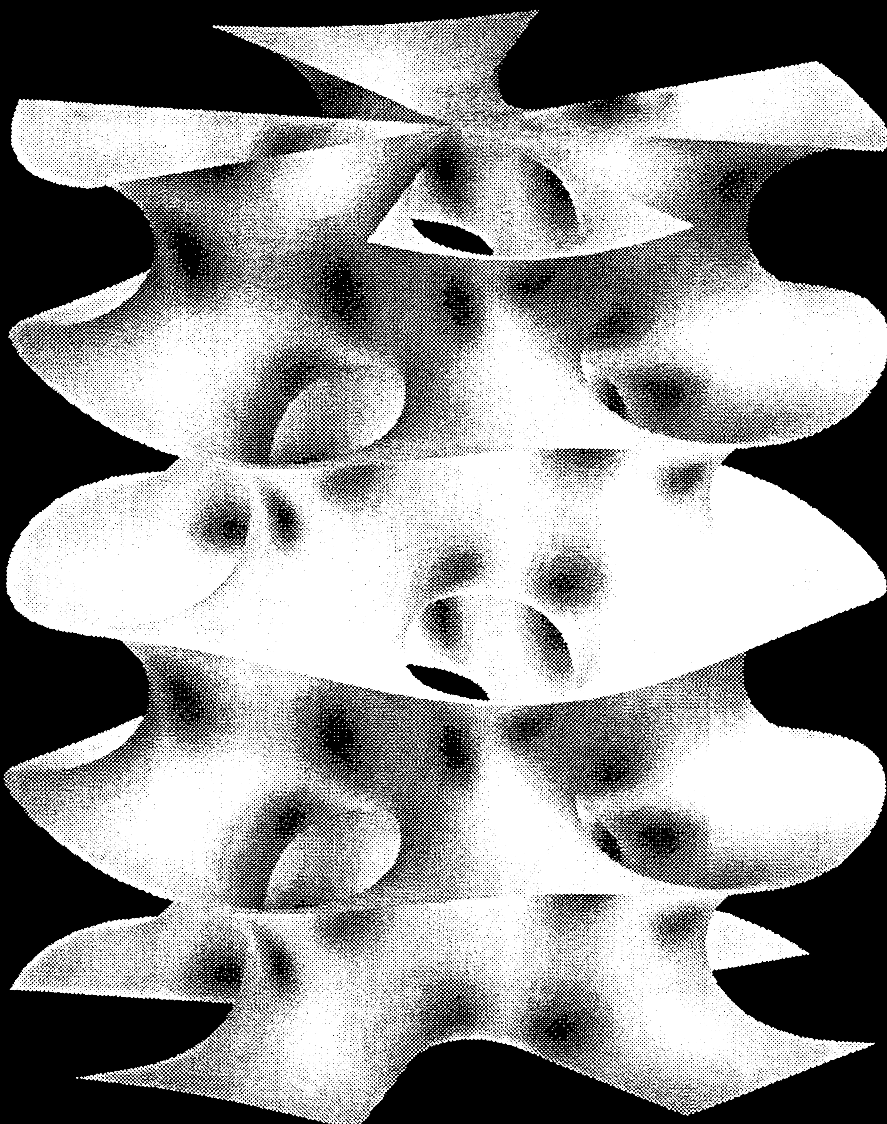


20.a

Another application allows us to classify the sum surface :

THEOREM 8.11 [M.-R.-4]. — *Let $M \subset \mathbb{R}^3/T$ be an m surface of finite topology, T a non trivial translation. If M has a helicoid end, then $M+M$ is a helicoid. if M has a planar end then $M+M$ is a point. if M has four Scherk type ends, then $M+M$ is a Scherk surface.*

Callahan, Hoffman and Meeks have proved a good structure theorem for singly periodic m–surfaces with more than one end.

THEOREM 8.12 [C.-H.-M.]. — *Let $M \subset \mathbb{R}^3$ be an m–surface with infinite symmetry group and more than one end. Then either M is a catenoid or M has the following properties :*

1) Sym(M) *contains an infinite cyclic subgroup S of finite index, generated by a screw motion S_θ.*

2) *M/S has finite topology precisely when M/S has finite total curvature.*

3) *there exists a plane whose intersection with M consists of a finite number of simple closed curves.*

As a corollary of their theorem, they prove that a doubly periodic m–surface in $\mathbb{R}^3$ has one end.

