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STOCHASTIC METHODS
AND DIFFERENTIAL GEOMETRY

by K. David ELWORTHY

1. Introduction

Stochastic methods, or "path integral techniques", have been used over the past 30 years, especially by mathematical physicists, to study differential operators $\mathcal{B}$ of the form $\mathcal{B}f = \frac{1}{2}\Delta f + A\cdot\nabla f + Vf$ in Euclidean spaces. These uses have varied from the heuristic, as in Feynman path integration [10], to the discussion of rather detailed problems, as for example in recent work by Carmona and Simon [5] or as described in Simon's monograph [25]. The techniques have often been particularly attractive giving a direct link between intuition and analysis.

These methods centre on the relationship between Brownian motion on $\mathbb{R}^n$ and the Euclidean Laplacian. An analogous set up for the Laplace-Beltrami operator on a Riemannian manifold gives similar intuition and insight into certain areas of differential geometry. This had lead [27] to an interaction between probability theory and differential geometry which can be stimulating to both disciplines as well as being very pretty mathematically. Here we are going to give examples to illustrate some of these points.

It will only be possible to touch on a few aspects : a more general view of the interaction can be obtained from the conference proceedings [12], [16], [27] ; see also [18], [22], and [24].

2. Conventions

The notation x_t is used for maps $\omega \mapsto x(t,\omega)$ and then we often write $f(x_t)$ instead of $f\circ x_t$ to denote a composition. Throughout, $d(\ ,\ )$ denotes the relevant Riemannian distance.

Many equalities have to be taken in the sense that they hold outside a set of measure zero.

3. Brownian motion and the Laplace-Beltrami operator

3.1. Let M be a C^∞ Riemannian manifold and Δ its Laplace-Beltrami operator. Let $B(M)$ denote the space of bounded measurable functions on M. Then [3] there is a unique family $\{P_t\}_{t>0}$ of bounded linear operators $P_t : B(M) \to B(M)$ such that for all $f \in B(M)$, all positive s and t, and all $x \in M$:

(i) $P_t f(x)$ is measurable in t ;

(ii) (positivity) $P_t f \geq 0$ when $f \geq 0$;

(iii) (semigroup) $P_s P_t f = P_{s+t} f$;

(iv) $P_t 1 \leq 1$;

(v) if f is C^2 with compact support then

$$P_t f(x) - f(x) = \int_0^t P_s \Delta f(x)\,ds \quad ;$$

(vi) (minimality) if $\{Q_t\}_{t>0}$ is another family satisfying (i) to (v) then $P_t f \leq Q_t f$ whenever $f \geq 0$.

For this family $\{P_t\}_{t>0}$ there is a kernel $p(t,x,y)$ smooth in $(t,x,y) \in \mathbb{R}(>0) \times M \times M$ such that

$$P_t f(x) = \int_M p(t,x,y) f(y)\,dy \qquad f \in B(M) ,$$

the integration being with respect to the usual Riemannian density of M. Also $P_t f$ is a classical solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) P_t f = 0 .$$

If $M = \mathbb{R}^n$ then

$$p(t,x,y) = (2\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{2t}\right) .$$

In general $P_t f$ can be obtained as the least upper bound of the classical solutions of the heat equation on the interiors of an increasing family of bounded domains of M with zero boundary values.

3.2. Let $M^+ = M \cup \{\infty\}$ be the one point compactification of M. Define

$$\bar{p} : \mathbb{R}(>0) \times M^+ \times M^+ \to \mathbb{R}$$

by

$$\bar{p}(t,x,y) = p(t,x,y)$$
$$\bar{p}(t,\infty,y) = 0 , \quad \bar{p}(t,\infty,\infty) = 1$$
$$\bar{p}(t,x,\infty) = 1 - \int_M p(t,x,y)\,dy = 1 - P_t(1)(x)$$

for x and y in M.

By a *Brownian motion* x on M starting at a point x_0 of M we mean a map

$$x : [0,\infty) \times \Omega \to M^+$$

for a measurable space $(\Omega,\mathcal{F})$, together with a probability measure P on $(\Omega,\mathcal{F})$ such that

(i) $x(0,\omega) = x_0$

(ii) each $x_t \equiv x(t,-)$ is measurable

(iii) each *sample path* $x(-,\omega) : [0,\infty) \to M^+$ is continuous

(iv) for all Borel sets $A_1,\dots,A_m$ in M^+ and times $0 < t_1 \le \dots \le t_m$:

$$P\{\omega\in\Omega : x(t_j,\omega)\in A_j \text{ for } 1\le j\le m\} = \int_{A_1\times\dots\times A_m} p(\Delta_0,x_0,x_1)\dots p(\Delta_{m-1},x_{m-1},x_m)\,dx_1\dots dx_m$$

where we have extended the measure on M to one on M^+ by giving $\{\infty\}$ unit mass. Also $\Delta_j \equiv \Delta_j t \equiv t_{j+1} - t_j$ with $t_0 = 0$.

The canonical example is with Ω the space of continuous paths $\sigma : [0,\infty) \to M^+$ satisfying $\sigma(0) = x_0$ and with $x_t : \Omega \to M^+$ the evaluation at t . The σ-algebra $\mathcal{F}$ is that generated by $\{x_t\}_{t\ge 0}$ and the measure P is then determined by (iv).

