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**Holomorphic vector bundles on $\mathbb{P}_n$**

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# HOLOMORPHIC VECTOR BUNDLES ON $\mathbb{P}_n$

by Michael SCHNEIDER

The classification of holomorphic (= algebraic) vector bundles on complex projective space $\mathbb{P}_n$ could be tried along the following lines :

I) Classify the topological complex vector bundles on $\mathbb{P}_n$ .

II) Determine which topological bundles admit an analytic structure.

III) Classify for fixed topological bundle all possible analytic structures.

This is a survey of some of the main results concerning I) - III) as well as a guide to the literature. We included only a few open problems. But in fact most of the work has still to be done.

Notation.- No distinction will be made between holomorphic vector bundles and locally free coherent analytic sheaves. $\mathcal{O}(1)$ is the line bundle having a holomorphic section vanishing precisely on a hyperplane. $E(k) := E \otimes \mathcal{O}(1)^{\otimes k}$ , $h^i(\mathbb{P}_n, E) := \dim_{\mathbb{C}} H^i(\mathbb{P}_n, E)$ for a vector bundle $E$ on $\mathbb{P}_n$ . The total Chern class of $E$ will be denoted by $c(E) = 1 + c_1(E) + \ldots + c_r(E)$ . The Chern classes $c_i(E) \in H^{2i}(\mathbb{P}_n, \mathbb{Z}) \simeq \mathbb{Z}$ will be regarded mostly as integers. The holomorphic tangent bundle of $\mathbb{P}_n$ will be denoted by $T_{\mathbb{P}_n}$ .

## 1. Topological classification

Let $\mathrm{Vect}^r_{top}(\mathbb{P}_n)$ be the isomorphism classes of topological complex vector bundles of rank $r$ on $\mathbb{P}_n$ . It is well known that $\mathrm{Vect}^r_{top}(\mathbb{P}_n) \simeq \mathrm{Vect}^n_{top}(\mathbb{P}_n)$ for all $r \geq n$ .

Schwarzenberger [53] noticed that the Chern classes of $E \in \mathrm{Vect}^r_{top}(\mathbb{P}_n)$ satisfy the condition

$$(S_n) \qquad \sum_{i=1}^{r} \binom{\delta_i}{k} \in \mathbb{Z} \qquad \text{for } 2 \leq k \leq n .$$

Here the $\delta_i$ are as usual related to the Chern class of $E$ by

$$c(E) = \prod_{i=1}^{r} (1 + \delta_i) .$$

The conditions $(S_n)$ for $r = 2$ are as follows :

$(S_2)$ no condition

$(S_3)$ $c_1c_2 \equiv 0 \ (2)$

$(S_4)$ $c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 \ (12)$

$(S_5)$ is equivalent to $(S_4)$.

For $r = 3$ one gets for instance $(S_3)$ : $c_3 \equiv c_1c_2 \ (2)$ .

A. Thomas [60] proved that the Schwarzenberger condition $(S_n)$ classifies stable bundles on $\mathbb{P}_n$ i.e.

$$\mathrm{Vect}^n_{\mathrm{top}}(\mathbb{P}_n) \simeq \{(c_1,\ldots,c_n) \in \mathbb{Z}^n : (c_1,\ldots,c_n) \text{ satisfy } (S_n)\} .$$

For $\mathbb{P}_2$ this gives

$$\mathrm{Vect}^r_{\mathrm{top}}(\mathbb{P}_2) \simeq \mathbb{Z} \times \mathbb{Z} \qquad \text{for } r \geq 2 .$$

For $\mathbb{P}_3$ there remains the classification of 2-bundles. This has been done by Atiyah and Rees [2]. They showed that for $c_1,c_2$ with $c_1c_2 \equiv 0 \ (2)$ and $c_1$ odd there exists exactly one 2-bundle with these $c_i$ as Chern classes. For $c_1$ even there are exactly two 2-bundles with these $c_i$ as Chern classes. These two bundles are distinguished by a certain mod 2 invariant $\alpha$ .

On $\mathbb{P}_4$ there remains the classification of bundles of rank 2 and 3 . Switzer [55], complementing the results of Atiyah and Rees, showed

$$\mathrm{Vect}^2_{\mathrm{top}}(\mathbb{P}_4) \simeq \{(c_1,c_2) \in \mathbb{Z} \times \mathbb{Z} : (S_4) \text{ is true}\} .$$

Switzer [55] recently pushed the classification of 2-bundles up to $\mathbb{P}_6$ . As a sample let us state his results on $\mathbb{P}_5$ because this is the first case where not all $c_1$ , $c_2$ satisfying the Schwarzenberger conditions arise as the Chern classes of a vector bundle of rank 2 . Set $\Delta = \dfrac{c_1^2 - 4c_2}{4}$ . Then for $c_1$ , $c_2$ satisfying $(S_5)$ there exists at least one 2-bundle with these $c_i$ as Chern classes if $c_1$ is odd or if $c_1$ is even and $\Delta^2(\Delta - 1) \equiv 0 \ (24)$ (if $c_1$ is even and $\Delta^2(\Delta - 1) \not\equiv 0 \ (24)$ there is <u>no</u> 2-bundle with these $c_i$ as Chern classes). For $c_2 \not\equiv c_1^2 \ (3)$ there exists exactly one 2-bundle and for $c_2 \equiv c_1^2 \ (3)$ there are exactly three 2-bundles.

## 2. Construction of holomorphic vector bundles on $\mathbb{P}_n$

In this section we will give some general procedures to construct holomorphic bundles. These will be applied to show that all topological vector bundles on $\mathbb{P}_n$ , $n \leq 3$ , admit an analytic structure.

Let us start by recalling that all line bundles on $\mathbb{P}_n$ are of the form $\mathcal{O}(k)$ , $k \in \mathbb{Z}$ . To convince the reader that the difficulties arise only if rank and dimension are bigger than 1 we include a short proof of the fact that all holomorphic vector bundles on $\mathbb{P}_1$ split into line bundles (see [19]).

THEOREM (Grothendieck [21]).- Any holomorphic vector bundle $E$ on $\mathbb{P}_1$ is of the form $E = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$ .

Proof. The proof is by induction on $r = \operatorname{rk} E$ . We may assume $r \geq 2$ . Choose $k \in \mathbb{Z}$ minimal with $H^0(E(k)) \neq 0$ ( $k$ exists by Serre's results on the cohomology of coherent sheaves on $\mathbb{P}_n$ ). We may assume $k = 0$ . Any nonzero $\sigma \in H^0(E)$ has zeroes only in codimension 2 . Hence a nonzero $\sigma \in H^0(E)$ gives a trivial line subbundle of $E$

$$(*) \qquad 0 \longrightarrow \mathcal{O} \xrightarrow{\sigma} E \longrightarrow F \longrightarrow 0 .$$

By induction we have $F \simeq \mathcal{O}(a_2) \oplus \ldots \oplus \mathcal{O}(a_r)$ . From (*) one gets the exact sequence

$$\longrightarrow H^0(E(-1)) \longrightarrow H^0(F(-1)) \longrightarrow H^1(\mathcal{O}(-1)) = 0 .$$

This shows $H^0(F(-1)) = 0$ and therefore $a_i \leq 0$ for all $i$ . The obstruction to split (*) lies in $H^1(F^*) = \bigoplus_i H^1(\mathcal{O}(-a_i)) = 0$ , since $a_i \leq 0$ for all $i$ .

