ROBERT D. EDWARDS

Characterizing infinite dimensional manifolds topologically

Séminaire N. Bourbaki, 1980, exp. nº 540, p. 278-302
<http://www.numdam.org/item?id=SB_1978-1979__21__278_0>

CHARACTERIZING INFINITE DIMENSIONAL MANIFOLDS TOPOLOGICALLY

[after Henryk TORUŃCZYK]

by Robert D. EDWARDS

§ 1. Introduction

In the last twenty-or-so years there has been remarkable progress made in the study of the topology of manifolds, both finite dimensional and infinite dimensional. The infinite dimensional theory has reached a particularly satisfactory state because it is now quite complete, with no loose ends to speak of (the same cannot be said for the finite dimensional theory). Before starting on the topic at hand, it may be worthwhile to recall some of the basic aspects of infinite dimensional manifold theory.

We will focus on two types of infinite dimensional manifolds : Hilbert cube manifolds and Hilbert space manifolds. A Hilbert cube manifold [respectively, Hilbert space manifold], abbreviated I^∞-manifold [ℓ_2-manifold], is a separable metric space each point of which has a neighborhood homeomorphic to the Hilbert cube I^∞ [to Hilbert space ℓ_2] (these definitions are amplified in § 3).

Hilbert space manifolds were the first infinite dimensional manifolds to be successfully understood topologically, with most of the activity taking place in the 1960's. The basic results are (see [An_4] and [Ch_2] for further details, and credits) :

1. Existence of triangulations. For any (locally finite separable metric) simplicial complex K, the product $K \times \ell_2$ is a ℓ_2-manifold, and furthermore any ℓ_2-manifold is so expressible.

2. Homotopy equivalence of triangulations. A map $f : K_1 \times \ell_2 \longrightarrow K_2 \times \ell_2$ is homotopic to a homeomorphism $\Leftrightarrow$ the induced map $K_1 = K_1 \times 0 \hookrightarrow K_1 \times \ell_2 \xrightarrow{f} K_2 \times \ell_2 \xrightarrow{proj} K_2$ is a homotopy equivalence. Furthermore, any two homotopic homeomorphisms of ℓ_2-manifolds are isotopic (= homotopic through homeomorphisms).

3. Promoting maps to embeddings. Any map of ℓ_2-manifolds is homotopic to a closed emdedding such that the image has a product neighborhood of the form image $\times\, c\ell_2$ (where $c\ell_2$ denotes the cone on ℓ_2 , which is in fact homeomorphic to ℓ_2). Furthermore, any two such embeddings which are homotopic are ambient isotopic ($\equiv$ there is an isotopy of the ambient manifold, starting at the identity, which carries the one

embedding to the other).

Hilbert cube manifolds have similar properties ; for example, Statement I above remains true with ℓ_2 replaced by I^∞ , and so does Statement III , if one restricts the statement to proper maps and proper homotopies, to take account of local compactness. Surprisingly, however, Statement II is false even for proper maps and proper homotopies of I^∞-manifolds (this unexpected result was established by T. Chapman in 1972). It must be replaced by

2'. Simple-homotopy equivalence of triangulations. A proper map $f : K_1 \times I^\infty \longrightarrow K_2 \times I^\infty$ is homotopic to a homeomorphism $\Leftrightarrow$ the induced map $K_1 = K_1 \times 0 \hookrightarrow K_1 \times I^\infty \xrightarrow{f} K_2 \times I^\infty \xrightarrow{proj} K_2$ is a proper simple-homotopy equivalence (*). Furthermore, any two homeomorphisms of I^∞-manifolds are isotopic $\Leftrightarrow$ their tori represent the same elements in parametrized simple-homotopy theory (as in [Hat]). (Note : homotopic homeomorphisms of I^∞-manifolds are always concordant, that is, their disjoint union, which defines a homeomorphism of manifold $\times \{0,1\}$, extends to a homeomorphism of manifold $\times [0,1]$.)

As a closing chapter in the development of the basic theory of infinite dimensional manifolds, it was established that the class of manifold factors consisted exactly of absolute neighborhood retracts (ANR's) (as had been conjectured by Borsuk). More precisely, given a (separable complete metric) space X , then $X \times \ell_2$ is a ℓ_2-manifold $\Leftrightarrow$ X is an ANR [To_1], and $X \times I^\infty$ is a I^∞-manifold $\Leftrightarrow$ X is a locally compact ANR [Ed].

§ 2. Statements of results

All spaces in this article, including those in statements of theorems and corollaries, are assumed to be separable complete metric.

The topic of this article is some recent definitive work of Henryk Toruńczyk.

The subject of infinite dimensional manifold topology has to a large extent been developed by attempts to settle whether certain naturally defined infinite dimensional spaces were manifolds. Some examples are : the space Maps(X,M) of continuous maps of a compact space X to a manifold M ; the space 2^I of all closed subsets of a closed interval I (equipped with the Hausdorff metric) ; or the space gotten by taking a countable product of absolute retracts (AR's) . Each of these spaces can be shown, without too much trouble, to be an ANR . Are they infinite dimensional manifolds ?

(*) The notion of simple-homotopy equivalence, defined and developed by J.-H. Whitehead, is a finer notion than homotopy equivalence, for many non-simply-connected complexes. The concept is intimately tied up with the fundamental group, and lies at the root of many of the esoteric phenomena in manifold topology.

All of the above examples (at least in special cases) were shown to be infinite dimensional manifolds by various ad hoc methods, during the period when the basic theory of the subject was being worked out. The recent work of Toruńczyk provides (among other things) new, briefer proofs of all of these results. What Toruńczyk has done is to find a natural, readily verifiable property, which a given ANR may or may not have, which determines whether the ANR is an infinite dimensional manifold. He found two such properties, one for locally compact ANR's and one for non-locally-compact ANR's . We discuss these properties in the order of their discovery.

The locally compact case : I^∞-manifold. In this case Toruńczyk identified the the following key property. A (metric) space X has the disjoint-cells property (*) if given any two maps $f_1, f_2 : D^n \longrightarrow X$ from an n-cell to X , $0 \leq n < \infty$, and given $\varepsilon > 0$, there are two maps $g_1, g_2 : D^n \longrightarrow X$, with $\mathrm{dist}(f_i, g_i) < \varepsilon$, such that $g_1(D^n) \cap g_2(D^n) = \emptyset$. (This "general position" property is discussed further in § 5.) Toruńczyk's main theorem can be stated this way :

TORUŃCZYK'S APPROXIMATION THEOREM : I^∞-MANIFOLD CASE [To_2].- A map $f : M \longrightarrow X$ from a I^∞-manifold M to a space X is approximable by homeomorphisms $\Leftrightarrow$ f is a proper fine homotopy equivalence (i.e., X is an ANR and f is a cell-like map) and X has the disjoint-cells property.

The implication $\Rightarrow$ is known ; for example, the fact that I^∞- manifolds have the disjoint-cells property is a straightforward application of general position in a collection of Hilbert cubes which cover the manifold. The implication $\Leftarrow$ is new.