3.3. Let $\xi : \Omega \to \mathbb{R}\cup\{\infty\}$ be the *lifetime* or *explosion time* of x :

$$\xi(\omega) = \inf\{t : x(t,\omega) = \infty\} .$$

Then $x(t,\omega) = \infty$ if $t \ge \xi(\omega)$. If $\xi \equiv \infty$ or equivalently if $P_t 1 = 1$ for all $t > 0$ we will say that M is *stochastically complete.* Since Brownian motions never hit submanifolds of codimension at least two ([11], vol. 2, Chapter 11), such manifolds can be removed without upsetting stochastic completeness. Thus it does not imply completeness. Conversely it is not implied by completeness [3]. However there is the following result, proved analytically by S.-T. Yau ([31], [8]). It is discussed more in § 8 below.

THEOREM.— *Every complete Riemannian manifold with Ricci curvature bounded below is stochastically complete.*

3.4. When $M = \mathbb{R}^n$ we will consider only Brownian motions starting at the origin 0 . It will be convenient to choose one of them, z say, once and for all. This then fixes an increasing family $\{\mathcal{F}_t\}_{t\ge 0}$ of sub σ-algebras of $\mathcal{F}$: we let $\mathcal{F}_t$ be the smallest σ-algebra which contains all the sets of P-measure zero in $\mathcal{F}$ and with respect to which the maps $z_s : \Omega \to \mathbb{R}^n$ are measurable for $0 \le s \le t$.

With respect to this family a map $x : \mathbb{R}(\ge 0)\times\Omega \to M^+$ is *adapted* (is an *adapted process*) if x_t is $\mathcal{F}_t$-measurable for each $t \ge 0$. When $M = \mathbb{R}^n$ it is a *martingale* (strictly speaking an $\mathcal{F}_*$-martingale) if x_t is integrable for each t and

$$\mathbb{E}\{x_t \mid \mathcal{F}_s\} = x_s \qquad s \le t .$$

Here

$$\mathbb{E}\{- \mid \mathcal{A}\} : L^1(\Omega,\mathcal{F},P;\mathbb{R}^n) \longrightarrow L^1(\Omega,\mathcal{A},P;\mathbb{R}^n)$$

denotes the *conditional expectation* operator with respect to a sub σ-algebra $\mathcal{A}$ of $\mathcal{F}$: it is the unique continuous linear map which restricts to the orthogonal projection of $L^2(\Omega,\mathcal{F},P;\mathbb{R}^n)$ onto $L^2(\Omega,\mathcal{A},P;\mathbb{R}^n)$. When $\mathcal{A} = \Omega$ we obtain the *expectation* : $\mathbb{E}(f \mid \Omega) = \int_\Omega f(\omega)P(d\omega) \equiv \mathbb{E}(f)$; i.e. it reduces to the integral over Ω .

A map $\xi : \Omega \to [0,\infty]$ is a *stopping time* if for each $t \geq 0$ we have

$$\{\omega \in \Omega : t < \xi(\omega)\} \in \mathcal{F}_t .$$

When an adapted process x has continuous sample paths it is easy to see that the first exit time $T(U)(\omega)$ of $x(-,\omega)$ from an open set U is a stopping time. In particular the lifetime ξ of an adapted Brownian motion x is a stopping time and if we set

$$[0,\xi) \times \Omega = \{(t,\omega) \in [0,\infty) \times \Omega : t < \xi(\omega)\}$$

we can consider x as a map with values in M :

$$x : [0,\xi) \times \Omega \to M .$$

These two ways of looking at x will be used interchangeably and without comment.

3.5. A typical partition Π of an interval $[0,t]$ will be denoted by $0 = t_0 < t_1 \leq \ldots \leq t_m = t$, and then we will set $\text{mesh}\,\Pi = \max_j \Delta_j t$ and $\Delta_j z = z_{t_{j+1}} - z_{t_j}$. With this notation Lévy's characterization of a 1-dimensional Brownian motion z starting from 0 is essentially ([16] page xii, or [28])

(i) $z_0 = 0$

(ii) z has continuous sample paths

(iii) z_t is square integrable for each t

(iv) z is a martingale

(v) $$\lim_{\text{mesh}\,\Pi \to 0} \sum_{j=0}^{m-1} (\Delta_j z)^2 = t$$

where the limit is taken in L^2 and over all partitions Π of $[0,t]$.

For n-dimensional Brownian motion z the components $z^1,\ldots,z^n$ are independent 1-dimensional Brownian motions. The analogous formula to (v) is

(v)' $$\lim_{\text{mesh}\,\Pi \to 0} \sum_{j=0}^{m-1} \Delta_j z \otimes \Delta_j z = \text{diag}\ t$$

where $\text{diag}\ t = t \sum_{i=1}^{n} e_i \otimes e_i \in \mathbb{R}^n \otimes \mathbb{R}^n$ for the standard basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$.

3.6. Next we give a characterization of Brownian motion on M along the lines of the Stroock-Varadhan approach [28] : Suppose ξ is a stopping time and $x : [0,\xi) \times \Omega \to M$ with $x(0,\omega) = x_0 \in M$ has continuous sample paths, is adapted, and is *maximal* i.e. $x(t,\omega) \to \infty$ as $t \to \xi(\omega)$ whenever $\xi(\omega) < \infty$. Then x is a Brownian motion iff

$$f(x_t) - f(x_0) - \frac{1}{2}\int_0^t \Delta f(x_s)ds$$

is a martingale whenever $f : M \to \mathbb{R}$ is C^2 with compact support (so we set $f(\infty) = 0 = \Delta f(\infty)$).