Hence (*) splits and we get

$$E \simeq \mathcal{O} \oplus \mathcal{O}(a_2) \oplus \ldots \oplus \mathcal{O}(a_r) .$$

### 2.1. Vector bundles of rank $n-1$ on $\mathbb{P}_n$

Tango [58] constructed indecomposable holomorphic $(n-1)$-bundles on $\mathbb{P}_n$ for each $n \geq 3$ using the following generalization of a general position argument of Serre's.

PROPOSITION 2.1.- Let $E$ be a holomorphic vector bundle on $\mathbb{P}_n$ generated by global sections. If $c_i(E) = 0$ for some $i \leq r = \operatorname{rk} E$ then $E$ has a trivial subbundle of rank $r - i + 1$ .

COROLLARY 1.- For $n \geq 3$ there is an indecomposable $(n-1)$-bundle on $\mathbb{P}_n$ .

Proof. $\Omega^1(2)$ is generated by global sections. Let

$$\varphi : H^0(\mathbb{P}_n, \Omega^1(2)) \times \mathbb{P}_n \longrightarrow \Omega^1(2)$$

be the canonical surjection and put $E = (\ker \varphi)^*$ . One calculates $c_n(E) = 0$ . Hence $E$ has a trivial subbundle such that the quotient $F$ is of rank $n-1$ . The indecomposability of $F$ can be proved by inspecting its cohomology groups.

COROLLARY 2.- For $n$ odd there is a $(n-1)$-bundle $N$ on $\mathbb{P}_n$ with Chern class

$$c(N) = 1 + h^2 + h^4 + \dots + h^{n-1} .$$

Here $h = c_1(\mathcal{O}(1))$ is the canonical generator of $H^2(\mathbb{P}_n, \mathbb{Z})$ .

<u>Proof</u>. $\Omega^1(2)$ is generated by global sections and $c_n(\Omega^1(2)) = 0$ for $n$ odd. This shows the existence of a trivial line subbundle of $\Omega^1(2)$ . This gives a surjection

$$T(-1) \longrightarrow \mathcal{O}(1) .$$

Let $N$ be the kernel of this map. Then

$$\begin{aligned} c(N) &= c(T(-1))(1+h)^{-1} \\ &= (1-h)^{-1}(1+h)^{-1} \\ &= 1 + h^2 + h^4 + \dots + h^{n-1} . \end{aligned}$$

<u>Remarks</u>.- 1) $N$ is the Null-correlation bundle.

2) The tangent bundle $T_{\mathbb{P}_n}$ is indecomposable.

3) Maruyama [38] has shown that for each $r > n$ there exist indecomposable $r$-bundles on $\mathbb{P}_n$ if $n \geq 2$.

## 2.2. <u>Subvarieties of</u> $\mathbb{P}_n$ <u>of codimension</u> 2 <u>and holomorphic vector bundles of rank</u> 2

In this section we will explain the connection of locally complete intersection subvarieties of codimension 2 and holomorphic bundles of rank 2 . This correspondence essentially goes back to Serre [49] and has been rediscovered and reformulated many times [28], [9], [18], [23], [25]. Here we follow mainly Hartshorne's presentation.

Let $E$ be a holomorphic 2-bundle on $\mathbb{P}_n$ and suppose $E$ has a holomorphic section $\sigma$ vanishing in codimension 2 only (this can always be achieved by replacing $E$ by $E(k)$ with $k \in \mathbb{Z}$ minimal with respect to $H^0(E(k)) \neq 0$ ). Then $Y = \{\sigma = 0\}$ is of codimension 2 and locally a complete intersection. $Y$ is in general neither reduced nor irreducible. The Koszul complex of $\sigma$ is

$$0 \longrightarrow \det E^* \longrightarrow E^* \longrightarrow J_Y \longrightarrow 0 .$$

This implies

$$E^*|Y \simeq J/J^2 .$$

Hence $E$ is an extension of the normal bundle $N_{Y|\mathbb{P}_n} = (J/J^2)^*$ of $Y$ in $\mathbb{P}_n$ to the whole of $\mathbb{P}_n$ . Inserting

$$E^* \simeq E \otimes \det E^*$$

into the Koszul complex gives.

$$0 \longrightarrow \mathcal{O} \xrightarrow{\sigma} E \longrightarrow J_Y \otimes \det E \longrightarrow 0 .$$

It is clear that

$$c_2(E) = \text{dual of } Y .$$

Hence $\qquad c_2(E) = \deg Y .$

The interesting point is the reversal of this procedure. Take a locally complete intersection $Y \subset \mathbb{P}_n$ of codimension 2. We would like to construct a 2-bundle $E$ together with a $\sigma \in H^0(\mathbb{P}_n, E)$ giving $Y = \{\sigma = 0\}$. By what we have seen it is natural to try getting $E^*$ as extension of $J_Y$ by some line bundle.

PROPOSITION 2.2.1.- Let $Y$ be a locally complete intersection of codimension 2 in $\mathbb{P}_n$, $n \geq 3$. Assume that $\det N_{Y|\mathbb{P}_n} \simeq \mathcal{O}_Y(k)$. Then there exists a holomorphic 2-bundle $E$ on $\mathbb{P}_n$ with a holomorphic section $\sigma \in H^0(\mathbb{P}_n, E)$ such that

$$Y = \{\sigma = 0\} .$$

In particular $c_1(E) = k$, $c_2(E) = \deg Y$.

<u>Proof</u>. The extensions of $J_Y$ by $\mathcal{O}(-k)$ are classified by $\mathrm{Ext}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k))$. The exact sequence

$$0 \longrightarrow H^1(\mathbb{P}_n, \underline{\mathrm{Hom}}(J_Y, \mathcal{O}(-k))) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k)) \longrightarrow H^0(\mathbb{P}_n, \underline{\mathrm{Ext}}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k))) \longrightarrow$$
$$\longrightarrow H^2(\mathbb{P}_n, \underline{\mathrm{Hom}}(J_Y, \mathcal{O}(-k)))$$

gives for $n \geq 3$ an isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k)) \xrightarrow{\sim} H^0(\mathbb{P}_n, \underline{\mathrm{Ext}}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k)))$$

since $\underline{\mathrm{Hom}}(J_Y, \mathcal{O}(-k)) = \mathcal{O}(-k)$ and $H^i(\mathbb{P}_n, \mathcal{O}(-k)) = 0$ for $1 \leq i \leq n-1$ and all $k \in \mathbb{Z}$. Using

$$\begin{aligned} \underline{\mathrm{Ext}}^1(J_Y, \mathcal{O}(-k)) &\xrightarrow{\sim} \underline{\mathrm{Ext}}^2(\mathcal{O}_Y, \mathcal{O}(-k)) \\ &\simeq \underline{\mathrm{Ext}}^2(\mathcal{O}_Y, \mathcal{O}(-n-1)) \otimes \mathcal{O}(-k+n+1) \\ &\simeq \omega_Y \otimes \mathcal{O}_Y(-k+n+1) \qquad \text{see [22]} \\ &\simeq \mathcal{O}_Y(-n-1) \otimes \det N \otimes \mathcal{O}_Y(-k+n+1) \\ &\simeq \mathcal{O}_Y , \end{aligned}$$

one finally gets an isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k)) \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y) .$$

The canonical section $\xi$ in $H^0(Y, \mathcal{O}_Y)$ therefore gives an extension

$$0 \longrightarrow \mathcal{O}(-k) \longrightarrow \mathfrak{F} \longrightarrow J_Y \longrightarrow 0$$

of $J_Y$ by $\mathcal{O}(-k)$ through a coherent sheaf. Since $\xi$ locally generates each stalk of $\underline{\mathrm{Ext}}^1_{\mathcal{O}}(J_Y, \mathcal{O}(-k))$ it follows from [49] that $\mathfrak{F}$ is locally free. $E := \mathfrak{F}^*$ is the desired bundle.