The preceding theorem nicely complements the following earlier landmark theorem (see Appendix 2) :

MILLER-WEST THEOREM.- For any locally compact ANR X , there is a proper fine homotopy equivalence (i.e. cell-like map) $f : M \longrightarrow X$ from a I^∞-manifold M onto X .

In fact, their Mapping Cylinder Neighborhood Theorem shows that whenever a locally compact ANR X is embedded as a closed negligible subset of a I^∞-manifold (e.g. $I^\infty \times [0,\infty)$ serves this rôle universally), then X has a I^∞-manifold closed mapping cylinder neighborhood there.

Combining these two theorems, Toruńczyk obtained the sweeping

HILBERT CUBE MANIFOLD CHARACTERIZATION THEOREM (Toruńczyk).- A space X is a I^∞-manifold $\Leftrightarrow$ X is a locally compact ANR with the disjoint-cells property.

(*) This is my own terminology ; Toruńczyk has not to my knowledge formally named this property. The name above is chosen because of the similarity to the analogous disjoint (2-)disc property, identified independently by J. Cannon, which seems to determine whether a finite dimensional locally compact ANR homology m-manifold, $m \geq 5$, is in fact a topological m-manifold (announced by F. Quinn).

As a corollary of the Approximation Theorem, one has :

COROLLARY (Chapman's Approximation Theorem [Ch_1]).- A map $f : M \longrightarrow N$ of I^∞-manifolds is approximable by homeomorphisms $\Leftrightarrow$ f is a proper fine homotopy equivalence (i.e., a cell-like map).

As corollaries of the Characterization Theorem, one has :

COROLLARY (Edwards [Ed]).- For any space X , the product $X \times I^\infty$ is a I^∞-manifold $\Leftrightarrow$ X is a locally compact ANR .

Proof.- The implication $\Rightarrow$ is clear, since X is a retract of the hypothesized I^∞-manifold (hence ANR) $X \times I^\infty$. The implication $\Leftarrow$ follows because $X \times I^\infty$ clearly has the disjoint-cells property, say by using the fact that I^∞ admits retractions, arbitrarily close to the identity, onto disjoint faces.

COROLLARY (Schori-West [S-W] ; Curtis-Shori [C-S]).- Suppose X is a compact connected metric space containing more than one point. Let 2^X [respectively, C(X)] denote the space, provided with the Hausdorff metric, of all closed [resp., closed and connected] subsets of X . Then

(1) 2^X is homeomorphic to I^∞ $\Leftrightarrow$ X is locally connected, that is, X is a Peano continuum, and

(2) C(X) is homeomorphic to I^∞ $\Leftrightarrow$ X is a Peano continuum and X contains no free arcs.

The classical case of part (1) of this corollary, conjectured by Borsuk and solved by Schori-West, is the case $X = I$. It is a pleasant exercise to verify that 2^I has the disjoint-cells property.

COROLLARY.- A countably infinite product of nontrivial ANR's is a I^∞-manifold $\Leftrightarrow$ each ANR is locally compact and all but finitely many are compact AR's.

A milestone special case of this corollary was the case where each ANR is a triod (= the cone on three points), solved by R.-D. Anderson.

There are many other corollaries too, of a more technical nature, answering questions which earlier had been raised by workers in this field. Further details can be found in [To_2].

The non-locally-compact case : ℓ_2-manifolds. In this case Toruńczyk identified a key property analogous to the one in the locally compact case. A (metric) space X has the discrete-cells property (*) if given any map $f : D \longrightarrow X$ from the disjoint union of cells $D \equiv +_{n=0}^{\infty} D^n$ to X , and given $\varepsilon > 0$, there is a map $g : D \longrightarrow X$,

(*) See preceding footnote.

with $\text{dist}(f,g) < \varepsilon$, such that the images of the D^n's under g comprise a disjoint, discrete (hence closed) collection of compacta in X . (See § 7 for further discussion of this property.) Toruńczyk's main theorem in this case can be stated this way (paralleling the statement in the I^∞-manifold case) :

TORUŃCZYK'S APPROXIMATION THEOREM : ℓ_2-MANIFOLD CASE [To_3].- A map $f : M \twoheadrightarrow X$ from a ℓ_2-manifold M onto a space X is approximable by homeomorphisms $\Leftrightarrow$ f is a fine homotopy equivalence and X has the discrete-cells property.

As before, the implication $\Rightarrow$ is known and straightforward ; the implication $\Leftarrow$ is new.

The preceding theorem nicely complements Toruńczyk's earlier, important

ANR $\times\ \ell_2$ THEOREM [To_1] (*).- For any space X , the product $X \times \ell_2$ is a ℓ_2-manifold $\Leftrightarrow$ X is an ANR .

Combining these two theorems, Toruńczyk obtained the

HILBERT SPACE MANIFOLD CHARACTERIZATION THEOREM (Toruńczyk).- A space X is a ℓ_2- manifold $\Leftrightarrow$ X is an ANR with the discrete-cells property.

As a corollary of the Approximation Theorem, one has :

COROLLARY.- A surjective map $f : M \twoheadrightarrow N$ of ℓ_2-manifolds is approximable by homeomorphisms $\Leftrightarrow$ f is a fine homotopy equivalence.

As corollaries of the Characterization Theorem, one obtains (details in [To_3]) :

COROLLARY.- Suppose X is an infinite compact space and M is a non-0-dimensional manifold (M may be finite dimensional or a I^∞-manifold or a ℓ_2-manifold). Then the space $\text{Maps}(X,M)$, topologized with the sup metric, is a ℓ_2-manifold.

COROLLARY.- A countable product of AR's, infinitely many of which are non-compact, is homeomorphic to ℓ_2 .

COROLLARY.- Suppose $f : M \twoheadrightarrow X$ is a proper map from a ℓ_2-manifold M onto an ANR X such that $\text{id}(X)$ is approximable by maps of the form $X \xrightarrow{g} M \xrightarrow{f} X$. Then X is a ℓ_2-manifold.

(*) At this point, the parallelism between the locally compact and non-locally-compact cases seems to be fading. But it is worth noting that the Miller-West theorem can be proved from basic principles in this ℓ_2-manifold case just as easily as in the I^∞-manifold case, to show that given any (separable complete metric) ANR X , there is a fine homotopy equivalence $f : M \to X$ from a ℓ_2-manifold M onto X (see Appendix 2). Consequently, the parallelism of the cases can be maintained, and Toruńczyk's ANR $\times\ \ell_2$ theorem can then be deduced as a corollary.

Note.- Toruńczyk calls such a map f a r*-map ; examples are proper fine homotopy equivalences and proper retractions. As an example of the Corollary, one has that if $X \times I^{\infty}$ is a ℓ_2-manifold, then so is X .

COROLLARY (Kadec-Anderson).- Any separable infinite dimensional Fréchet space is homeomorphic to ℓ_2 .

A Fréchet space is a locally convex complete-metrizable topological vector space. The all-important case of this corollary, that $(-1,1)^{\infty} \approx \ell_2$, was settled by Anderson (see § 7). Some care has to be taken to avoid circular reasoning in this corollary ; see [To_3, Appendix].