To relate this to our definition define

$$P_t f(x_0) = \int_\Omega f(x(t,\omega))P(d\omega) .$$

4. Stochastic integrals and stochastic differential equations

4.1. Let z be our Brownian motion on $\mathbb{R}^n$ and for some $a \geq 0$ consider an *adapted* map $G : [a,\infty) \times \Omega \to \mathbb{L}(\mathbb{R}^n;\mathbb{R}^m)$; the values being in the space of linear maps of $\mathbb{R}^n$ into $\mathbb{R}^m$ furnished with the norm $\|A\| = \sqrt{\text{trace } A^*A}$. The *stochastic integral*

$$\int_a^t G_s dz_s : \Omega \to \mathbb{R}^m \qquad t \geq a$$

can be defined by approximating G by adapted simple functions, provided

$$\int_a^t \|G(s,\omega)\|^2 ds < \infty \quad \text{almost all } \omega \in \Omega .$$

If G_s lies in L^2 for each s , is continuous in s into L^2 almost everywhere on $[a,t]$, and if $\|G_s\|_{L^2}$ is bounded for $a \leq s \leq t$ then the integral is an L^2 limit of Riemann sums using partitions Π of $[a,t]$:

$$\int_a^t G_s dz_s = \lim_{\text{mesh}\,\Pi \to 0} \sum_{j=0}^{m-1} G_{t_j} \Delta_j z .$$

Here it is important that the evaluation, G_{t_j} of G is taken at the initial point t_j of the interval $[t_j,t_{j+1}]$, [19]. The integral is only defined up to equivalence. However it is possible to choose a version for each t so that as t varies we obtain an adapted process with continuous sample paths. This will always be done in what follows. The need for a special definition of these integrals is because for almost all $\omega \in \Omega$ the sample paths of z are not of bounded variation in any interval of $\mathbb{R}(\geq 0)$, cf. (v) above.

A straightforward reference for stochastic integration is [2] ; [11] and [15] have more material ; parts of [19] may be found helpful and also the reviews [16], [28] ; the standard reference [20] develops the general theory.

4.2. The main properties for our purposes are as follows : The Euclidean norm is used on $\mathbb{R}^m$.

(i) (Estimates). If $\mathbb{E}\int_a^t \|G(s,\omega)\|^2 ds < \infty$ then $\int_a^t G_s dz_s$ is in L^2 and satisfies

$$\mathbb{E}\,\|\int_a^t G_s dz_s\|^2 = \mathbb{E}\int_a^t \|G_s\|^2 ds$$

and (by (ii) below and the martingale inequality) if $\delta > 0$ and $T \geq a$

$$P\{\omega : \sup_{a\leq t\leq T} \|\int_a^t G_s(\omega) dz_s(\omega)\| > \delta\} \leq \frac{1}{\delta^2}\mathbb{E}\int_a^T \|G_s\|^2 ds$$

(ii) (Martingale property). If $\mathbb{E}\int_a^t \|G(s,\omega)\|^2 ds < \infty$ then

$$\mathbb{E}\{\int_a^t G_r dr \mid \mathcal{F}_s\} = \int_a^s G_r dr \qquad a \leq s \leq t .$$

(iii) (Itô formula). If $\theta : \mathbb{R}^m \to \mathbb{R}^p$ is C^2 and

$$x_t = x_a + \int_a^t G_s dz_s + \int_a^t H_s ds$$

where $x_a : \Omega \to \mathbb{R}^m$ is $\mathcal{F}_a$-measurable and $H : [a,\infty) \times \Omega \to \mathbb{R}^m$ is adapted with

$$\int_a^t \|H(s,\omega)\| ds < \infty \qquad a \le t < \infty ,$$

almost surely. Then

$$\theta(x_t) = \theta(x_a) + \int_a^t D\theta(x_s)G_s dz_s + \int_a^t D\theta(x_s)H_s ds + \frac{1}{2}\int_a^t \text{trace } D^2\theta(x_s)(G_s,G_s)ds$$

where by the trace term we mean $\sum_{i=1}^n D^2\theta(x_s)(G_s e_i, G_s e_i)$.

Of these (ii) looks plausible by considering the corresponding Riemann sums. For the Itô formula : given a partition Π of $[a,t]$ we have

$$\theta(x_t) - \theta(x_a) = \sum_{j=1}^{m-1} \{\theta(x_{t_{j+1}}) - \theta(x_{t_j})\}$$

$$= \sum_{j=1}^{m-1} \{D\theta(x_{t_j})\Delta_j x + \int_0^1 (1-\alpha)D^2\theta(\alpha x_{t_{j+1}} + (1-\alpha)x_{t_j})(\Delta_j x, \Delta_j x)d\alpha\} .$$

The approximation $\Delta_j x \approx G_{t_j}\Delta_j z + H_{t_j}\Delta_j t$ can be substituted in. Terms involving $\Delta_j t \otimes \Delta_j z$ and $(\Delta_j t)^2$ converge to zero while those involving $\Delta_j z \otimes \Delta_j z$ give the trace term in the formula using 3.5 (v)' above.