Remarks.- 1) Barth, Larsen and Ogus [36], [45] have shown that $\mathrm{Pic}(\mathbb{P}_n) \xrightarrow{\sim} \mathrm{Pic}(Y)$ for $n \geq 6$ and nonsingular $Y$. Thus each nonsingular submanifold $Y \subset \mathbb{P}_n$, $n \geq 6$, of codimension 2 gives a holomorphic vector bundle of rank 2 on $\mathbb{P}_n$.

2) The above construction does not work without further considerations on $\mathbb{P}_2$. But if $k \leq 2$ the group $H^2(\mathbb{P}_2, \mathcal{O}(-k))$ still vanishes and the proposition 2.2.1 remains valid in that case. For arbitrary $k$ see [51], [18].

Let us apply this proposition to produce many holomorphic 2-bundles on $\mathbb{P}_2$ and $\mathbb{P}_3$.

Examples.

1) Take $Y$ to be the union of $d$ simple points in $\mathbb{P}_2$. Then $\det N_{Y|\mathbb{P}_2} = \mathcal{O}_Y(2)$ and we get a holomorphic 2-bundle $E$ on $\mathbb{P}_2$ with $c_1 = 2$ and $c_2 = d$. This shows the existence of 2-bundles with $c_1 = 0$, $c_2 \geq 0$.

2) Take $Y$ to be the union of $d$ disjoint lines in $\mathbb{P}_3$. Then $\det N_{Y|\mathbb{P}_3} = \mathcal{O}_Y(2)$ and we get a 2-bundle with $c_1 = 2$, $c_2 = d$. Normalizing gives $c_1 = 0$, $c_2 \geq 0$ arbitrary.

3) Take $Y$ to be the union of $r$ disjoint nonsingular conics in $\mathbb{P}_3$. Then $\det N_{Y|\mathbb{P}_3} \simeq \mathcal{O}_Y(3)$ and we get a 2-bundle with $c_1 = 3$, $c_2 = 2r$. This shows the existence of 2-bundles with $c_1 = -1$, $c_2 \geq 0$ even.

4) Horrocks [28]

Let $p \geq 2$ be an integer and $m_1, \dots, m_r \in \mathbb{Z}$ with $0 < m_i < p$. Choose $r$ disjoint lines $L_i \subset \mathbb{P}_3$ and give them a nilpotent structure through $J_{L_i} = (x^{m_i}, y^{p-m_i})$. Here $x$, $y$ are equations for $L_i$. Take $Y$ to be the union of these fattened lines. Then $\det N_{Y|\mathbb{P}_3} \simeq \mathcal{O}_Y(p)$ and we get a 2-bundle with

$$c_1 = p \ , \ c_2 = \sum_{i=1}^{r} m_i(p - m_i) \ .$$

A short calculation shows that all $c_1$, $c_2 \in \mathbb{Z}$ with $c_1 c_2 \equiv 0 \ (2)$ are of this form (modulo twisting). Therefore all $c_1$, $c_2$ with $c_1 c_2 \equiv 0 \ (2)$ are the Chern classes of a holomorphic 2-bundle on $\mathbb{P}_3$.

Atiyah and Rees [2] showed that for a holomorphic 2-bundle $E$ with even $c_1$ the $\alpha$-invariant can be given by

$$\alpha(E) = h^0(E_{norm}(-2)) + h^2(E_{norm}(-2)) \quad \mathrm{mod.}\ 2 \ .$$

Here $E_{norm}$ denotes $E(-c_1/2)$ for $c_1$ even and $E((-(c_1+1))/2)$ for $c_1$ odd. Note that $h^2(E_{norm}(-2)) = h^1(E_{norm}(-2))$ by Serre-duality.

It takes some arithmetic [2] to show that by the above Horrocks construction one can achieve both values of $\alpha$ . This implies

$$\mathrm{Vect}^2_{\mathrm{hol}}(\mathbb{P}_3) \longrightarrow \mathrm{Vect}^2_{\mathrm{top}}(\mathbb{P}_3)$$

is surjective.

5) Take $Y$ to be the disjoint union of a plane nonsingular cubic curve and a nonsingular elliptic space curve of degree $d$ . $Y$ gives a 2-bundle on $\mathbb{P}_3$ with Chern classes $c_1 = 4$ , $c_2 = d + 3$ . A short calculation shows $\alpha = 1$ . Normalizing one gets the invariants

$$c_1 = 0 \ , \quad c_2 = d + 1 \ , \quad \alpha = 1 \ .$$

Note that in Example 2) one has $\alpha = 0$ .

6) Horrocks, Mumford [32]
These authors show the existence of a 2-bundle on $\mathbb{P}_4$ which comes from an abelian surface $Y \subset \mathbb{P}_4$ . Suppose you have shown the embedding of an abelian surface $Y$ into $\mathbb{P}_4$ . The exact sequence

$$0 \longrightarrow \mathcal{O}^2_Y \longrightarrow T_{\mathbb{P}_4}|Y \longrightarrow N_{Y|\mathbb{P}_4} \longrightarrow 0$$

gives

$$\det N_{Y|\mathbb{P}_4} = \mathcal{O}_Y(5) \quad \text{and} \quad \deg Y = 10 \ .$$

Hence we get a 2-bundle with $c_1 = 5$ , $c_2 = 10$ . This is essentially the only known indecomposable 2-bundle on $\mathbb{P}_4$ .

<u>Problem 1</u>. Are there any holomorphic 2-bundles on $\mathbb{P}_n$ , $n \geq 5$ , which do not split into line bundles ?

Let us close this section by some remarks on the connection of 3-bundles on $\mathbb{P}_n$ and locally complete intersections $Y \subset \mathbb{P}_n$ of codimension 2 .