Toruńczyk extended these results to Hilbert space manifolds of higher weights, too (that is, to non-separable Hilbert space manifolds), but that will not be discussed here (see [To_3]).

§ 3. Definitions and some basic facts

We again emphasize that all spaces in this article, including those in the statements of theorems and corollaries, are separable complete metric (one exception : the space Maps(W,X) defined below may be neither separable nor metrizable when W is noncompact). We are most interested in the compact versions of theorems, and so in particular in the Hilbert cube manifold theorems ; the other cases are included mainly for completeness of exposition.

The following definitions are all quite standard, and need be consulted only as required.

The basic compact manifold of this article is the Hilbert cube I^{∞} , which is defined as the countable product of closed unit intervals, $I^{\infty} = [-1,1]^{\infty} = \times_{i=1}^{\infty} [-1,1]$. A natural metric on I^{∞} is $\mathrm{dist}(x,y) = \Sigma_{i=1}^{\infty} |x_i - y_i|/2^i$ for $x,y \in I^{\infty}$. Metrically one should think of I^{∞} as an infinite dimensional brick, with the later sides getting shorter and shorter (alternatively one could define $I^{\infty} = \times_{i=1}^{\infty}[-1/2^i , 1/2^i]$ and $\mathrm{dist}(x,y) = \Sigma_{i=1}^{\infty}|x_i - y_i|$, but this is notationally more cumbersome). The interior of I^{∞} is $\mathrm{int}\, I^{\infty} = (-1,1)^{\infty}$, and the boundary of I^{∞} is $\partial I^{\infty} = I^{\infty} - \mathrm{int}\, I^{\infty}$. (These terms are justified solely by analogy with finite dimensional cubes, for the Hilbert cube is homogeneous. This basic fact is discussed in Appendix 1.)

The model non-locally-compact manifold in this article is Hilbert space ℓ_2 , the space of square summable sequences of real numbers with the usual metric. Since only the topological properties of ℓ_2 are used in this article, it could just as well be replaced by $\mathrm{int}\, I^{\infty}$, to which it was shown homeomorphic by R.-D. Anderson (see Corollary above).

A map $f : W \longrightarrow X$ is approximable by a map g (usually having certain additional properties) if, for any target-majorant map $\varepsilon : X \longrightarrow (0,\infty)$, such a map $g : W \longrightarrow X$ can be found so that for each $w \in W$, $\mathrm{dist}(f(w),g(w)) < \varepsilon(f(w))$. (It is important to keep in mind in the non-proper-mapping case that closeness here is being measured by target-majorant maps, not by source-majorant maps, which are more stringent.) If W is compact, or merely has relatively compact image in X , then this is ordinary uniform approximation. One could as well define approximations by using open covers of X instead of maps $\{\varepsilon : X \longrightarrow (0,\infty)\}$ (as is done in [$\mathrm{To}_{2,3}$]). A map $f : W \longrightarrow X$ is a near-homeomorphism if it is approximable by homeomorphisms.

For spaces W and X , the set $\mathrm{Maps}(W,X)$ is topologized by letting a neighborhood basis of $f : W \longrightarrow X$ consist of sets of the form $N(f,\varepsilon) = \{g \in \mathrm{Maps}(W,X) \mid \forall w \in W , \mathrm{dist}(f(w),g(w)) < \varepsilon(f(w))\}$ for all possible target-majorant maps $\varepsilon : X \longrightarrow (0,\infty)$. If W is compact (and X is separable complete metric) then $\mathrm{Maps}(W,X)$ is separable complete metric, with metric given by the sup norm. If W and X are noncompact, then in general $\mathrm{Maps}(W,X)$ is non-metrizable, but still a simple limit argument establishes that $\mathrm{Maps}(W,X)$ retains the Baire property, that the intersection of a countable collection of open dense subsets is dense.

A map $f : W \longrightarrow X$ is proper if the preimage of each compact set is compact. This can be shown to be equivalent to saying that f is closed and each point-inverse $f^{-1}(x)$ is compact. If X is locally compact, the proper maps in $\mathrm{Maps}(W,X)$ comprise an open-closed subset.

An absolute neighborhood retract (ANR) is a space which, when embedded as a closed subset of $\mathrm{int}\, I^\infty$ or ℓ_2 (recall any separable complete metric space can be so embedded), then some neighborhood of the image retracts to the image. The members of this important class of spaces have many interesting properties and characterizations (see e.g. [Hu]) ; for example, a finite dimensional space is an ANR it is locally contractible. An absolute retract (AR) is a contractible ANR .

A map $f : W \longrightarrow X$ is a fine homotopy equivalence if for any target-majorant map $\varepsilon : X \longrightarrow (0,\infty)$ there is a map $g : X \longrightarrow W$ such that $fg : X \longrightarrow X$ is homotopic to $\mathrm{id}(X)$ through maps in $N(\mathrm{id}(X),\varepsilon)$ and $gf : W \longrightarrow W$ is homotopic to $\mathrm{id}(W)$ through maps in $N(\mathrm{id}(W),\varepsilon f)$. A near-homeomorphism is an example of a fine homotopy equivalence. A fine homotopy equivalence (or a near-homeomorphism, for that matter) need not be onto, for example $(0,1) \hookrightarrow [0,1)$ (or $(0,1) \times \ell_2 \hookrightarrow [0,1) \times \ell_2$). However in this article all such maps will be surjective. The image of an ANR under a fine homotopy equivalence is an ANR (see e.g. [Hu, Theorem IV 6.3a,c, p. 139]).

A map $f : W \longrightarrow X$ is cell-like if it is proper, surjective and if each point-inverse $f^{-1}(x)$ is cell-like, i.e., has the shape of a point, i.e., is contractible in any neighborhood of itself when embedded as a subset of I^∞ .

A cell-like map of ANR's is a fine homotopy equivalence, but the converse fails, e.g. consider projection : $\ell_2 \times \ell_2 \longrightarrow \ell_2$. For locally compact ANR's, the class of cell-like maps and the class of proper fine homotopy equivalences coincide (these basic facts can be found in [La] and [Hav]). Nevertheless in this article we stress the latter notion, because cell-like maps are too restrictive a class in the non-locally-compact setting.

Given a map $f : W \longrightarrow X$, its <u>mapping cylinder</u> is the disjoint union $Cyl(f) = W \times [0,1) + X$ topologized by letting $W \times [0,1)$ with the product topology be an open subset, and by letting a basic neighborhood of $x \in X$ be of the form $f^{-1}(U) \times (t,1) + U$, where U is a neighborhood of x in X and $t < 1$. If W and X are locally compact and f is proper, then $Cyl(f)$ is a quotient of $W \times [0,1]$, with the quotient topology (in more general cases, the quotient topology may be finer (e.g. non-first-countable) than the above metrizable topology described for $Cyl(f)$).