4.3. When $\tau : \Omega \to [a,\infty]$ is a stopping time we set

$$\int_a^{t\wedge\tau} G_s dz_s = \int_a^t \chi_s^\tau G_s dz_s$$

where

$$\chi^\tau : [a,\infty) \times \Omega \to \{0,1\}$$

is the characteristic function of $[a,\tau) \times \Omega$.

4.4. Closely related to Lévy's characterization of Brownian motion in 3.5 is the fact that when $m = 1$ and $a = 0$ there is a Brownian motion B on $\mathbb{R}$, (depending on G) with

$$B_{\sigma(t)} = \int_0^t G_s dz_s \qquad 0 \le t < \infty$$

where

$$\sigma(t) = \int_0^t \|G_s\|^2 ds$$

4.5. If $X : \mathbb{R}^m \to \mathbb{L}(\mathbb{R}^n;\mathbb{R}^m)$ and $Y : \mathbb{R}^m \to \mathbb{R}^m$ are globally Lipschitz then for each $x_0 \in \mathbb{R}^m$ there is a unique adapted process x_t with continuous sample paths

$$x : \mathbb{R}(\ge 0) \times \Omega \to \mathbb{R}^m$$

satisfying the stochastic integral equation

$$x_t = x_0 + \int_0^t X(x_s)dz_s + \int_0^t Y(x_s)ds \qquad 0 \le t < \infty .$$

Using the Itô formula one can obtain an invariant definition of *stochastic differential equations* on a C^2 manifold N , [15], [18], [12] : given a C^1 vector field Y on N and a C^2 vector bundle map $X : \underline{\mathbb{R}}^n \to TM$ over the identity, where $\underline{\mathbb{R}}^n$

denotes the trivial $\mathbb{R}^n$ bundle over N, for each x_0 in N there is a unique adapted process with continuous sample paths

$$x : [0,\xi) \times \Omega \to N$$

for some stopping time ξ, which is maximal and such that for all C^2 maps $f : N \to \mathbb{R}^p$ we have

$$(1) \qquad f(x_t) = f(x_0) + \int_0^t df \circ X(x_s)dz_s + \int_0^t df \circ Y(x_s)ds + \frac{1}{2}\int_0^t \mathscr{L}_X^2(x_s)ds$$

on $\{\omega : t < \xi(\omega)\}$, where

$$\mathscr{L}_X^2(f) : N \to \mathbb{R}^p$$

is given by $\mathscr{L}_X^2(f) = \sum_{i=1}^n X_i(X_i(f))$ for the vector fields $X_i \equiv X(-)e_i$ on M. Such x will be called a (maximal) *solution* of the *stochastic differential equation*

$$dx = X \circ dz + Yds .$$

If N were $\mathbb{R}^n$ this would correspond to the integral equation

$$(2) \qquad x_t = x_0 + \int_0^t X(x_s)dz_s + \int_0^t Y(x_s)ds + \frac{1}{2}\int_0^t \operatorname{trace} DX(x_s)(X(x_s)-,-)ds .$$

The trace term is the so called "Stratanovich" term needed to get an invariant definition. Our differential equation is a "Stratanovich" equation which is why we have used $X \circ dz$ rather than Xdz. The existence of the solution x is obtained by solving equations like (2) obtained from charts of N.

The "Itô formula" (1) has other forms. When N has an affine connection it can be written

$$(3) \qquad f(x_t) = f(x_0) + \int_0^t df \circ X(x_s)dz_s + \int_0^t df \circ Y(x_s)ds + \frac{1}{2}\int_0^t \sum_{i=1}^n df \circ \nabla X_i(X_i(x_s))ds$$
$$+ \frac{1}{2}\int_0^t \operatorname{trace} \nabla df(X(x_s)-,X(x_s)-)ds , \quad \text{on } \{\omega : t < \xi(\omega)\}$$

Letting $S_i(r,y_0)$ denote the integral curve at time r of X_i, starting at y_0, we have

$$(4) \qquad f(x_t) = f(x_0) + \int_0^t df \circ X(x_s)dz_s + \int_0^t df \circ Y(x_s)ds$$
$$+ \frac{1}{2}\int_0^t \sum_{i=1}^n \frac{d^2}{dr^2} f \circ S_i(r,x_s)\big|_{r=0} ds \quad \text{on } \{\omega : t < \xi(\omega)\} .$$

From 3.6 and 4.2 (ii) we see that when N is a Riemannian manifold x will be a Brownian motion if $X(y) : \mathbb{R}^n \to T_yN$ is an orthogonal projection at each point and $Y + \frac{1}{2}\sum_i \nabla X_i(X_i) = 0$.