9. SOME PROBLEMS, CONJECTURES AND RELATED RESULTS

Perhaps there are many m–surfaces in $\mathbb{R}^3$ of finite topology and infinite total curvature. For the moment, the only known example is the helicoid. Are there any others? Mark Soret has proved there are no others near the helicoid (graphs over the helicoid in an ε–tubular neighborhood of the helicoid [M.-S.]).

A less general question is to decide if the helicoid and plane are the only simply connected m–surfaces in $\mathbb{R}^3$. We know this to be the case if the surface has an infinite symmetry group; theorem 8.10.

Maybe every infinite total curvature m–surface in $\mathbb{R}^3$ has an infinite symmetry group (I doubt it) in which case the answer would be affirmative. All of the examples of infinite total curvature m–surfaces we know today are constructed using symmetries. There is no good reason (as far as I am concerned) to believe there are no others. It is likely that one can add exactly one handle (maybe more) to the helicoid to create an m–surface of infinite total curvature and non periodic.

There is an important difference when an m–surface in $\mathbb{R}^3$ has more than one end. We have seen in V, that this enables us to find planar or catenoid ends in the complement of M. Then Hoffman and Meeks proved that at most two annular ends of such M can have infinite total curvature (6.1). This leads Hoffman and Meeks to conjecture :

The finite total curvature conjecture [H.-M.-3] :

An annular end of an m–surface in $\mathbb{R}^3$, with at least two ends has finite total curvature.

This is related to the Nitsche conjecture : a minimal surface that meets every horizontal plane in a Jordan curve is a catenoid. Nitsche proved this assuming the Jordan curves are star shaped [N].

Now Meeks and I have proved (6.2) that an annular end of an m–surface in $\mathbb{R}^3$ with at least two ends, is either of finite total curvature or contains a subend which meets every horizontal plane, in the upper halfspace of $\mathbb{R}^3$, in a Jordan curve (after a Euclidean motion of the surface).

Hence the finite total curvature conjecture is a consequence of an affirmative answer to the following conjecture of Meeks and me :

The generalized Nitsche conjecture [M.-R.-3] :

Let A be a minimal annular end such that $A \cap \{x_3 = c \geq 0\}$ is a Jordan curve for every $c \geq 0$. Then A has finite total curvature.

Notice that this question concerns one holomorphic function g in the punctured disc and the problem is whether the origin is an essential singularity. Since A (or a subend of A) can be conformally parametrized by D^* with $x_3 = K\ \ell n|z|$ the Weierstrass data of A is of the form $(g, \frac{1}{zg})$. It seems difficult to relate the singularity of g at the origin with the property that A is embedded. Erik Toubiana and I have constructed examples of immersed annuli meeting every horizontal plane transversally and of infinite total curvature (g has an essential singularity [R.-T.-1]). We have even constructed such immersions in a slab of $\mathbb{R}^3$. From time to time, I find myself working on the Nitsche conjecture using techniques from complex analysis (concerning essential singularities), but not using A embedded. Fortunately, this doesn't happen to me very often.

An affirmative answer to the generalized Nitsche conjecture would imply that finite topology, m–surfaces with more than one end can be parametrized by meromorphic data on a compact Riemann surface. All of the examples of properly embedded m–surfaces in $\mathbb{R}^3$, that we presently know, do have this property : all of the infinite total curvature examples we know are periodic and have quotients of finite topology. We saw in VIII that the generalized Weierstrass representation is meromorphic on a compact Riemann surface.

What are the m-surfaces in $\mathbb{R}^3$ with exactly one end, topologically an annulus. For the moment, we know of only the plane and the helicoid, but as I said earlier, it is likely one can add a handle to a helicoid. Perhaps one can realise all compact surfaces, of arbitrary genus, with one puncture. Let us call an annular end algebraic if it is conformally a punctured disk and $\frac{dg}{g}$ and η extend meromorphically to the puncture. Is every finite topology m–surface in $\mathbb{R}^3$ algebraic? Is a properly embedded minimal annular end algebraic? Can one at least decide if it is conformally a punctured disk? I can prove that a minimally immersed annulus whose total curvature grows polynomially (*i.e.* $\int_{D_r} |K| \leq cr^n$, where D_r is a geodesic disk of radius r) is conformally a punctured disk. This growth condition should imply algebraic. An interesting related problem is to study minimal surfaces whose intersection with every plane $x_3 =$ constant, is one properly embedded real line. Is such a surface conformally $\mathbb{C}$? Is it a plane or a helicoid?