Probably the single most important concept in infinite dimensional manifold topology is that of negligibility, so recognized by R.-D. Anderson. A subset Y of a space X is <u>negligible</u> (*) in X if $id(X)$ is (arbitrarily closely) approximable by a map taking X into $X - Y$. Clearly this is equivalent to saying that any map $f : W \longrightarrow X$ is approximable by a map $g : W \longrightarrow X - Y$. The model example to think of here is when Y is a subset of $Y_\partial \subset X$, where Y_∂ has a product open neighborhood $Y_\partial \times [0,1)$ in X . Basic examples are when X is a finite dimensional manifold and Y is a subset of its boundary, or when $X = I^\infty$ and Y is a subset of a face of I^∞ . Also, any subset Y of $\text{int } I^\infty$ is negligible in I^∞ , because for any $\varepsilon > 0$ there is an ε-retraction of I^∞ onto some face. This discussion shows that any point of I^∞ is negligible. More generally, any subset of ∂I^∞ is negligible in I^∞ (because I^∞ can be ε-retracted into $\text{int } I^\infty$), and in fact it can be shown that any compact subset of $\text{int } I^\infty$ (or ℓ_2) is negligible in $\text{int } I^\infty$ (or ℓ_2). This concept is discussed further in the next section.

A useful remark which relates the preceding definitions is the following : A map $f : W \longrightarrow X$ of ANR's is a fine homotopy equivalence $\Longleftrightarrow$ X is negligible in $Cyl(f)$. The proof is basically a matter of chasing definitions.

The notation $W \approx X$ denotes that W is homeomorphic to X .

An outstanding reference for all of the basic material of this article, at least in the locally compact case, is [Ch_2] (one expects that in the non-locally-compact case, this will be matched by [To_4]).

(*) Following Anderson [An_2], such a subset Y is usually called a Z-<u>set</u> these days. We hope no one will be offended by our using in this article the more meaningful term <u>negligible</u>, even if it has been used in a slightly different (but strongly related) context elsewhere [An_3].

§ 4. Prerequisites for the Approximation Theorem : I^∞-manifold case

In the next three sections we present the locally compact, I^∞-manifold case of Toruńczyk's Approximation Theorem (in general our attention is restricted to the compact case, which is the model for all other cases). In this section we discuss the two main tools used in Toruńczyk's proof.

Negligible embedding and isotopy theorems. The following results were developed primarily by R.-D. Anderson in the late 1960's. (Interestingly, the analogous results in finite dimensional manifolds, concerning embeddings and isotopies of tame subsets in the trivial dimension range, were also being developed independently about that time.)

NEGLIGIBLE APPROXIMATION THEOREM.- Suppose $f : W \longrightarrow M$ is a proper map of a locally compact space W into a I^∞-manifold M. Then there is a proper negligible embedding $g : W \longrightarrow M$ arbitrarily close to f.

Discussion of proof.- Negligible embedding means the image is negligible in M. The model case is when W is compact and $M = I^\infty$. It is a basic exercise in point-set topology that any map $f : W \longrightarrow I^\infty$ can be approximated by an embedding, which certainly can be chosen to have image in $[-1+\varepsilon, 1-\varepsilon]^\infty \subset \text{int } I^\infty$ for some small $\varepsilon > 0$, in which case the embedding is negligible in I^∞. □

It turns out that any two negligible embeddings $f_0, f_1 : W \longrightarrow I^\infty$ of a compactum W are equivalent, in that there exists a homeomorphism $h : I^\infty \longrightarrow I^\infty$ such that $hf_0 = f_1$. This cornerstone result may be regarded is the first nontrivial theorem in the subject. A summary of the usual proof is as follows (see [Ch_2, II] for details) :

(1) If W is a negligible compact subset of I^∞, then there is a homeomorphism of I^∞ carrying W into int I^∞. This is accomplished by moving W off of the faces of I^∞ one at a time, by smaller and smaller homeomorphisms whose composition converges to a homeomorphism (cf. Appendix 1).

(2) The assertion is true if $f_i(W) \subset \text{int } I^\infty$, $i = 0,1$. This is established by using the so-called Klee trick, first moving f_0 [respectively, f_1] so that all the even coordinates [resp., odd coordinates] of points in $f_0(W)$ [resp., $f_1(W)$] are 0, and then moving f_0 to f_1 by moving each to the "graph" $\{\text{"}(f_0(w), f_1(w))\text{"} \mid w \in W\} \subset \text{int } I^\infty$ by natural homeomorphisms.

In this preceding discussion, it is often important to have control on how far the homeomorphism h moves points ; for example, if $f_0, f_1 : W \longrightarrow I^\infty$ are nearby negligible embeddings, can the homeomorphism h be chosen close to the identity ? The following fundamental theorem provides such control.

NEGLIGIBLE HOMOTOPY-ISOTOPY THEOREM.- Suppose $f_t : W \longrightarrow M$, $0 \leq t \leq 1$, is a proper homotopy of a locally compact space W into a I^∞-manifold M such that f_o, f_1 are negligible embeddings. Then for any $\varepsilon > 0$ there is an ambient isotopy (i.e. homotopy of homeomorphisms) $h_t : M \longrightarrow M$, $0 \leq t \leq 1$, with h_o = identity and $h_1 f_o = f_1$, such that for each $z \in M$ either $h_t(z) = z$ for all t , or else there is a $w = w(z) \in W$ such that the path-image $\{h_t(z) \mid 0 \leq t \leq 1\}$ lies in the ε-neighborhood of the path-image $\{f_t(w) \mid 0 \leq t \leq 1\}$.

Note.- The proof in § 6 requires only the homeomorphism h_1 , but it is noteworthy that an entire isotopy exists.

Discussion of proof.- What is interesting is that the theorem follows quite easily from the unregulated equivalence-of-negligible-embeddings result discussed above, basically by means of a simple conjugation trick. Details are given in [Ch_2, IV and esp. 9.1]. □

Bing Shrinking Criterion. The Bing Shrinking Criterion is a tool introduced by R.-H. Bing in [Bi] for detecting whether certain maps are approximable by homeomorphisms. It is embodied in the

SHRINKING THEOREM.- A proper surjective map $\pi : W \longrightarrow X$ of locally compact metric spaces is approximable by homeomorphisms $\Leftrightarrow$ the following Bing Shrinking Criterion holds : Given any majorant map $\varepsilon : X \longrightarrow (0,\infty)$, there is a homeomorphism $h : W \longrightarrow W$ such that

(1) for each $w \in W$, $\mathrm{dist}(\pi h(w), \pi(w)) < \varepsilon(\pi(w))$, and

(2) for each $x \in X$, $\mathrm{diam}\, h(\pi^{-1}(x)) < \varepsilon(x)$.