4.6. If our stochastic differential equation, z was replaced by another Brownian motion $\tilde{z}$ on $\mathbb{R}^n$ using a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},P)$ then the corresponding solution $\tilde{x}$ to $d\tilde{x} = X \circ d\tilde{z} + Yd_t$ would have the same distributions as x; i.e. for all $0 < t_1 \le \dots \le t_m$ and Borel sets $A_1,\dots,A_m$ in N we have

$$P\{\tilde{\omega} \in \tilde{\Omega} : \tilde{x}(t_j,\tilde{\omega}) \in A_j \text{ for } j = 1 \text{ to } m\} = P\{\omega \in \Omega : x(t_j,\omega) \in A_j \text{ for } j = 1 \text{ to } m\} .$$

5. The stochastic development

5.1. For our Riemannian manifold M there is no canonical choice of coefficients X , Y to give a stochastic differential equation whose solutions are Brownian motions on M . However if we consider the orthonormal frame bundle $\pi : O(M) \to M$ there is a suitable $X : \mathbb{R}^n \to TO(M)$ determined by the Levi-Civita connection, namely

$$X(v)e = h_v v(e) \qquad v \in O(M) \ , \quad e \in \mathbb{R}^n$$

where $h_v : T_{\pi(v)}M \to T_v O(M)$ is the horizontal lift and v is considered as an isometry $v : \mathbb{R}^n \to T_{\pi(v)}M$. Given $x_0 \in M$ choose $u_0 \in \pi^{-1}(x_0)$. Let $u : [0,\xi) \times \Omega \to O(M)$ be the maximal solution to $du = X \circ dz$ with $u(0,\omega) = u_0$. Set $x_t = \pi \circ u_t$. Using 4.5 (4) and the fact that each $\pi \circ S_i(t,y)$ is a geodesic in M we see that x_t is a Brownian motion. For convenience we will use this model of Brownian motion, and in particular this notation, from now on.

This construction goes back to the beginnings of differential geometry. When σ is a piecewise C^1 path in $T_{x_0}M$ and $v_t(\sigma)$ is the solution of

$$\frac{d}{dt} v_t(\sigma) = X(v_t(\sigma))u_0^{-1} \frac{d\sigma}{dt} \qquad v_0(\sigma) = u_0$$

then $\pi \circ v_t(\sigma)$ is the *Cartan development* of σ . Physically it is the track left on M by the point of contact of M with its tangent plane $T_{x_0}M$ as it "rolls without slipping" on $T_{x_0}M$ along the path σ . The stochastic version was formulated this way by Eells and Elworthy following earlier work by Gangolli (see [12] for references, related details are in [15], [18]).

In this classical case $v_t(\sigma)$ is the horizontal lift of $\pi v_t(\sigma)$. Thus in the stochastic situation we have not only obtained a Brownian motion on M but also its "horizontal lift" u_t . We will use this to look at the heat flow on differential forms, treating only 1-forms for simplicity.

6. The heat flow on differential forms

For $\omega \in \Omega$ and $V_0 \in T_{x_0}M$ define a vector field $V(t,\omega)$ along $x(t,\omega)$ by insisting that $V(0,\omega) = V_0$ and that $u(t,\omega)^{-1}V(t,\omega)$ satisfies

$$\frac{d}{dt}\{u(t,\omega)^{-1}V(t,\omega)\} = -\frac{1}{2} u(t,\omega)^{-1}\mathrm{Ric}(V(t,\omega),-)^{\#}$$

where Ric denotes the Ricci tensor and $\mathrm{Ric}(V(t,\omega),-)^{\#} \in T_{x(t,\omega)}M$ corresponds to $\mathrm{Ric}(V(t,\omega),-) \in T^*_{x(t,\omega)}M$.

Let ϕ be a C^2 1-form on M . We can lift ϕ and $V(t,\omega)$ to $\tilde{\phi} : O(M) \to (\mathbb{R}^n)^*$ and $\tilde{V}(t,\omega) : O(M) \to \mathbb{R}^n$ so that $\phi(V(t,\omega)) = \tilde{\phi}(u(t,\omega))\tilde{V}(t,\omega)$. Applying the Itô formula together with the Weitzenböck formula

$$\Delta\phi = \text{trace } \nabla^2\phi - \text{Ric}(-,\phi^{\#})$$

(note our sign convention for the Laplace-Beltrami operator on forms : $\Delta = -(d\delta + \delta d)$) we obtain

(1) $$\phi(V_t) = \phi(V_o) + \int_o^t \nabla\phi(u_s dz_s)V_s + \frac{1}{2}\int_o^t \Delta\phi(V_s)ds \quad \text{on} \quad \{\omega : t < \xi(\omega)\} .$$

Observe that

(2) $$\|V_t\|^2_{x_t} = \|V_o\|^2_{x_o} - \int_o^t \text{Ric}(V_s,V_s)ds .$$

Consequently if Ric is bounded below on M and ϕ is bounded we can set

$$P_t\phi(V_o) = \int_{\{\omega:t<\xi(\omega)\}} \phi(V(t,\omega))P(d\omega) .$$

In fact $\xi \equiv \infty$ by Yau's theorem, 3.3, and if also $\Delta\phi$ is bounded we have

(3) $$P_t\phi(V_o) = \phi(V_o) + \int_o^t P_s(\Delta\phi)(V_o)ds .$$

This follows immediatly from 6.1 (1) and 4.2 (ii) when $\nabla\phi$ is bounded : if it is not we can nevertheless take the limit $n \to \infty$ of integrations over sets $\{\omega : t < \tau^n(\omega)\}$ where τ^n is the first exit time of x from $\{y : d(x_o,y) < n\}$, $n = 1,2,\ldots$

If we allow V_o (and x_o) to vary P_t determines a semi-group on the space of bounded measurable one-forms with $\frac{1}{2}\Delta$ as differential generator. In fact

THEOREM.— *Suppose that* ψ_t *is a* C^2 *1-form on* M *for* $0 \le t \le T < \infty$ *which satisfies the heat equation* $\frac{\partial\psi_t}{\partial t} = \frac{1}{2}\Delta\psi_t$. *Assume that* M *is complete and the Ricci curvature of* M *is bounded below. Then if* ψ_t *is bounded on* M *for* $0 \le t \le T$ *we have*

$$\psi_t = P_t\psi_o \qquad 0 \le t \le T .$$

In particular such a solution is uniquely determined by its initial value ψ_o .