PROPOSITION 2.2.2 (Van de Ven, Vogelaar [64]).- Let $Y$ be a locally complete intersection of codimension 2 in $\mathbb{P}_n$ , $n \geq 3$ . Suppose there is a holomorphic line bundle $L$ on $Y$ together with holomorphic sections $\sigma_1$ , $\sigma_2 \in H^0(Y,L)$ such that $\{\sigma_1 = 0\} \cap \{\sigma_2 = 0\} = \emptyset$ . If furthermore $\det N_{Y|\mathbb{P}_n} \otimes L^* \simeq \mathcal{O}_Y(k)$ then there is a holomorphic 3-bundle $E$ on $\mathbb{P}_n$ with

$$c_1(E) = k \ , \quad c_2(E) = \deg Y \ , \quad c_3(E) = \deg(\sigma_i = 0) \ .$$

<u>Remark</u>.- One gets $E$ as an extension

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow E \longrightarrow J_Y(k) \longrightarrow 0 \ .$$

As an application it is shown that all $c_1$ , $c_2$ , $c_3 \in \mathbb{Z}$ with $c_3 \equiv c_1c_2$ (2) occur as the Chern classes of a holomorphic 3-bundle on $\mathbb{P}_3$ . Combining with 4) one obtains the surjectivity of the map

$$\mathrm{Vect}^r_{hol}(\mathbb{P}_3) \longrightarrow \mathrm{Vect}^r_{top}(\mathbb{P}_3)$$

for all $r$ .

## 2.3. Monads

The description of holomorphic vector bundles on $\mathbb{P}_n$ by monads is due to Horrocks [27], [29], [31] and was recently put into a general frame by Beilinson [11]. In specific cases they have been studied by Barth, Hulek, Drinfeld and Manin [5], [8], [33], [12].

DEFINITION 2.3.1.- A monad is a complex of holomorphic vector bundles

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

which is exact except possibly at $B$ .

Remark.- $E := \ker b / \mathrm{im}\, a$ is a holomorphic vector bundle with $\mathrm{rk}\, E = \mathrm{rk}\, B - \mathrm{rk}\, A - \mathrm{rk}\, C$ and Chern class

$$c(E) = c(B)\, c(A)^{-1}\, c(C)^{-1} .$$

The following version of the Beilinson construction I learned from Verdier.

THEOREM 2.3.2 (Beilinson [11]).- Let $E$ be a holomorphic vector bundle on $\mathbb{P}_n$ . There exists a spectral sequence with

$$E_1^{pq} = H^q(\mathbb{P}_n, E \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(p) ,$$

$$E_\infty^{pq} = 0 \quad \text{for } p+q \neq 0$$

and a filtration of $E$ whose associated graded module is $\bigoplus_p E_\infty^{p,-p}$ .

Proof. Let $\mathbb{P}_n = \mathbb{P}(V)$ , $V$ a complex vector space of dimension $n+1$ . Consider the canonical exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathbb{P}(V) \times V \longrightarrow Q \longrightarrow 0 .$$

Here $Q = T(-1)$ and $H^0(\mathbb{P}_n, Q) = V$ . On $\mathbb{P}_n \times \mathbb{P}_n$ we look at $Q \boxtimes \mathcal{O}(1) := \mathrm{pr}_1^* Q \otimes \mathrm{pr}_2^* \mathcal{O}(1)$ . There is a canonical section $\sigma \in H^0(\mathbb{P}_n \times \mathbb{P}_n, Q \boxtimes \mathcal{O}(1)) = V \otimes V^*$ corresponding to $\mathrm{id}_V$ . This section vanishes precisely and transversally at the diagonal $\Delta$ of $\mathbb{P}_n \times \mathbb{P}_n$ . Hence we have the Koszul complex

$$0 \longrightarrow \Omega^n(n) \boxtimes \mathcal{O}(-n) \longrightarrow \ldots \longrightarrow \Omega^1(1) \boxtimes \mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_n \times \mathbb{P}_n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 .$$

This gives

$$\mathbb{R}^i\, \mathrm{pr}_{2*}(C^\bullet \otimes \mathrm{pr}_1^* E) = \begin{cases} 0 & \text{for } i \neq 0 \\ E & \text{for } i = 0 , \end{cases}$$

where $C^\nu = \Omega^{-\nu}(-\nu) \boxtimes \mathcal{O}(\nu)$ for $\nu \leq 0$ and $C^\nu = 0$ for $\nu > 0$ . The spectral sequence for the hypercohomology of $\mathrm{pr}_{2*}$ now gives the result.

Remark.- Interchanging $pr_1$ with $pr_2$ in the above proof gives a spectral sequence with

$$E_1^{pq} = H^q(\mathbb{P}_n , E(p)) \otimes \Omega^{-p}(-p)$$

satisfying the same properties as the one in the theorem.

Applications (compare [8] and [31] for a different approach)

1) Let $E$ be a holomorphic $r$-bundle on $\mathbb{P}_2$ with $H^o(\mathbb{P}_2 , E(-1)) = H^o(\mathbb{P}_2 , E^*(-1)) = 0$. Then $E$ is the cohomology of a monad

$$H^1(E(-2)) \otimes \mathcal{O}(-1) \longrightarrow H^1(E \otimes \Omega^1) \otimes \mathcal{O} \longrightarrow H^1(E(-1)) \otimes \mathcal{O}(1) .$$

If $c_1(E) = 0$, then $h^1(E(-2)) = h^1(E(-1)) = c_2(E)$ by Riemann-Roch. In case $E$ is orthogonal or symplectic (i.e. we have a nondegenerate symmetric or skew bilinear form on $E$ ), one can give the bundles in terms of linear algebra. Let $H$ and $K$ be complex vector spaces of dimension $n$ and $2n + r$. $K$ should be equipped with an orthogonal or symplectic nondegenerate form. $GL(H) \times O(K)$ acts on the linear mappings $L(H,K)$ by

$$(f,g) . \varphi = g \varphi f^{-1} .$$

Using the above description of bundles by monads it is easy to show that the isomorphism classes of orthogonal (symplectic) holomorphic $r$-bundles on $\mathbb{P}_2 = \mathbb{P}(V)$ with $H^o(\mathbb{P}_2 , E(-1)) = 0$ and $c_2(E) = n$ correspond one to one to the orbits of $GL(H) \times O(K)$ on the set of all linear maps $\alpha : V \longrightarrow L(H,K)$ with

(i) $\alpha(v)$ is injective for all $v \neq 0$

(ii) $\alpha(v)(H)$ is for all $v \in V$ a totally isotropic subspace of $K$.

Remark.- $H^o(E) = 0$ is equivalent to the surjectivity of the map $H \otimes V \longrightarrow K$ induced by $\alpha$.

2) Let $E$ be a holomorphic $r$-bundle on $\mathbb{P}_2 = \mathbb{P}(V)$ with $H^o(\mathbb{P}_2 , E) = H^o(\mathbb{P}_2 , E^*(-1)) = 0$. Then $E$ comes from a monad

$$H^1(E(-2)) \otimes \mathcal{O}(-1) \xrightarrow{a} H^1(E(-1)) \otimes \Omega^1(1) \xrightarrow{b} H^1(E) \otimes \mathcal{O} .$$

One can make explicit the maps $a$ and $b$ [37] :

for $z \in V^* = \Gamma(\mathbb{P}_2 , \mathcal{O}(1))$ denote the maps

$$H^1(E(-2)) \longrightarrow H^1(E(-1)) \quad \text{and} \quad H^1(E(-1)) \longrightarrow H^1(E)$$

given by the multiplication with $z$ by $\alpha(z)$ and $\beta(z)$. At the point $x \in \mathbb{P}_2$ the map $a$ is given by

$$(z' \wedge z'') \otimes h \longrightarrow z'' \otimes \alpha(z')h - z' \otimes \alpha(z'')h .$$

Here $z' , z'' \in \Omega^1(1)_x$ (note that $\mathcal{O}(-1) = \det \Omega^1(1)$ ). The map $b$ is given at $x \in \mathbb{P}_2$ by

$$z \otimes k \longmapsto \beta(z)k .$$

The injectivity of $a$ is equivalent to :

for each nonzero $h \in H^1(E(-2))$ the map $z \longmapsto \alpha(z)h$ from $V^*$ to $H^1(E(-1))$ has rank at least $2$ .