We are primarily interested in the case where W and X are compact, in which case ε may as well be constant. Toruńczyk's proof makes use of the implication $\Leftarrow$. The reverse implication is mentioned here only for completeness ; it is quickly proved by letting $h = g_o^{-1} g_1$ for two successively chosen homeomorphisms g_o, g_1 approximating π . Concerning the implication $\Leftarrow$, it is worth presenting here a slick Baire category proof (which is not the way the proof was originally discovered and developed). Suppose W and X are compact. In the Baire space $\mathrm{Maps}(W,X)$, let $\mathcal{E}$ be the closure of the set $\{\pi h^{-1} \mid h : W \longrightarrow W$ is a homeomorphism$\}$. The Bing Shrinking Criterion amounts to saying that for any $\varepsilon > 0$, the open subset of ε-maps in $\mathcal{E}$ ($\equiv$ maps having all point-inverses of diameter $< \varepsilon$), denoted $\mathcal{E}_\varepsilon$, is dense in $\mathcal{E}$. Hence $\mathcal{E}_o \equiv \cap_{\varepsilon > 0} \mathcal{E}_\varepsilon$ is dense in $\mathcal{E}$, since $\mathcal{E}$ is a Baire space. Since $\mathcal{E}_o$ consists of homeomorphisms, this shows that $\pi \in \mathcal{E}$ is approximable by homeomorphisms. The general locally compact case is deducible by the same proof, or one can deduce it from the compact case by a clever one-point-compactification argument (see [Ch_2, § 26]).

As simple applications of this theorem we present here two results, both of which are used later in the article.

STABILITY PROPOSITION.- Suppose M is a I^∞-manifold, and $0 \leq n \leq \infty$ is any integer. Then the projection map $\pi : M \times I^n \longrightarrow M$ is approximable by homeomorphisms.

Proof.- We examine only the case $M = I^\infty$ and $n = 1$; the general case is a simple extension of this. Given the projection $\pi : I^\infty \times [-1,1] \longrightarrow I^\infty$, then according to the Shrinking Theorem, it suffices to show, given any $\varepsilon > 0$, that there is a homeomorphism $h : I^\infty \times [-1,1] \longrightarrow I^\infty \times [-1,1]$, with $\mathrm{dist}(\pi h,\pi) < \varepsilon$, such that for any $z \in I^\infty$, $\mathrm{diam}(h(z \times [-1,1])) < \varepsilon$. To construct h, we simply choose m so large that the diameter of the m-th coordinate of I^∞ is $< \varepsilon$, and we let $h =$ "$\theta \times$ identity", where θ is a homeomorphism of the 2-disc $[-1,1]_m \times [-1,1]$ gotten by rotating it 90°, taking the "long" segments point $\times [-1,1]$ onto "short" segments $[-1,1]_m \times$ point. That is, h changes only the $[-1,1]_m$ and $[-1,1]$ coordinates of any point in $I^\infty \times [-1,1]$, by applying θ to them, and h leaves unchanged all the remaining coordinates. □

MAPPING CYLINDER PROPOSITION (J. West).- Suppose $f : M \longrightarrow X$ is a proper fine homotopy equivalence (i.e., a cell-like map) from a I^∞-manifold M onto a locally compact ANR X. Then the mapping cylinder $\mathrm{Cyl}(f)$ is a I^∞-manifold ; in fact, the natural quotient map $\pi : M \times [0,1] \longrightarrow \mathrm{Cyl}(f)$ is approximable by homeomorphisms.

A simple corollary of this is that $cI^\infty \approx I^\infty$, where cI^∞ is the cone on I^∞, because cI^∞ is the mapping cylinder of the trivial map $I^\infty \longrightarrow$ point.

Proof.- To keep notation simple, we restrict attention to the case where M and X are compact. To show that $\pi : M \times [0,1] \longrightarrow \mathrm{Cyl}(f)$ is approximable by homeomorphisms, it suffices, according to the Shrinking Theorem, to construct a homeomorphism $h : M \times [0,1] \longrightarrow M \times [0,1]$ such that $\mathrm{dist}(\pi h,\pi) < \varepsilon$ and for each $x \in X$, $\mathrm{diam}(h(f^{-1}(x) \times 1)) < \varepsilon$, where $\varepsilon > 0$ is given. We construct h as follows. Since f is a fine homotopy equivalence, there is a homotopy $\alpha_t : M \longrightarrow M$, $0 \leq t \leq 1$, starting at $\alpha_0 = \mathrm{id}(M)$, such that for each t, $f\alpha_t$ is arbitrarily close to f and such that α_1 factors through X, $\alpha_1 : M \xrightarrow[f]{} X \xrightarrow[g]{} M$. We can perturb the homotopy an arbitrarily small amount to make $\alpha_1 : M \longrightarrow M$ an embedding (but it may no longer factor through X). By the Negligible Homotopy-Isotopy Theorem applied in the ambient manifold $M \times [0,1]$ to the two homotopic negligible embeddings $\alpha_0,\alpha_1 : M \longrightarrow M \times 1 \subset M \times [0,1]$, there is a homeomorphism $h : M \times [0,1] \longrightarrow M \times [0,1]$ such that $h \mid M \times 1 = \alpha_1\alpha_0^{-1}$ and such that πh is close to π. This is the desired shrinking homeomorphism h. (The reader can fill in the ϵ's .) □

§ 5. A reformulation of the Disjoint-Cells Property

The following result of Toruńczyk establishes a key condition which is equivalent to the disjoint-cells property.

PROPOSITION.- Suppose X is a locally compact ANR. Then X has the disjoint-cells property $\Leftrightarrow$ any proper map $f : W \longrightarrow X$ from a locally compact space W to X is approximable by a negligible embedding $g : W \longrightarrow X$.

Proof.- The implication $\Leftarrow$ is trivial, for letting $W = D^n + D^n$ (+ denotes disjoint union) and letting $f = f_1 + f_2 : W \longrightarrow X$, then any embedding approximating f offers the desired conclusion. In our discussion of the implication $\Rightarrow$, we treat only the case where W and X are compact. We begin by observing that the disjoint-cells property implies the disjoint-Hilbert-cubes property, using the fact that given any map $f : I^\infty \longrightarrow X$, the map $I^\infty \xrightarrow[\pi]{} I^n \xrightarrow[f|]{} X$ is arbitrarily close to f for n sufficiently large, where $I^n = I^n \times 0 \times \ldots \subset I^\infty$ and $\pi : I^\infty \longrightarrow I^n$ is projection. In the separable space $\mathrm{Maps}(I^\infty, X)$ let $\{\varphi_i : I^\infty \longrightarrow X \mid i = 1,2,\ldots\}$ be a countable dense set with the additional property that each map φ_i appears in the listing infinitely often. It is a routine matter, applying the disjoint-Hilbert-cubes property to the pair of maps φ_1, φ_2, then to the pair $(\text{new})\varphi_2, \varphi_3$, then to the pair $(\text{new})\varphi_1, (\text{new})\varphi_3$, etc., to get as a limit a new collection of maps $\{\psi_i : I^\infty \longrightarrow X\}$, with $\mathrm{dist}(\psi_i, \varphi_i) < 1/i$, such that the ψ_i's have pairwise disjoint images. The main point here is that when any pair φ_i, φ_j has been rechosen to have disjoint images, then their disjointness can be maintained under all further rechoosings, simply by choosing the subsequent approximations sufficiently close. By the infinite repetition condition in the list of φ_i's, the ψ_i's remain a dense collection.