<u>Proof</u>.— It suffices to show that $\psi_T = P_T\psi_o$. For this take $V_o \in T_{x_o}M$ and V_t as above. Set $\phi_t = \psi_{t-T}$ for $0 \le t \le T$. Then $\frac{\partial\phi_t}{\partial t} = -\frac{1}{2}\Delta\phi_t$ and formula (1), modified to take into account the time dependence of ϕ_t , yields

$$\phi_t(V_t) = \phi_o(V_o) + \int_o^t \nabla\phi_s(u_s dz_s)V_s$$

whence

$$\psi_o(V_T) = \psi_T(V_o) + \int_o^t \nabla\phi_s(u_s dz_s)V_s .$$

The result follows on integrating over Ω as in the proof of (3).

The uniqueness of bounded solutions to the heat equation on differential forms under the conditions of the theorem was proved by Dodziuk [8] using analytical methods. See also [26]. The approach we have used to discuss $P_t\phi$ was given by Airault [1] following work by Itô, Dynkin, Eells-Malliavin, and Malliavin ; see [13]. Equation (2) shows very clearly the role that can be played by positive Ricci curvature.

7. A comparison theorem : Harmonic manifolds

7.1. Comparison theorems for solutions of stochastic differential equations have been used effectively, especially by Malliavin and Debiard, Gaveau and Mazet ; see [18], [7]. The following is a very simple special case of one taken from Malliavin [18]. See also [14].

THEOREM.— *Let* x *be a Brownian motion on* M . *Suppose* $p : M \to \mathbb{R}(\geq 0)$ *satisfies*

(i) p *is* C^2 *on* $p^{-1}[(0,\infty)]$

(ii) $p(x_t) \neq 0$ *for* $t \geq 0$, *almost surely*

(iii) $\|\nabla p(x_t)\|_{x_t} \neq 0$ *for* $t \geq 0$, *almost surely.*

Set

$$a(y) = \frac{\nabla p(y)}{\|\nabla p\|_y^2} \qquad y \in M$$

and

$$\sigma_t = \int_0^t \|\nabla p(x_s)\|_{x_s}^2 ds \ .$$

Consider locally Lipschitz maps $a^+ : (0,\infty) \to \mathbb{R}$ *and* $a^- : (0,\infty) \to \mathbb{R}$ *satisfying*

$$a^+(\xi) \geq \sup\{a(y) : p(y) = \xi\}$$

and

$$a^-(\xi) \leq \inf\{a(y) : p(y) = \xi\}$$

Then there is a real valued Brownian motion B *defined on* $(\Omega,\mathcal{F},P)$ *such that the solutions* ξ^+ *and* ξ^- *to*

$$\xi_t^{\pm} = p(x_0) + \int_0^t a^{\pm}(\xi_s^{\pm})ds + B_t$$

in $(0,\infty)$ *satisfy*

$$\xi^-(\sigma(t,\omega),\omega) \leq p(x(t,\omega)) \leq \xi^+(\sigma(t,\omega),\omega)$$

during the lifetimes of the processes.

To believe this take x and u as given by the stochastic development. By Itô's formula

$$p(x_t) = p(x_0) + \int_0^t dp(u_s dz_s) + \frac{1}{2}\int_0^t \Delta p(x_s)ds \quad \text{on} \quad \{\omega : t < \xi(\omega)\} ,$$

and as mentioned in 4.4 the stochastic integral $\int_0^t dp(u_s dz_s)$ can be written as B_{σ_t} for B as required.

7.2. A similar argument but for a simpler case was used by Dominique Michel [21] when she proved that compact simply connected globally harmonic manifolds are strongly harmonic : at that time an open problem in differential geometry, (see [4] for definitions and discussion). Here is her main step :

THEOREM.— *Assume that* M *is complete and stochastically complete, and that* $m_0 \in M$ *satisfies :*

(i) *the cut locus* $C(m_0)$ *of* m_0 *has capacity zero (e.g.* $C(m_0)$ *has codimension at least two)*

(ii) *there is a function* $f : (0,\infty) \to \mathbb{R}$ *such that if* $r(m)$ *denotes the distance of* m *from* m_o *then*

$$\Delta r(m) = f(r(m)) \qquad m \in M - C(m_o) \ , \quad m \neq m_o \ .$$

Then $p(t,m,m_o)$ *depends only on* $r(m)$ *when* $m \in M - C(m_o)$.

Proof.— Take Brownian motions x_t and y_t on M with $r(x_o) = r(y_o) > 0$. The assumption (i) on $C(m_o)$ implies that x_t and y_t never hit $C(m_o)$, and so can be considered as non explosive processes on $M - [C(m_o) \cup \{m_o\}]$, assuming $\dim M > 1$.