Now let $E$ be of rank $2$ and $c_1(E) = -1$ . Serre-duality gives a symmetric nondegenerate form on $H^1(E(-1))$ and an isomorphism $H^1(E(-2))^* \simeq H^1(E)$ . In this case $\beta(z) = \alpha(z)^t$ , $z \in V^*$ . From this one can deduce as in 1) a bijective correspondence (see [37]) between the isomorphism classes of holomorphic 2-bundles $E$ on $\mathbb{P}_2$ with $c_1(E) = -1$ , $H^0(E) = 0$ , $c_2(E) = n$ and the orbits of $GL(H) \times O(K)$ on the set of all linear maps $\alpha : V^* \longrightarrow L(H,K)$ satisfying

(i) $\alpha(z')^t\alpha(z'') = \alpha(z'')^t\alpha(z')$ for $z'$ , $z'' \in V^*$

(ii) the map $z \longmapsto \alpha(z)h$ from $V^*$ to $K$ is for all nonzero $h \in H$ of rank at least $2$ .

Here $H$ and $K$ are complex vector spaces of dimension $n - 1$ and $n$ . Furthermore $K$ is equipped with a nondegenerate symmetric bilinear form.

The case $c_1(E) = 0$ is different. Here Serre-duality gives

$$H^1(\mathbb{P}_2, E(-2))^* \simeq H^1(\mathbb{P}_2, E(-1))$$

and for $z \in V^*$ the map

$$\alpha(z) : H^1(E(-2)) \longrightarrow H^1(E(-2))^*$$

is symmetric. It takes some work (see [5], [37]) to show that the isomorphism classes of 2-bundles $E$ with $c_1(E) = 0$ , $H^0(E) = 0$ and $c_2(E) = n$ are in bijective correspondence with the orbits of $GL(H)$ acting on the set of all linear maps $\alpha : V^* \longrightarrow S^2H^*$ satisfying

(i) the map $z \longmapsto \alpha(z)h$ from $V^*$ to $H^*$ is for all nonzero $h \in H$ of rank at least 2

(ii) there is a base $(z_0, z_1, z_2)$ of $V^*$ such that $\alpha(z_0)$ is invertible and the map $H \longrightarrow H^*$ given by $\alpha(z_1)\alpha(z_0)^{-1}\alpha(z_2) - \alpha(z_2)\alpha(z_0)^{-1}\alpha(z_1)$ is of rank $2$ .

Here $H$ is a complex vector space of dimension $n$ $(\geq 2)$ . Monads of this type have been used by Barth [5] to classify stable 2-bundles on $\mathbb{P}_2$ with $c_1 = 0$ .

3) Let $E$ be a holomorphic r-bundle on $\mathbb{P}_3$ with $H^0(E(-1)) = 0$ , $H^1(E(-2)) = 0$ ("instanton condition"), $E \simeq E^*$ and $c_2(E) = n$ . Then $E$ comes from a monad

$$H^1(E(-3) \otimes T) \otimes \mathcal{O}(-1) \longrightarrow H^1(E \otimes \Omega^1) \otimes \mathcal{O} \longrightarrow H^1(E(-1)) \otimes \mathcal{O}(1) .$$

In particular this shows that $H^1(\mathbb{P}_3, E(-\nu)) = 0$ for all $\nu \geq 2$ . Using the notation of the first application one gets in the some way a bijection between isomorphism classes of orthogonal (symplectic) r-bundles on $\mathbb{P}_3 = \mathbb{P}(V)$ satisfying the conditions $H^o(E(-1)) = 0$ , $H^1(E(-2)) = 0$ , $c_2(E) = n$ and the orbits of $GL(H) \times O(K)$ acting on the linear maps $\alpha : V \longrightarrow L(H,K)$ with

(i) $\alpha(v) : H \longrightarrow K$ is injective for all $v \neq 0$

(ii) $\alpha(v)(H)$ is for all $v \in V$ a totally isotropic subspace of $K$ .

Remark.- The condition $H^o(P_3,E) = 0$ is equivalent to the surjectivity of the map $H \otimes V \longrightarrow K$ induced by $\alpha$ .

Monads of this type have been used to describe instantons [1], [22].

4) Let $E$ be a holomorphic r-bundle on $\mathbb{P}_3$ with $H^o(E) = H^1(E(-2)) = 0$ and $E \simeq E^*$ . Then $E$ comes from a monad

$$H^2(E(-3)) \otimes \mathcal{O}(-1) \longrightarrow H^1(E(-1)) \otimes \Omega^1(1) \longrightarrow H^1(E) \otimes \mathcal{O} .$$

## 3. Stable bundles

DEFINITION 3.1.- A holomorphic r-bundle $E$ on $\mathbb{P}_n$ is said to be stable if for all proper coherent subsheaves $\mathfrak{F}$ of $E$ of rank $s$ we have the inequality

$$\frac{c_1(\mathfrak{F})}{s} < \frac{c_1(E)}{r} .$$

If we have only " $\leq$ " instead of " $<$ " then $E$ is called semi-stable. A bundle which is not semi-stable is usually called unstable.

Remarks.- 1) This definition is due to Mumford and Takemoto [56]. Recently Gieseker [17] suggested a slightly different definition. He calls $E$ stable if

$$\frac{p_{\mathfrak{F}}(m)}{s} < \frac{p_E(m)}{r}$$

for $m >> 0$ . Here $p_{\mathfrak{F}}(m) = \chi(\mathbb{P}_n, \mathfrak{F}(m))$ is the Hilbert polynomial of $\mathfrak{F}$ . With this definition one generally gets more stable but fewer semi-stable bundles than before.

2) It is straightforward [56] that stable bundles $E$ are always simple, i.e. $H^o(E^* \otimes E) = \mathbb{C}$ , and therefore indecomposable.

3) $T_{\mathbb{P}_n}$ is stable [35].

PROPOSITION 3.2 [4].- The stable 2-bundles on $\mathbb{P}_n$ are precisely the simple ones.

Proof. Assume $E$ to be simple. We can choose $k \in \mathbb{Z}$ minimal with $H^o(E(k)) \neq 0$ . Take a nonzero $\sigma \in H^o(E(k))$ and put $Y = \{\sigma = 0\}$ . $Y$ is of codimension $2$ and we get an exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\sigma} E(k) \longrightarrow J_Y(c_1(E) + 2k) \longrightarrow 0 .$$

If $c_1 + 2k \leq 0$ we get a "non-trivial" endomorphism of $E(k)$ by composing

$$E(k) \longrightarrow J_Y(c_1 + 2k) \hookrightarrow \mathcal{O}(c_1 + 2k) \hookrightarrow \mathcal{O} \xrightarrow{\sigma} E(k) .$$

Hence $c_1 + 2k > 0$ .