What are the genus zero m–surfaces in $\mathbb{R}^3$? The only examples we know are the plane, the helicoid, and the singly periodic Riemann examples. Meeks has conjectured that if the surface is also periodic then it is one of the these three examples [M.-1].

What are the genus zero m–surfaces in the other flat 3–manifolds than $\mathbb{R}^3$? In $T^2 \times \mathbb{R}$ we believe the only such examples lift to a Scherk surface in $\mathbb{R}^3$ (notice Scherck has infinite genus in $\mathbb{R}^3$). This was proved by Meeks and me when such a surface has 4 ends [M.-R.-1] and Wei extended this to 6 ends [Wei].

In $\mathbb{R}^3/S_\theta$, $\theta = 0$ theorem 8.9 of Perez–Ros says the only genus zero finite topology example with helicoid type ends is the helicoid. What are all the genus zero examples in $\mathbb{R}^3/S_\theta$? Notice the Riemann example has genus zero in $\mathbb{R}^3$ and genus one in $\mathbb{R}^3/T$.

A (too) general question is to classify the genus g finite topology m–surfaces in $T^2 \times \mathbb{R}$ or $\mathbb{R}^3/S_\theta$. For $g = 0$ or 1, I believe the problem is presently within our grasp. Certainly the same problem in $\mathbb{R}^3$ is beyond our means for the moment. Until recently, the only doubly periodic examples we knew were coverings of the Scherk surface or the Karcher saddles, together with their families constructed by Meeks and me [M.-R.-1]. Then F. Wei found a very beautiful example (using conjugate Plateau techniques or Weierstrass representation) to construct a genus two doubly periodic example with two top ends and two bottom ends, all parallel, different from the other known examples. Wei's surface had no lines as in the Scherk's surface and Karcher saddle [Wei], figure 21-a. Using Wei's idea, Karcher was able to add a handle to Scherk's surface (so the new surface has the same end behavior as the Scherk surface and is of genus one; personal communication), figure 21-b. Rabah Souam has proved that neither Wei's nor Karcher's surface could exist if one tried to keep the four vertical lines on the surface (thesis; Paris VII).

D. Hoffman conjectures that if M is a finite total curvature m–surface in $\mathbb{R}^3$ then the number of ends of M is less than or equal to the genus of M plus two. He believes that to add an end to an embedded minimal surface of finite total curvature in $\mathbb{R}^3$, one must increase the genus (contrary to the Riemann example).

Another interesting subject to pursue is the relationship between the intrinsic isometries of an m–surface M (i.e. it's symmetry group) and the ambiant isometries leaving M invariant (its isometry group). When M is an m–surface in $\mathbb{R}^3$, Meeks conjectures that every symmetry of M extends to an isometry of $\mathbb{R}^3$. Meeks and I have proved this for doubly periodic m–surfaces; in fact we proved more (rigidity) : let $f_1 : M \longrightarrow \mathbb{R}^3$ be a doubly periodic m–surface and suppose $f_2 : M \longrightarrow \mathbb{R}^3$ is another isometric minimal immersion of M, then there is an isometry ϕ of $\mathbb{R}^3$ such that $\phi f_2 = f_1$, [M.-R.-1]. Choi, Meeks and White have proved that an m–surface in $\mathbb{R}^3$ with more than one end is rigid [C.-M.-W.].

Singly periodic m–surfaces in $\mathbb{R}^3$ are not rigid (the helicoid) however it is true that their symmetry group equals their isometry group when M/S_θ has finite topology [M.-1].

Meeks has also conjectured that a non simply connected m–surface M in $\mathbb{R}^3$ is rigid (maybe the helicoid is the only non rigid m–surface in $\mathbb{R}^3$?) : any other isometric proper minimal immersion of M is congruent to M [M.-1].