Applying the Itô formula to r restricted to the complement of $C(m_o) \cup \{m_o\}$ we obtain

$$r(x_t) = r(x_o) + \int_o^t dr(u_s dz_s) + \frac{1}{2}\int_o^t f(r(x_s))ds \ .$$

Since $\|dr\| \equiv 1$ we have $\sigma_t \equiv 1$ and the stochastic integral is just a one dimensional Brownian motion B^x say. Thus $\xi_t \equiv r(x_t)$ satisfies

$$\xi_t = r(x_o) + B_t^x + \frac{1}{2}\int_o^t f(\xi_s)ds \ .$$

Similarly there is a one dimensional Brownian motion B^y such that $\eta_t \equiv r(y_t)$ satisfies

$$\eta_t = r(y_o) + B_t^y + \frac{1}{2}\int_o^t f(\eta_s)ds \ .$$

Since $r(x_o) = r(y_o)$ it follows from 4.6 that η and ξ have the same distributions : in particular if $\varepsilon > 0$

$$P\{\omega \in \Omega : \xi(t,\omega) < \varepsilon\} = P\{\omega \in \Omega : \eta(t,\omega) < \varepsilon\}$$

i.e.

$$P\{\omega \in \Omega : r(x(t,\omega)) < \varepsilon\} = P\{\omega \in \Omega : r(y(t,\omega)) < \varepsilon\} \ .$$

But this proves the theorem since

$$p(t,x_o,m_o) = \lim_{\varepsilon \to 0} \frac{P\{\omega \in \Omega : r(x(t,\omega)) < \varepsilon\}}{\text{Volume}\{m \in M : r(m) < \varepsilon\}}$$

and similarly for $p(t,y_o,m_o)$.

8. A criterion for $P_t 1 = 1$: Yau's theorem

As in the last paragraph set $r(m) = d(m,m_o)$. If the Ricci tensor of M is bounded below then for given $\varepsilon > 0$ the Laplacian Δr is bounded above on the set of points m in $M - C(m_o)$ with $r(m) \geq \varepsilon$, [30]. If we could ignore $C(m_o)$ as in the proof of 7.2 the comparison theorem would easily yield Yau's non-explosion result 3.3. In any case we have

Lemma.— Suppose $\alpha : M \to \mathbb{R}(\geq 0)$ *is* C^2 *and satisfies*

(i) $\alpha(y) \to \infty$ *as* $y \to \infty$ *in* M

(ii) *if* $A_n = \{y : \alpha(y) < n\}$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \sup\{\Delta\alpha(y) : y \in A_n\} \leq 0 \ .$$

Then M *is stochastically complete.*

Proof.— Set $k(n) = \sup\{\Delta\alpha(y) : y \in A_n\}$. Let τ_n be the first exit time of our Brownian motion x from $A(n)$ and set $\Omega_t^n = \{\omega : t < \tau_n(\omega)\}$. By Itô's formula

$$\alpha(x_{t\wedge\tau_n}) = \alpha(x_0) + \int_0^{t\wedge\tau_n} d\alpha(u_s dz_s) + \frac{1}{2}\int_0^{t\wedge\tau_n} \Delta\alpha(x_s)ds$$

giving

$$\mathbb{E}\alpha(x_{t\wedge\tau_n}) \leq \alpha(x_0) + \frac{1}{2}k(n)t .$$

However

$$\mathbb{E}\alpha(x_{t\wedge\tau_n}) \geq n(1 - P(\Omega_t^n))$$

giving

$$1 - P(\Omega_t^n) \leq \frac{1}{n}\alpha(x_0) + \frac{1}{2}\frac{k(n)}{n} t .$$

Therefore

$$1 - P(\bigcup_{n=1}^{\infty} \Omega_t^n) \leq 0$$

and the result follows by the maximality of x .

From the lemma we can immediately deduce Yau's result if we are willing to use the smoothing theory of Greene and Wu to obtain a C^∞ exhaustion function α on M with $\Delta\alpha$ bounded above, given that the Ricci tensor is bounded below (see theorem 4 of [29]). Their function α would even be Lipschitz.

For an alternative probabilistic approach, with strengthened conclusions, see [13] part II, § 3.

9. Zero one laws and harmonic maps

9.1. Let Z denote the space of continuous paths $\sigma : [0,\infty) \to M^+$ and Q the σ-algebra of subsets of Z generated by the evaluation maps $\{ev_t : Z \to M^+\}_{t\geq 0}$. For $h \geq 0$ define

$$\theta_h : Z \to Z$$

by

$$\theta_h(\sigma)(t) = \sigma(t+h)$$

and set

$$\mathfrak{J} = \{A \in Q : \theta_h(A) = A \text{ for all } h \geq 0\} .$$

We will say that M satisfies the *zero one law on* $\mathfrak{J}$ if for any Brownian motion x on M and any $A \in \mathfrak{J}$ we have $P\{\omega \in \Omega : x(-,\omega) \in A\}$ equal either to zero or to one. This is equivalent to the constancy of all bounded harmonic functions $f : M \to \mathbb{R}$; (see [17] final paragraphs). In fact for A in $\mathfrak{J}$ define $f_A : M \to \mathbb{R}$ by

$$f_A(x_0) = \int_\Omega \chi_A(x(-,\omega))P(d\omega)$$

where χ_A is the characteristic function of A . For $h > 0$, because $\theta_h(x(-,\omega))$ is a Brownian motion on M , starting at (the random point) x_h

$$P_h f_A(x_0) = \int_\Omega f_A(x_h) P(d\omega) = \int_\Omega \chi_A(\theta_h(x(-,\omega))) P(d\omega) = f_A(x_0)$$

by the invariance of A . Thus f_A is harmonic. Furthermore because it is harmonic $f_A(x_t)$ is a bounded martingale and the martingale convergence theorem implies that $f_A(x(t,\omega))$ converges almost surely to $\chi_A(x(-,\omega))$ as $t \to \infty$. Consequently if all bounded harmonic functions on M are constant we must have

$$\chi_A(x,(-,\omega)) = f_A(x(t,\omega)) = 0 \text{ , or } 1 \text{ , almost surely ,}$$

and the 0-1 law holds.