Now let $\mathcal{O}(\ell)$ be a subsheaf of $E$ . By minimality of $k$ we get $-\ell \geq k$ and therefore $\ell < c_1/2$ . This shows the stability of $E$ .

Remark.- It is easy to see that a 2-bundle $E$ on $\mathbb{P}_n$ is stable if and only if $H^0(\mathbb{P}_n, E_{norm}) = 0$ . For 3-bundles with $c_1 = 0$ stability is equivalent to $H^0(E) = H^0(E^*) = 0$ .

Problem 2. Give a similar criterion of stability for bundles of higher rank.

Schwarzenberger has shown [52] that Riemann-Roch implies that the Chern classes of a stable 2-bundle on $\mathbb{P}_2$ have to satisfy $c_1^2 - 4c_2 < 0$ (for a semi-stable 2-bundle one has $c_1^2 - 4c_2 \leq 0$ ). In fact $c_1^2 - 4c_2 = -4$ cannot occur for a stable 2-bundle on $\mathbb{P}_2$ [38].

It is a general fact, proved by Maruyama [43], that the restriction of a semi-stable $r$-bundle on $\mathbb{P}_n$ , $r < n$ , to a general byperplane is semi-stable again (Barth [4] showed the same to be true for stable 2-bundles on $\mathbb{P}_n$ , $n \geq 3$ , with the exception of the Null-correlation bundle). Hence for a semi-stable 2-bundle $E$ on $\mathbb{P}_n$ we have

$$c_1^2 - 4c_2 \leq 0$$

and for stable 2-bundles one necessarily has

$$c_1^2 - 4c_2 < 0 .$$

Problem 3. Determine similar necessary conditions for stable (semi-stable) holomorphic bundles of higher rank.

We show next how stability of a 2-bundle $E$ on $\mathbb{P}_n$ coming from a locally complete intersection $Y \subset \mathbb{P}_n$ of codimension 2 is reflected by $Y$ .

Let $Y \subset \mathbb{P}_n$ , $n \geq 2$ , be a locally complete intersection of codimension 2 and $\det N_{Y|\mathbb{P}_n} = \mathcal{O}_Y(k)$ . Then we can find an extension $E$ of $N_{Y|\mathbb{P}_n}$ as in 2.2.1.

PROPOSITION 3.3 (see [25]).- $E$ is stable if and only if $k > 0$ and $Y$ is not contained in any hypersurface of degree $d \leq k/2$ .

Proof. We have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow J_Y(k) \longrightarrow 0 .$$

If $E$ is stable then $c_1(E) = k > 0$ .

Assume $k$ to be even. The sequence

$$0 \to \mathcal{O}(-k/2) \to E_{norm} \to J_Y(k/2) \to 0$$

gives

$$H^o(E_{norm}) \xrightarrow{\sim} H^o(J_Y(k/2)) .$$

Stability of $E$ is equivalent to $H^o(E_{norm}) = 0$ . Therefore $H^o(J_Y(k/2)) = 0$ , which is equivalent to the fact that $Y$ is not contained in any hypersurface of degree $\leq k/2$ . Assume on the other hand $k > 0$ and $H^o(J_Y(k/2)) = 0$ . This gives $H^o(E_{norm}) = 0$ which is the stability of $E$ . The case $c_1$ odd is treated in a similar way.

Using this criterion we re-examine the examples of 2.1.

Examples.- 1) If $E$ comes from $d$ simple points in $\mathbb{P}_2$ , $E$ is stable if and only if the points do not all lie on a line. This shows the existence of stable 2-bundles on $\mathbb{P}_2$ with $c_1 = 0$ , $c_2 \geq 2$ .

2) If $E$ comes from $d$ disjoint lines in $\mathbb{P}_3$ then $E$ is stable if and only if these lines are not contained in a plane. This is the case for $d \geq 2$ . This gives stable 2-bundles on $\mathbb{P}_3$ with $c_1 = 0$ , $c_2 \geq 1$ , $\alpha = 0$ .

3) For bundles coming from disjoint nonsingular conics we have the some result as in 2). One gets stable bundles with $c_1 = -1$ , $c_2 \geq 2$ even.

4) If $E$ comes from a plane cubic and a disjoint elliptic curve of degree $d$ then $E$ is stable if $d \geq 4$ . This gives stable bundles on $\mathbb{P}_3$ with

$$c_1 = 0 , \quad c_2 \geq 5 , \quad \alpha = 1 .$$

5) The 2-bundle of Horrocks and Mumford on $\mathbb{P}_4$ is stable since an abelian surface $Y$ can neither lie in some $\mathbb{P}_3$ (because of $\pi_1(Y) \neq 0$ ) nor in some hyperquadric $Q$ (consider normal bundles).

Here we wish to draw the attention of the reader to an example of a stable 3-bundle on $\mathbb{P}_5$ constructed by Horrocks [30] using representation theory.

Let us close this section by giving the following

Conjecture.- Each 2-bundle on $\mathbb{P}_n$ , $n \geq 5$ , which is not stable is a direct sum of line bundles.

In [20] a "proof" for this was given even for $n \geq 4$ . Unfortunately there is a gap in that paper.

The conjecture has nice consequences [20], [50] :

1) Each topological 2-bundle on $\mathbb{P}_n$ , $n \geq 5$ , which is not the direct sum of two line bundles and satisfying $c_1^2 - 4c_2 \geq 0$ cannot have an analytic structure. By [46],

[54], [55] there are many topological 2-bundles with $c_1^2 - 4c_2 \geq 0$ and which do not split.

2) Each holomorphic 2-bundle on $\mathbb{P}_5$ which can be extended topologically to $\mathbb{P}_n$, $n$ arbitrarily large, is the direct sum of line bundles.

This sharpened the theorem of Barth and Van de Ven [9] on Babylonian vector bundles (see also [48], [61]).

3) Each nonsingular submanifold $Y \subset \mathbb{P}_n$ of codimension 2 is a complete intersection if $n \geq 6$ and $n \geq \frac{1}{3}\sqrt{\deg(Y)} + 1$. This would improve some results in [3].

One can even show, for example, that a nonsingular 4-dimensional submanifold $Y \subset \mathbb{P}_6$ is a complete intersection if $\deg Y \leq 514$.

4) Furthermore one could improve the results of Barth and Van de Ven in [10].

## 4. Moduli of stable bundles

So far we commented the points I and II of the introduction. To deal with III one would like to introduce on the set of isomorphism classes of stable holomorphic r-bundles on $\mathbb{P}_n$ with fixed topological type a "good" analytic structure.