Perhaps the notion of rigidity should be restricted to isometric minimal embeddings of M (not immersions). Then the helicoid is probably rigid.

Meeks has extended the finite total curvature theorem 7.1 to finite genus doubly periodic surfaces. He proved an m–surface in $T^2 \times \mathbb{R}$ of finite genus has finite total curvature [M.-1]. Does this remain true in $\mathbb{R}^3/S_\theta$?

In what generality does the maximum principle at infinity remain valid? Can one remove the hypothesis ∂M_1, ∂M_2 compact? The minimum distance between M_1 and M_2 (assumed disjoint) should not be realizable at *interior points at infinity*.

In the same spirit, Antonio Ros asked me the following question : suppose M is an m–surface in $\mathbb{R}^3$; can an end of M be an accumulation point of other ends of M ? More precisely, can there be a divergent sequence x_n on an end A of M and a sequence $y_n \in M - A$ such that $\text{dist}(x_n, y_n) \longrightarrow 0$, as $n \longrightarrow \infty$?

A problem that arises when studying m–surfaces in $\mathbb{R}^3$ of finite total curvature is the following : do all the catenoid ends have the same axis? This is unknown even for three catenoid ends.

There has been important work done by Celso Costa on the problem of classifying m–surfaces in $\mathbb{R}^3$ of finite total curvature : he classified those of total curvature -12π [Cost.-3]. His proof uses very difficult calculations in elliptic function theory. It would be very interesting to understand this from another point of view.

There has been much important and beautiful recent work done on minimal surfaces that I have not discussed. I consider my most important ommission the theorem of Frohman and Meeks that two one–ended m–surfaces in $\mathbb{R}^3$ of the same genus are ambiently isotopic [F.-M.].

There is also the very beautiful work of Fujimoto on values of the Gauss map [Fuj.-1,2]; but is is not clear to me this has anything to do with the surface being embedded or not.

Meeks and White have studied the space of minimal submanifolds of $\mathbb{R}^3$ bounded by two convex Jordan curves C_1 and C_2. When C_1 and C_2 are in parallel planes they proved there are 0, 1 or 2 minimal annuli with boundary $C_1 \cup C_2$ [M.-Wh.].

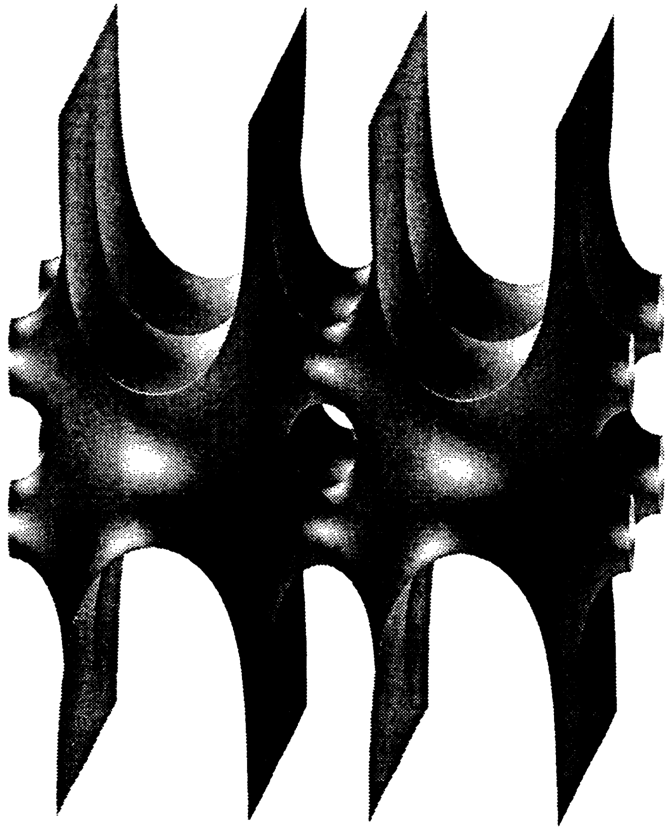
Finally, let me mention the problem of how, and when, can one desingularize a minimal variety : given two m–surfaces M_1, M_2 in $\mathbb{R}^3$, when is there an m–surface M that is close to $M_1 \cup M_2$ outside of a neighborhood of $M_1 \cup M_2$? In many examples, the desingularization M looks like a string of handles along $M_1 \cup M_2$. Here are some examples. Scherks singly periodic surface is the desingularization of two orthogonal planes. Karcher's singly periodic generalization of this Scherk surface is the desingularization of n planes meeting along an axis; figure 18-a.