At this point, to illustrate the main idea of the proof most quickly, let us suppose that X is contractible, and let us establish the weaker conclusion that there is an approximating map $g : W \longrightarrow X$ with negligible image. We can suppose that $W \subset I^\infty$. Since X is a contractible ANR, the given map $f : W \longrightarrow X$ extends to a map $f_\# : I^\infty \longrightarrow X$. Then we can take $g = \psi_k \mid W$, where $\psi_k \in \{\psi_i\}$ is chosen to approximate $f_\#$. We assert that g has negligible image in X. The point is, we can assume in this special case that X is a retract of I^∞, say by a retraction $r : I^\infty \longrightarrow X$, and so letting $\psi_\ell \in \{\psi_i\} - \psi_k$ be a map approximating r, we get a map $\psi_\ell \mid X : X \longrightarrow X - g(W)$ which is arbitrarily close to $\mathrm{id}(X)$.

In the general compact case, one can first establish as a lemma that any two given maps $f_i : K_i \longrightarrow X$, $i = 1,2$, of finite simplicial complexes to X can be approximated by maps $g_i : K_i \longrightarrow X$, $i = 1,2$, whose images are disjoint. This follows from the disjoint-cells property, applying it successively to pairs of vertex stars of K_1 and K_2, which can be assumed to be embedded in high dimensional cells. In fact, using the same principle, each g_i can be chosen to have the additional property that disjoint (closed) stars of K_i have disjoint images under g_i. Now let $\{\varphi_i : K_i \longrightarrow X \mid i = 1,2,\ldots\}$ be a collection of maps of finite simplicial complexes to X, such that (as above) each φ_i appears infinitely often in the list, and such that for any given finite simplicial complex K (of which there are only countably many, since each is a subcomplex of some large finite dimensional simplex), the subcollection $\{\varphi_i \mid K_i = K\}$ is dense in $\mathrm{Maps}(K,X)$. As above, one can get a collection of approximations $\{\psi_i : K_i \longrightarrow X\}$, with $\mathrm{dist}(\psi_i,\varphi_i) < 1/i$, such that all of the images are disjoint, and such that each ψ_i keeps disjoint stars of K_i disjoint. Given $f : W \longrightarrow X$, the desired embedding $g : W \longrightarrow X$ is gotten as a limit, $g = \mathrm{limit}_{j\to\infty} f_j : W \longrightarrow X$, where each f_j is chosen to approximate f_{j-1} (starting with $f_o = f$) and where f_j is of the form $f_j = \psi_{i(j)}\eta_j$, where $\eta_j : W \longrightarrow K_j$ is a nerve map from W to the nerve K_j of a fine finite open cover of W, and where $\psi_{i(j)} \in \{\psi_i\}$. Each f_j can be chosen arbitrarily close to f_{j-1}, and each f_j can be chosen to be a $1/j$-map (that is, a map each point-inverse of which has diameter $< 1/j$). Sufficiently rapid convergence of the f_j's therefore will ensure that g is an embedding. Furthermore, by suitable choice of the $i(j)$'s, g can be made to have the additional property that $g(W) \cap \bigcup_{i=1}^{\infty} \psi_i(K_i) = \emptyset$, for at the j-th stage f_j can be chosen so that $f_j(W) \cap \bigcup_{i=1}^{j} \psi_i(K_i) = \emptyset$, and subsequent f_j's can be chosen to stay bounded away from $\bigcup_{i=1}^{j} \psi_i(K_i)$. Thus, by an argument like that used earlier, $g(W)$ is negligible in X. This completes our discussion of the Proposition. □

§ 6. Proof of Toruńczyk's Approximation Theorem : I^∞-manifold case

Toruńczyk's original proof used as a starting point the fact that $X \times I^\infty$ is a I^∞-manifold [Ed], which in turn had been deduced by starting with the Miller-West Theorem. Since both of these proofs (Toruńczyk's and ours) use the same sort of Bing shrinking argument, it is natural to expect them to be combinable into a single, direct argument. This is done here. We give details only for the compact case ; the general case is identical, except that constant ε's should be replaced by majorant ε's . Only the prerequisites discussed in §§ 4, 5 are used in this section.

Suppose, then, that $f : M \longrightarrow X$ is a fine homotopy equivalence from a compact I^∞-manifold M onto a space X (necessarily an ANR ; see § 3). (We will not invoke the disjoint-cells property until it is required.) In attempting to show that f is approximable by homeomorphisms, it is convenient to work with the following commutative square :

$$\begin{array}{ccc} M \times [0,1] & \xrightarrow[\text{near-homeo by § 4}]{\text{quotient map}} & \mathrm{Cyl}(f) \\ \text{proj} \Big\downarrow \begin{array}{l}\text{near-}\\ \text{homeo}\\ \text{by § 4}\end{array} & & \pi \Big\downarrow \begin{array}{l}\text{mapping}\\ \text{cylinder}\\ \text{projection}\end{array} \\ M & \xrightarrow[f]{} & X \end{array}$$

In § 4, it was shown that the maps $M \times [0,1] \to M$ and $M \times [0,1] \to \mathrm{Cyl}(f)$ are approximable by homeomorphisms. Consequently f is approximable by homeomorphisms if and only if π is.

Hence, our goal is to show that if X has the disjoint-cells property, then the mapping cylinder projection $\pi : \mathrm{Cyl}(f) \to X$ is approximable by homeomorphisms.

The following result is the basic tool of the proof (it is stated in its non-compact form).

PROPOSITION.- Suppose $f : M \to X$ is a proper fine homotopy equivalence (i.e. cell-like map) from a I^∞-manifold M onto an ANR X. Suppose $Y \subset X$ is a negligible closed subset. Then the decomposition $\{\pi^{-1}(y) \mid y \in Y\}$ of $\mathrm{Cyl}(f)$ is shrinkable. Consequently there exists a near-homeomorphism $g : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$, with πg arbitrarily close to π, such that g agrees with π over Y (that is, $g \mid g^{-1}(Y) = \pi \mid \pi^{-1}(Y)$).

Note.- We point out that $\pi^{-1}(Y)$ is not necessarily negligible in $\mathrm{Cyl}(f)$ (e.g. consider $f : I^\infty \times [0,1] \xrightarrow[\text{projection}]{} [0,1] \xrightarrow[\text{retraction}]{} [0,1/2] = X$ and $Y = \{1/2\}$) ; if it were, the proof would be simpler.