Consider the functor

$$\Sigma(c_1,\dots,c_r) : \underline{\mathrm{An}} \longrightarrow \underline{\mathrm{Ens}}$$

from analytic spaces to sets given by

$$\Sigma(c_1,\dots,c_r)(S) := \{\text{bundles } E \text{ on } \mathbb{P}_n \times S \text{ of fixed rank with } E(s) \text{ stable and } c_i(E(s)) = c_i \text{ for } i = 1,\dots,r \text{ and } s \in S\}/\sim .$$

Here $E_2 \sim E_1$ if $E_2 \simeq pr_S^*(L) \otimes E_1$ for a holomorphic line bundle $L$ on $S$.

$\Sigma$ is contravariant in an obvious way.

DEFINITION 4.1.- $M = M(c_1,\dots,c_r) \in \underline{\mathrm{An}}$ is a coarse moduli space for $\Sigma(c_1,\dots,c_r)$ if there is a morphism of functors

$$\Sigma \longrightarrow \mathrm{Hom}(-, M)$$

with

$$\Sigma(pt) \xrightarrow{\sim} M .$$

Furthermore $M$ should be minimal with respect to these properties, i.e. if $N$ is another analytic space satisfying the above then there should be a unique morphism $M \longrightarrow N$ making the diagram

$$\begin{array}{ccc} \Sigma & \longrightarrow & \mathrm{Hom}(-, M) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}(-, N) & \end{array}$$

commutative.

If a coarse moduli space exists one has put in a functorial way an analytic structure onto the stable bundles on $\mathbb{P}_n$ with fixed Chern classes and fixed rank.

If $M$ represents $\Sigma$ then $M$ is said to be a fine moduli space. This is equivalent to the existence of a universal family over $M \times \mathbb{P}_n$ .

It seems much easier to construct a coarse moduli space $M$ in the analytic category than to do it in the algebraic category. In the algebraic category the existence was proved by Maruyama [39], [40], [41] by using Mumford's geometric invariant approach. Maruyama could not show that $M$ is always of finite type. For $n = 2$ and arbitrary rank this was shown to be true by Gieseker [17]. For arbitrary $n$ and rank $\leq 4$ it was verified recently by Maruyama [43].

These authors also study compactifications of $M$ and it turns out that one has not only to admit semi-stable bundles but also semi-stable torsion free coherent sheaves.

Our object here is only to mention some specific results for the moduli spaces $M$ of bundles over $\mathbb{P}_2$ and $\mathbb{P}_3$ .

By deformation theory the Zariski tangent space of $M$ at $m$ is $H^1(\mathrm{End}(E))$ if $E$ is the bundle corresponding to $m$ . If $H^2(\mathrm{End}(E)) = 0$ then $M$ is smooth at $m$ . In particular the moduli spaces of stable bundles on $\mathbb{P}_2$ are nonsingular. By Riemann-Roch we get

$$\dim M_{\mathbb{P}_2}(c_1, c_2, r) = (1 - r)c_1^2 + 2rc_2 - r^2 + 1 .$$

For rank 2 we get

$$\dim M_{\mathbb{P}_2}(c_1, c_2) = 4c_2 - c_1^2 - 3 .$$

Let us summarize the properties of $M_{\mathbb{P}_2}(c_1, c_2)$ .

THEOREM 4.2.- $M_{\mathbb{P}_2}(c_1, c_2)$ is a smooth, quasi-projective manifold of dimension $4c_2 - c_1^2 - 3$ . $M$ is connected and rational. $M$ is a fine moduli space if and only if $4c_2 - c_1^2 \not\equiv 0 \ (8)$ .

Remarks.- The rationality and connectedness was proved by Barth [5] for $c_1$ even and by Hulek [33] for $c_1$ odd using monads. Maruyama [42] showed that $M$ is connected, unirational (and in some cases rational) and that $M$ is a fine moduli space if $4c_2 - c_1^2 \not\equiv 0$ (8). Le Potier [37] proved the nonexistence of a universal family for $4c_2 - c_1^2 \equiv 0$ (8) using monads. He showed that in this case there are topological obstructions to the existence of the universal family. In doing this he calculated

$$\pi_1(M(0,c_2)) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{for } c_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_2(M(0,c_2)) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } c_2 = 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & \text{for } c_2 > 2\ ,\ c_2 \text{ even} \\ \mathbb{Z} & \text{for } c_2 \text{ odd .} \end{cases}$$

To conclude this section we give the simplest examples of moduli spaces on $\mathbb{P}_2$ and $\mathbb{P}_3$ which can be deduced quickly from the description of bundles by monads.

<u>Examples</u>.- 1) $M_{\mathbb{P}_2}(-1,1) = \{\Omega^1(1)\}$ .

This follows immediately from Application 2 of 2.3.

2) $M_{\mathbb{P}_2}(-1,2) = S^2\mathbb{P}_2 \setminus \Delta$ (see [37]).

The application 2 of 2.3 shows that

$$M(-1,2) = \{\alpha : V^* \longrightarrow \mathbb{C}^2 \text{ linear and surjective}\} \text{ modulo the action of } \mathbb{C}^* \times O(\mathbb{C}^2)\ .$$

Here $\mathbb{C}^2$ is equipped with a nondegenerate symmetric bilinear form. A linear algebraic calculation identifies the righthand side to $(\mathbb{P}(V) \times \mathbb{P}(V)) \setminus \Delta$ modulo $\mathbb{Z}/2\mathbb{Z}$ . This finally gives $M(-1,2) \simeq S^2\mathbb{P}_2 \setminus \Delta$ .

3) $M_{\mathbb{P}_2}(0,2) = \{\text{nonsingular conics in } \mathbb{P}_2\}$ , [5].

By application 2 of 2.3 one has

$$M(0,2) = \operatorname{Isom}(V^*, S^2H^*) / GL(H)\ .$$

$H$ is of dimension $2$ . Let $C := \{q \in S^2(H^*) : \det q = 0\}$ ; for $\alpha \in \operatorname{Isom}(V^*, S^2H^*)$ the inverse image $\alpha^{-1}(C)$ will be a nonsingular conic. $\alpha'$ , $\alpha \in \operatorname{Isom}(V^*, S^2H^*)$ with $\alpha^{-1}(C) = \alpha'^{-1}(C)$ differ by an automorphism $\gamma \in \operatorname{Aut}(S^2H^*)$ with $\gamma(C) = C$ . But these $\gamma$'s come from automorphisms of $H$ . This proves our claim.

4) $M_{\mathbb{P}_3}(0,1) = PGL(3,\mathbb{C})/Sp(2,\mathbb{C})$ (see [4]).

By application 3 of 2.3 we have

$$M(0,1) = \operatorname{Isom}(\mathbb{C}^4,\mathbb{C}^4)/\mathbb{C}^* \times Sp(2,\mathbb{C}) = PGL(3,\mathbb{C})/Sp(2,\mathbb{C})\ .$$

In particular $PGL(3)$ operates transitively on $M(0,1)$ . The Null-correlation bundle belongs to $M(0,1)$ .

Hartshorne [25] gives a description of $M_{\mathbb{P}_3}(0,2)$ . In particular $M_{\mathbb{P}_3}(0,2)$ is still connected. For $c_2 \geq 3$ the space $M_{\mathbb{P}_3}(0,c_2)$ will be divided into 2 components by the $\alpha$-invariant. The following example due to Barth and Hulek [8] (see also [25]) shows that $M_{\mathbb{P}_3}(0,c_2)$ is reducible if $c_2$ is odd and at least 5.