A helicoid and its rotation about it's axis, meet along the axis. Karcher's examples desingularize this (and in general, n helicoids meeting along their axis) desingularize this by a string of handles along the axis; figure 18-a.

Costa's finite total curvature m–surface, with 3–ends, can be thought of as the desingularization of the vertical catenoid and the horizontal plane passing through the waist circle; figure 1. The higher genus examples of Hoffman and Meeks with 3–ends are a better illustration of this (figure 2) : one places a string of handles around the circle of intersection of the catenoid and the horizontal plane.

When $M_1 \cap M_2$ is a Jordan curve C, a necessary condition for desingularization appears to be : $\int_C n_1 \cdot n_2 = 0$, where n_1 is the normal to C in M_1 and n_2 the normal to M_2 along C.

How to make sense of this is not at all clear. How can one do *minimal surgery* on $M_1 \cup M_2$?



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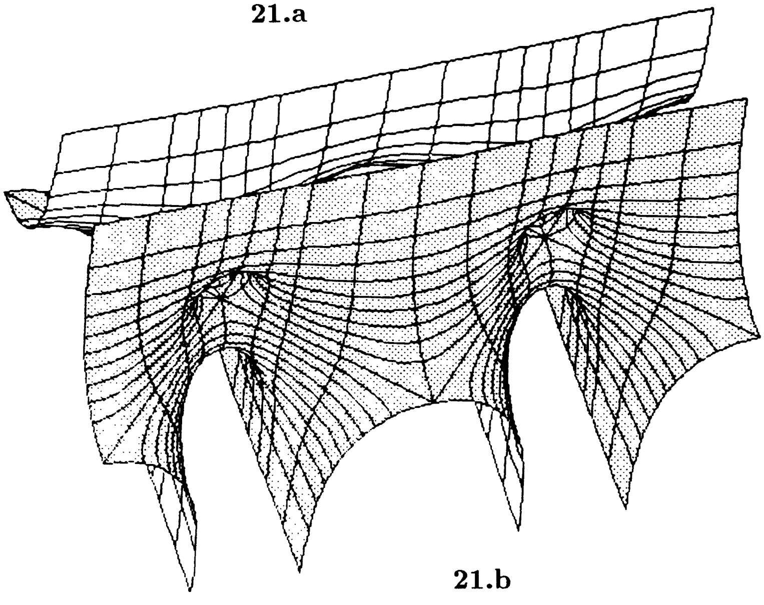

21.b

BIBLIOGRAPHY

[B.Do C.] J.L. Barbosa and M. Do Carmo. On the size of a stable minimal surface in $\mathbb{R}^3$. *American Journal of Mathematics* 98(2) : 515-528, 1976.

[C.-H.-M] M. Callahan, D. Hoffman, and W. H. Meeks III. The structure of singly-periodic minimal surfaces. *Inventiones Math.* 99 : 455-481, 1990.

[Cost.-1] C. Costa. *Imersões minimas en* $\mathbb{R}^3$ *de gênero un e curvatura total finita.* PhD thesis, IMPA, Rio de Janeiro, Brazil, 1982.

[Cost.-2] C. Costa. Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends. *Bull. Soc. Bras. Mat.* 15 : 47-54, 1984.

[Cost.-3] C. Costa. Uniqueness of minimal surfaces embedded in $\mathbb{R}^3$ with total curvature -12π. *Journal of Differential Geometry* 30(3) : 597-618, 1989.

[Cour.] R. Courant. *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces.* Interscience Publishers, Inc., New York, 1950.

[Darb.] G. Darboux. *Leçons sur la théorie générale des surfaces et les applications géometriques du calcul infinitésimal.* Gauthier-Villars, Paris, 1st part, 2nd edition, 1914.

[Do C.-P.] M. Do Carmo and C.K. Peng. Stable minimal surfaces in $\mathbb{R}^3$ are planes. *Bulletin of the AMS* 1 : 903-906, 1979.