Proof.- We restrict attention to the compact case. Let $q : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f) / \{\pi^{-1}(y) \sim y \mid y \in Y\}$ denote the quotient map from $\mathrm{Cyl}(f)$ onto the reduced mapping cylinder. To shrink the decomposition, we show that the Bing Shrinking Criterion is satisfied, that is, given $\varepsilon > 0$, there exists a homeomorphism $h : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$, with $\mathrm{dist}(qh,q) < \varepsilon$, such that for each $y \in Y$, $\mathrm{diam}\, h(\pi^{-1}(y)) < \varepsilon$. To achieve this, we first construct a homeomorphism $h_1 : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$, with $\mathrm{dist}(qh_1,q)$ arbitrarily small, such that h_1 moves $\pi^{-1}(Y)$ off of $M \times 0$ in $\mathrm{Cyl}(f)$. By the negligibility of Y in X, there is a homotopy $\alpha_t : X \to X$, $t \in [0,1]$, arbitrarily close to $\mathrm{id}(X)$, such that $\alpha_0 = \mathrm{id}(X)$ and $\alpha_1(X) \subset X - Y$. Since f is a fine homotopy equivalence, there is an "approximate lift" of α_t, say $\tilde{\alpha}_t : M \to M$, with $f\tilde{\alpha}_t$ close to $\alpha_t f$, such that $\tilde{\alpha}_0 = \mathrm{id}(M)$ and $\tilde{\alpha}_1(M) \subset M - f^{-1}(Y)$. We can assume, by approximation, that $\tilde{\alpha}_1$ is an embedding. Applying the Negligible Homotopy-Isotopy Theorem in the I^∞-manifold $\mathrm{Cyl}(f)$ to the negligible embeddings $\tilde{\alpha}_0, \tilde{\alpha}_1 : M \to M = M \times 0 \subset \mathrm{Cyl}(f)$, we obtain the desired homeomorphism $h_1 : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$ such that h_1 covers $\tilde{\alpha}_0 \tilde{\alpha}_1^{-1}$. Now, let $b > 0$ be so small that $M \times [0,b] \cap h_1(\pi^{-1}(Y)) = \emptyset$, and let $\rho : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$ be the natural homeomorphism which changes only the $[0,1]$ coordinate in $\mathrm{Cyl}(f)$, such that $\rho(M \times [0,b]) = M \times [0,1-\delta]$ for some small $\delta > 0$. Then $h = \rho h_1$ is the desired shrinking homeomorphism of the Proposition.

Consequently, by the Shrinking Theorem, the quotient map q is a near-homeomorphism, and in fact an approximating homeomorphism q_* can be chosen so that $q_* | Y =$ "id(Y)" (this is so either by construction, observing that the homeomorphism h above is the identity on Y, or it can be justified because Y is negligible in $\mathrm{Cyl}(f)$ and $q(Y)$ is negligible in $q(\mathrm{Cyl}(f))$). Then the near-homeomorphism $g = q_*^{-1} q$ satisfies the Proposition. □

Returning to the proof of the Theorem, we observe that in order to show that the mapping cylinder projection $\pi : \mathrm{Cyl}(f) \longrightarrow X$ is approximable by homeomorphisms, it suffices to construct a near-homeomorphism (instead of homeomorphism) $h_* : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$ satisfying the conditions of the Bing Shrinking Criterion, for then any homeomorphism approximating h_* will also satisfy the conditions of the Bing Shrinking Criterion.

The near-homeomorphism $h_* : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$ will be constructed as a composition $h_* = \tau\mu^{-1} g$, where πg, $\pi\mu$ (hence $\pi\mu^{-1}$) and $\pi\tau$ are each close to π, where g is a near-homeomorphism to be provided by the Proposition, and where μ and τ are certain auxiliary homeomorphisms of $\mathrm{Cyl}(f)$ (note pictures below).

<u>Construction of</u> μ. Let $\mu_1 : M \longrightarrow X$ be a negligible embedding approximating the given map $f : M \longrightarrow X$ (see § 5 ; here is the point where the disjoint-cells property of X is used). Since μ_1 can be taken as the terminal map of a homotopy $\mu_t : M \longrightarrow \mathrm{Cyl}(f)$, $t \in [0,1]$, where $\mu_0 : M = M \times 0 \subset \mathrm{Cyl}(f)$ and where $\pi\mu_t$ is arbitrarily close to $\pi\mu_0$ for each t, the Negligible Homotopy-Isotopy Theorem provides a homeomorphism $\mu : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$, with $\pi\mu$ close to π, such that $\mu | M \times 0 = \mu_1$.

<u>Construction of</u> g. Let $g : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$ be a near-homeomorphism provided by the Proposition above, for the negligible subset $Y \equiv \mu_1(M) = \mu(M \times 0)$ in X. Note that $\mu^{-1} g$ takes Y onto $M \times 0$.

<u>Construction of</u> τ. We make the following key

<u>Observation</u>.- For each $x \in X$, if $\mu^{-1} g(\pi^{-1}(x)) \cap M \times 0 \neq \emptyset$, then $\mu^{-1} g(\pi^{-1}(x))$ is a single point. This follows because $g | g^{-1}(\mu(M \times 0)) = \pi | \pi^{-1}(\mu(M \times 0))$. As a consequence of this observation and elementary continuity considerations, we see that for any $x \in X$, if any point of $\mu^{-1} g(\pi^{-1}(x))$ lies sufficiently close to $M \times 0$ in $\mathrm{Cyl}(f)$, then all of $\mu^{-1} g(\pi^{-1}(x))$ will lie close to $M \times 0$. Hence, we can find an infinite sequence $1 = t_0 > t_1 > t_2 > \ldots > 0$ of points in $(0,1]$

chosen in order of increasing index, such that for each $x \in X$, the set $\mu^{-1}g(\pi^{-1}(x))$ intersects at most one level $M \times t_i$ in Cyl(f). For some large n (the largeness depending on the smallness of the original ε in the Bing Shrinking Criterion), let $\tau : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$ be the natural "semi-linear" sliding homeomorphism which changes only the $[0,1]$ coordinate, such that for each $0 \leq j \leq n$, $\tau(M \times t_j) = M \times (n+1-j)/(n+1)$.

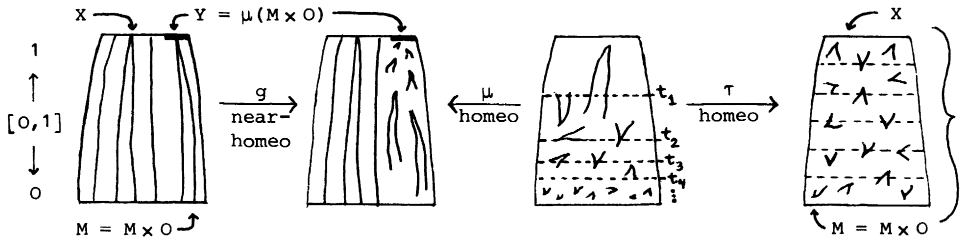

Figure 1 : The near-homeomorphism $h_* = \tau\mu^{-1}g : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f)$. The cones ($V$'s) in the successive frames indicate what the images of the $\pi^{-1}(x)$'s, $x \in X$, might look like.

The near-homeomorphism $h_* = \tau\mu^{-1}g$ has the desired properties. This completes the proof of the Approximation Theorem.

§ 7. Proof of Toruńczyk's Approximation Theorem : ℓ_2-manifold case

The proof follows the same outline as the proof in § 6, so we confine ourselves here to pointing out what adaptations are required. Recall that all spaces are assumed to be separable complete metric.