9.2. If M is complete with non-negative Ricci curvature all bounded harmonic functions on M are constant : see [7] for a probabilistic proof, and so are all positive harmonic functions [30]. A special class of manifolds with the 0-1 law on $\mathfrak{J}$ are those with recurrent Brownian motions or equivalently with no non-constant positive superharmonic functions [3], [13]. These include all complete manifolds with finite volume : see [6].

9.3. Now let (M,g) and (N,h) be C^∞ Riemannian manifolds and $f : M \to N$ a C^∞ map. Then f is *harmonic* if in local coordinates

$$\Delta f^\gamma + g^{ij}\Gamma^\gamma_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} = 0$$

where i , j refer to coordinates in M and α , β , γ refer to coordinates in N , with Γ the Christoffel symbol of N and Δ the Laplacian of M . For $p \in M$ let $\lambda_1(p) \geq \dots \geq \lambda_r(p) \geq 0$ be the eigenvalues of the first fundamental form $(T_p f)^* h$ on $T_p M$ using an orthonormal base with respect to g , and repeated according to multiplicity. Then f has *dilatation bounded by* K if

$$\frac{\lambda_1(p)}{\lambda_2(p)} \leq K$$

at all points p of M with $T_p f \neq 0$. Using the 0-1 law on $\mathfrak{J}$, W. Kendall [27] has proved the following :

THEOREM.— *Assume that* M *and* N *are complete and that for some positive* A , B *and* C

(i) $-B \leq \mathrm{Riem}^N \leq -A$

(ii) $-C \leq \mathrm{Ric}^M$

(iii) N *is simply connected*

(iv) *all bounded real valued harmonic functions on* M *are constant.*

Then every harmonic map $f : M \to N$ *of bounded dilatation is constant.*

This should be compared with an earlier result of Yau : he showed that the same holds for complete manifolds M and N provided M has non-negative Ricci curvature and N satisfies $\mathrm{Riem}^N \leq -A < 0$; see [9] section 5.9.

9.4. Kendall's proof is based on work by Prat [23], [22] concerning the behaviour of Brownian motion on simply connected manifolds N satisfying (i). Suppose $\dim M > 1$ and $\dim N > 1$ and that N satisfies (i) and (iii), both manifolds being complete. Take normal coordinates about a point n_o of N so that N is identified with $T_{n_o}N$. Set $r(p) = d(p,n_o)$ for $p \in N$. Then Prat showed that for Brownian motions y_t on N with $y_o \neq n_o$ the angular component

$$\theta_t = y_{t/r(y_t)} : \Omega \to S$$

where S is the unit sphere in $T_{n_o}N$, converges almost surely as t tends to infinity. To do this he used upper and lower estimates on Δr to show that $r(y_t)$ tends to infinity at a linear rate in t while $\sup\{d(y_n,y_t) : n \le t \le n+1\}$ grows sub-linearly with n. He can then apply the lemma :

Lemma.— Under the conditions on N *there are positive constants* α, β *such that for* p, q *in* N *if*

$$d(p,q) < \alpha \inf\{r(p),r(q)\}$$

then the geodesic distance in the unit sphere S *between* $r(p)^{-1}.p$ *and* $r(q)^{-1}.q$ *is dominated by*

$$d(p,q)\exp(-\beta \inf\{r(p),r(q)\}) .$$

In his proof Kendall examines the behaviour of θ_t when $y_t = f(x_t)$ for x a Brownian motion on M and f harmonic and of bounded dilatation. He shows that θ_t converges to a limit $\theta_\infty : \Omega \to S$ given (i), (ii) and (iii), and that if f were non-constant then θ_∞ would also be not almost surely constant. However for any Borel set U of S

$$\{\sigma \in C([0,\infty);M) : \lim_{t\to\infty} \frac{f(\sigma_t)}{rf(\sigma_t)} \in U\}$$

lies in $\mathfrak{J}$, and so this would contradict the 0-1 law. In outline he follows Prat's steps. By Itô's formula, since f is harmonic

$$r(y_t) = r(f(x_o)) + \int_o^t dr \circ Tf(u_s dz_s) + \frac{1}{2}\int_o^t \operatorname{trace} \nabla dr(T_{x_s}f(-),T_{x_s}f(-))ds .$$

If $\operatorname{trace} \nabla dr(T_m f,T_m f)/\|dr \circ Tf\|_m^2$, $m \in M$ were bounded it would be possible to use a comparison theorem. However this need not be, so the necessary estimates have to be proved ad hoc. Because of the bounded dilatation condition a convenient time scale turns out to be $\mu(t)$ where

$$\mu(t) = \int_o^t \lambda_1(x_s)ds .$$

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