[Doug.] J. Douglas, Solution of the problem of Plateau, *Trans. AMS* 33 : 263-321, 1931.

[F.-Oss.] R. Finn and R. Osserman. On the Gauss curvature of non-parametric minimal surfaces, *J. Anal. Math.* 12 : 351-364, 1964.

[F.C.] D. Fischer-Colbrie. On complete minimal surfaces with finite Morse index in 3-manifolds. *Inventiones Math.* 82 : 121-132, 1985.

[Fr.-M.] C. Frohman and W.H. Meeks III. The topological uniqueness of complete one-ended minimal surfaces and Heegard surfaces in $\mathbb{R}^3$, preprint.

[Fuj.-1] H. Fujimoto. On the number of exceptional values of the Gauss maps of minimal surfaces. *Journal of the Math. Society of Japan* 40(2) : 235-247, 1988.

[Fuj.-2] H. Fujimoto. Modified defect relations for the Gauss map of minimal surfaces. *Journal of Differential Geometry* 29 : 245-262, 1989.

[G.-T.] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of* *second order*. Springer-Verlag, New York, 2nd edition, 1983.

[H.-S.] R. Hardt and L.Simon. Boundary reguarity and embedded minimal solutions for the oriented Plateau problem. *Annals of Math.* 110 : 439-486, 1979.

[Heinz] E. Heinz. Über die Lösungen der Minimalflächengleichung. *Nachr. Akad. Wiss. Göttingen Math. Phys.* K1, II (1952) 51-56.

[H.-M.-1] D. Hoffman and W.H. Meeks III. A complete embedded minimal surface in $\mathbb{R}^3$ with genus one and three ends. *Journal of Differential Geometry* 21 : 109-127, 1985.

[H.-M.-2] D. Hoffman and W.H. Meeks III. Properties of properly embedded minimal surfaces of finite total curvature. *Bulletin of the AMS* 17(2) : 296-300, 1987.

[H.-M.-3] D. Hoffman and W.H. Meeks III. The asymptotic behavior of properly embedded minimal surfaces of finite topology. *Journal of AMS* 2(4) : 667-681, 1989

[H.-M.-4] D. Hoffman and W.H. Meeks III. The strong halfspace theorem for minimal surfaces. *Inventiones Math.* 101 : 373-377, 1990.

[H.-M.-5] D. Hoffman and W.H. Meeks III. Minimal surfaces based on the catenoid. *Amer. Math. Monthly, Special Geometry Issue* 97(8) : 702-730, 1990.

[H.-Wei] D. Hoffman and F. Wei. Adding handles to the helicoid, preprint.

[E.H.] E. Hopf. On an inequality for minimal surfaces $z = f(x, y)$, *J. Rat. Mech. Anal.* 2 : 519-522, 1953.

[Hub.] A. Huber. On subharmonic functions and differential geometry in the large. *Commentari Mathematici Helvetici* 32 : 181-206, 1957.

[J.-S.] H. Jenkins, J. Serrin. Variational problems of minimal surface type II, *Arch. Rat. Mech. Analysis* 21 : 321-342, 1966.

[J.-Xav.] L. Jorge, F. Xavier. A complete minimal surface in a slab of $\mathbb{R}^3$, *Annals of Maths*, 1980, 203-206.

[K.-1] H. Karcher. Construction of minimal surfaces. *Surveys in Geometry*, pages 1-96, 1989. University of Tokyo, 1989, and Lecture Notes No.12, SFB256, Bonn, 1989.

[K.-2] H. Karcher. Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.* 62 : 83-114,1988.

[K.-3] H. Karcher. The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions. *Manuscripta Math.* 64 : 291-357, 1989.

[K.-4] H. Karcher. Construction of higher genus embedded minimal surfaces. *Geom. and Top. of Sub. III* World Sc. 174-191, 1990.

[L.-R.] R. Langevin and H. Rosenberg. A maximum principle at infinity for minimal surfaces and applications. *Duke Math. Journal* 57 : 819-828, 1988.

[Lo.-Ros] F.J. Lopez and A. Ros. On embedded complete minimal surfaces of genus zero. *Journal of Differential Geometry* 33(1) : 293-300, 1991.