Consider the monad

$$\mathcal{O}(-m-1) \xrightarrow{a} \mathcal{O}(m) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-m) \xrightarrow{b} \mathcal{O}(m+1)$$

on $\mathbb{P}_3$. The map $a \in H^0(\mathcal{O}(2m+1) \oplus \mathcal{O}(m+1) \oplus \mathcal{O}(m+1) \oplus \mathcal{O}(1))$ has to be chosen such that the $a_i$ have no common zero. On $\mathcal{O}(m) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-m)$ take the sympletic form

$$q = \begin{pmatrix} & 0 & & 1 \\ & & 1 & \\ & -1 & & 0 \\ -1 & & & \end{pmatrix}$$

and put $b = a^t$.

The stable 2-bundles defined by these monads have Chern classes $c_1 = 0$, $c_2 = 2m + 1$.

This family of bundles depends effectively on

$$\#a\text{'s} - \dim(\mathbb{C}^* \times O(q))$$

parameters (compare 2.3).

One checks that $\dim O(q) = 4 + 2\binom{m+3}{3} + \binom{2m+3}{3}$ and thus gets that the family depends on

$$3m^2 + 10m + 8$$

parameters.

For $m \geq 2$ this number is bigger than $16m + 5 = 8c_2 - 3$ which is the dimension of the Zariski-open smooth part of bundles $E$ with $H^2(\mathbb{P}_3, \mathrm{End}(E)) = 0$.

<u>Questions</u>.- 1) Are $M_{\mathbb{P}_3}(0,3)$ and $M_{\mathbb{P}_3}(0,4)$ nonsingular and do they have only two components (given by $\alpha$ ) ?

2) What can be said about $M(0,c_2)$, $c_2$ even ?

3) Is the Zariski-open part of mathematical instanton bundles of $M_{\mathbb{P}_3}(0,c_2)$, i.e. the bundles $E$ with $H^1(\mathbb{P}_3, E(-2)) = 0$, nonsingular ?

## 5. Jumping lines and uniform bundles

If $E$ is a holomorphic r-bundle on $\mathbb{P}_n$ the restriction of $E$ to a projective line $L \subset \mathbb{P}_n$ is by the theorem of Grothendieck of the form

$$E|L \simeq \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r).$$

The integers $a_i$ depend on $L$ but are the same for the general line $L$. Lines for which $E|L$ is different from the generic form are called <u>jumping lines</u>. The set of jumping lines will be denoted by $S(E)$. It is a closed analytic subset of $Gr(1,n)$.

One of the main tools in studying stable 2-bundles on $\mathbb{P}_n$ is the theorem of Grauert and Mülich [18], [4].

THEOREM 5.1.- For a stable normalized 2-bundle $E$ on $\mathbb{P}_n$ the restriction of $E$ to the general line is

$$E|L \simeq \begin{cases} \mathcal{O} \oplus \mathcal{O} & \text{for } c_1 = 0 \\ \mathcal{O} \oplus \mathcal{O}(-1) & \text{for } c_1 = -1 . \end{cases}$$

To study stable bundles of higher rank it would be desireable to solve the following

Problem 4. Let $E$ be a stable $r$-bundle on $\mathbb{P}_n$ . Is it true that for the general line $L$ one has

$$E|L \simeq \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$$

with $a_1 \geq a_2 \geq \ldots \geq a_r$ , $a_{i-1} - a_i \leq 1$ for $i = 2,\ldots,r$ ?

For $r = 2$ it is true by the Grauert-Mülich theorem. For $r = 3$ and $n = 2$ it is true by [43].

For stable 2-bundles $E$ with $c_1$ even one can say more about $S(E)$ . The Grauert-Mülich theorem implies for a normalized stable 2-bundle $E$ on $\mathbb{P}_n$ :

$$S(E) = \{L : H^o(L , E(-1)|L) \neq 0\} .$$

Suppose now $n = 2$ and $c_1 = 0$ . The exact sequence

$$0 \longrightarrow H^o(E(-1)|L) \longrightarrow H^1(E(-2)) \xrightarrow{\alpha(L)} H^1(E(-1))$$

shows that

$$S(E) = \{L \in \mathbb{P}_2^* : \det \alpha(L) = 0\} ,$$

because $h^1(E(-2)) = h^1(E(-1)) = c_2(E)$ . Hence $S(E)$ is a curve of degree $c_2(E)$ . Barth [4] has shown that this remains true if $n > 2$ , i.e. $S(E)$ is a divisor of degree $c_2(E)$ in $Gr(1,n)$ .

For $c_1$ odd $S(E)$ is not a hypersurface. For example look at $E \in M_{\mathbb{P}_2}(-1,2) = S^2\mathbb{P}_2 \setminus \Delta$ . If $E$ corresponds to 2 different points $p_1$ , $p_2 \in \mathbb{P}_2$ then there is only one jumping line : the line containing $p_1$ and $p_2$ . In order to associate geometric objects to $M_{\mathbb{P}_2}(-1,c_2)$ Hulek [33] gives the following

DEFINITION 5.2.- Let $E$ be a normalized 2-bundle on $\mathbb{P}_2$ . A line $L \subset \mathbb{P}_2$ is called a <u>jumping line of the second kind</u> if $H^o(E|L^2) \neq 0$ . Here $L^2$ denotes the first infinitesimal neighborhood of $L$ in $\mathbb{P}_2$ .

Hulek shows that for stable 2-bundles on $\mathbb{P}_2$ with $c_1 = -1$ the set $C(E)$ of jumping lines of the second kind is a curve in $\mathbb{P}_2^*$ of degree $2c_2(E) - 2$ . Furthermore

$$S(E) \subset \text{Sing } C(E)$$

and in general one has equality.

Holomorphic bundles E on $\mathbb{P}_n$ with $S(E) = \emptyset$ are called uniform. Van de Ven [63] showed that a uniform 2-bundle on $\mathbb{P}_n$ either splits into line bundles or is of the form $T_{\mathbb{P}_2}(k)$, $k \in \mathbb{Z}$. This was generalized by Sato [47] to r-bundles on $\mathbb{P}_n$ with $r \leq n$. Elencwajg [14] proved that uniform 3-bundles E on $\mathbb{P}_2$ (and therefore on $\mathbb{P}_n$ for all n by Sato's result) are homogeneous, i.e. $\sigma^*E \simeq E$ for all $\sigma \in PGL(n)$. This gave much evidence to the old conjecture [51] that uniform bundles of arbitrary rank on $\mathbb{P}_n$ are homogeneous.

Recently Elencwajg [15] gave an example of a uniform 4-bundle on $\mathbb{P}_2$ which is not homogeneous. In fact he uses a monad of the type described in application 2 of 2.3.

Problem 5. Does every uniform unstable bundle on $\mathbb{P}_n$ split ?

For rank two this is true (and easy to see).

Finally we recommend to consult a recent problem list (26 problems) on vector bundles on $\mathbb{P}_n$ compiled by Hartshorne [26]. There one can especially find many problems related to instantons which we have almost completely neglected due to limited space and knowledge.

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