[M.-1] W.H. Meeks III. The geometry, topology and existence of periodic minimal surfaces, preprint.

[M.-2] W.H. Meeks III. *Lectures on Plateau's Problem.* Insituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, 1978.

[M.-3] W.H. Meeks III. The theory of triply-periodic minimal surfaces. *Indiana University Math. Journal* 39(3) : 877-936, 1990.

[M.-R.-1] W.H. Meeks III and H. Rosenberg. The global theory of doubly periodic minimal surfaces. *Inventiones Math.* 97 : 351-379, 1989.

[M.-R.-2] W.H. Meeks III and H. Rosenberg. The maximum principle at infinity for minimal surfaces in flat three-manifolds. *Commentari Mathematici Helvetici* 65 : 255-270, 1990.

[M.-R.-3] W.H. Meeks III and H. Rosenberg. The geometry and conformal structure of properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$, to appear in *Invent. Math.*

[M.-R.-4] W.H. Meeks III and H. Rosenberg. The geometry of periodic minimal surfaces, to appear in *Comment. Math. Helv.*

[M.-Wh.] W.H. Meeks III and B. White. Minimal surfaces bounded by convex curves in parallel planes. *Commentari Mathematici Helvetici* 66 : 263-278, 1991.

[M.-Y.] W.H. Meeks and S.T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.* 179 : 151-168, 1982.

[N.] J.C.C. Nitsche. A characterization of the catenoid. *Journal of Math. Mech.* 11 : 293-302, 1962.

[Oss.-1] R. Osserman. Global properties of minimal surfaces in E^3 and E^n. *Annals of Math.* 80(2) : 340-364, 1964.

[Oss.-2] R. Osserman. On the Gauss curvature of minimal surfaces. *Trans. AMS* 96 : 115-128, 1960.

[P.-Ros] J. Pérez and A. Ros. Some uniqueness and nonexistence theorems for embedded minimal surfaces, preprint.

[Rado-1] T. Rado. The problem of the least area and the problem of Plateau. *Math. Z.* 32 : 763-796, 1930.

[Rado-2] T. Rado. On the problem of Plateau. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, Berlin 1933.

[Reif.] R. Reifenberg. Solution for the Plateau problem for m-dimensional surfaces of varying topological type. *Acta Math.* 104 : 1-92, 1960.

[R.-T.-1] H. Rosenberg and E. Toubiana. A cylindrical type complete minimal surface in a slab of $\mathbb{R}^3$. *Bull. Sc. Math. III*, pages 241-245, 1987.

[R.-T.-2] H. Rosenberg and E. Toubiana. Complete minimal surfaces and minimal herissons. *Journal of Differential Geometry* 28 : 115-132, 1988.

[R.-S.E.] Sa Earp and H. Rosenberg. The Dirichlet problem for the minimal surface equation on unbounded planar domains. *Journal de Mathématiques Pures et Appliquées* 68 : 163-183, 1989.

[Sch.-1] R. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. *Journal of Differential Geometry* 18 : 791-809, 1983.

[Sch.-2] R. Schoen. *Estimates for Stable Minimal Surfaces in Three Dimensional Manifolds*, volume 103 of *Annals of Math. Studies*. Princeton University Press, 1983.

[Simon] L. Simon. Lectures on geometric measure theory. In *Proceedings of the Center for Mathematical Analysis*, volume 3, Canberra, Australia, 1983. Australian National University.

[Smale] N. Smale. A bridge principle for minimal and constant mean curvature submanifolds of $\mathbb{R}^n$. *Invent. Math.* 90 : 505-549, 1987.

[M.S.] M. Soret. Deformations de surfaces minimales. *Thèse Univ. Paris VII*, 1992.

[Souam] R. Souam. Stabilité et unicité des surfaces minimales. *Thèse Univ. Paris VII*, 1992.

[T.] E. Toubiana. On the uniqueness of the helicoid. *Ann. Inst. Four.* 38 : 121-132, 1988.

[Wei] F. Wei. Some existence and uniqueness theorems for doubly periodic minimal surfaces, to appear in *Invent. Math.*

[Wh.] B. White. Complete surfaces of finite total curvature. *Journ. Diff. Geom.* 26 : 315-326, 1987.

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