The Negligible Approximation Theorem and Negligible Homotopy-Isotopy Theorem (both in § 4) are still basic tools, and so must be established in the ℓ_2-manifold setting. In this new setting W is no longer taken to be locally compact, and $f : W \longrightarrow M$ is arbitrary, and the approximating negligible embedding $g : W \longrightarrow M$ is closed ; also the homotopy $f_t : W \longrightarrow M$ is arbitrary, and the negligible embeddings $f_0, f_1 : W \longrightarrow M$ should be closed. The proofs of these fundamental theorems are referenced in $[\mathrm{To}_3]$.

The Bing Shrinking Criterion and the Shrinking Theorem were generalized by Toruńczyk in the following natural but non-obvious manner.

SHRINKING THEOREM.- <u>A surjective map</u> $\pi : W \longrightarrow X$ <u>of complete metric spaces is approximable by homeomorphisms $\Leftrightarrow$ the following criterion holds : Given any majorant maps</u> $\varepsilon : X \longrightarrow (0,\infty)$ <u>and</u> $\eta : W \longrightarrow (0,\infty)$, <u>there is a majorant map</u> $\delta : X \longrightarrow (0,\infty)$ <u>and a homeomorphism</u> $h : W \longrightarrow W$ <u>such that</u>

(1) <u>for each</u> $w \in W$, $\text{dist}(\pi h(w), \pi(w)) < \varepsilon(\pi(w))$, <u>and</u>

(2) <u>for each</u> $x \in X$, $\text{diam } h(\pi^{-1}(N_{\delta(x)}(x)) < \inf \eta(h(\pi^{-1}(N_{\delta(x)}(x))))$.

(These statements look nicer when expressed using open covers instead of majorant maps.) The proof is a direct generalization of the usual proof. Using this theorem, it is an easy matter to establish that West's Mapping Cylinder Proposition (§ 4) holds for a fine homotopy equivalence $f : M \longrightarrow X$ from a ℓ_2-manifold M onto an ANR X , using the same proof. (However, the Stability Proposition (§ 4) ; with I^∞-manifold replaced by ℓ_2-manifold, is not to be proved yet. See below.)

Toruńczyk reformulated his discrete-cells property in the following manner.

PROPOSITION.- <u>Suppose</u> X <u>is an ANR</u> . <u>Then</u> X <u>has the discrete-cells property</u> $\Leftrightarrow$ <u>any map</u> $f : W \longrightarrow X$ <u>from a space</u> W <u>to</u> X <u>is approximable by a negligible closed embedding</u> $g : W \longrightarrow X$.

A proof can be given using arguments extending those in § 5.

Concerning the proof itself of the Approximation Theorem (as in § 6), the following points should be made. First, it is no longer trivial to prove that the projection $M \times [0,1] \longrightarrow M$ is a near-homeomorphism if M is a ℓ_2-manifold (cf. Stability Proposition, § 4). However, one can treat this as a special case of the main body of the proof, that $\pi : \text{Cyl}(f) \longrightarrow X$ is a near-homeomorphism, because the projection $M \times [0,1] \longrightarrow M$ is the mapping cylinder projection of $\text{Cyl}(\text{id}(M))$. In other words, the fact that the projection $M \times [0,1] \longrightarrow M$ is approximable by homeomorphisms is nothing more than a special case of the main goal, that $\pi : \text{Cyl}(f) \longrightarrow X$ is a near-homeomorphism when X has the discrete-cells property.

Concerning this main goal, perhaps the main points to be made in carrying over the arguments of § 6 to the ℓ_2-manifold setting are these. In the proof of the Proposition (where now $f : M \longrightarrow X$ is a fine homotopy equivalence from a ℓ_2-manifold M onto an ANR X), the "reduced" mapping cylinder ($\text{Cyl}(f)$ reduced over the closed subset Y of X), here denoted $\text{Cyl}(f)_Y$, is not topologized as a quotient, but rather as a complete metrizable space thus : writing $\text{Cyl}(f)_Y = (M - f^{-1}(Y)) \times [0,1) + X$, let $(M - f^{-1}(Y)) \times [0,1)$ with the product topology be an open subset ; let a basic neighborhood of $x \in X - Y$ be of the form $f^{-1}(U) \times (t,1) + U$, where U is a neighborhood of x in X and $t < 1$; and let a basic neighborhood of a point $y \in Y$ be of the form $f^{-1}(U) \times [0,1) + U$, where U is a neighborhood of y in X . The proof of the Proposition remains the same, showing that the natural map $q : \text{Cyl}(f) \longrightarrow \text{Cyl}(f)_Y$ is approximable by homeomorphisms. One minor change is that the constant b must now be a map $b : M \longrightarrow (0,1)$, chosen so that for each $z \in M$, $z \times [0,b(z)] \cap h_1(\pi^{-1}(Y)) = \emptyset$.

Similarly, in the main proof, the constants t_i must now be maps $t_i : M \longrightarrow (0,1]$, and the justification that they can be found is that given any map $t_{i-1} : M \longrightarrow (0,1]$, the set $\mu^{-1}g\pi^{-1}\pi g^{-1}\mu\{(z,t_{i-1}(z)) \mid z \in M\}$ is closed in Cyl(f) and is disjoint from $M \times 0$, and hence a smaller map $t_i : M \longrightarrow (0,1]$ can be chosen whose graph misses this set. Also, the number of the t_i's which are actually used now varies over different parts of M (just as in the locally compact case). These point-set topological details can be worked out by the patient reader, or can be consulted in [To_3].

Concerning Anderson's result that $\ell_2 \approx \text{int } I^\infty$, we note that it follows quickly from the preceding work, thus. By the Characterization Theorem, $\text{int } I^\infty$ is a ℓ_2-manifold. By the Approximation Theorem, each of the projections $\text{int } I^\infty \times \ell_2 \longrightarrow \ell_2$ and $\text{int } I^\infty \times \ell_2 \longrightarrow \text{int } I^\infty$, being fine homotopy equivalences, is a near-homeomorphism. Hence $\ell_2 \approx \text{int } I^\infty$.

APPENDIX 1. THE HOMOGENEITY OF THE HILBERT CUBE

One of the basic and surprising properties of the Hilbert cube is that it is <u>homogeneous</u>, that is, for all $x,y \in I^\infty$, there exists a homeomorphism $h : I^\infty \longrightarrow I^\infty$ such that $h(x) = y$. In order to illustrate the fundamental convergence-of-homeomorphisms fact which is used over and over again in infinite dimensional manifold topology, we recall briefly the proof of this homogeneity. Since it is easy to see that h exists for $x,y \in \operatorname{int} I^\infty$, it suffices to show that if $x \in \partial I^\infty$, then there is a homeomorphism $h : I^\infty \longrightarrow I^\infty$ such that $h(x) \in \operatorname{int} I^\infty$. One defines h as $h = \lim_{n\to\infty} \psi_n \cdots \psi_2 \psi_1$, where the ψ_j's are defined thus : Let $i(1)$ be the first ± 1 coordinate of x (that is, $x_{i(1)} = \pm 1$, and $x_j \in (-1,1)$ for $j < i(1)$). For some large $m = m(1) > i(1)$, define θ_1 to be a homeomorphism of the 2-disc $[-1,1]_{i(1)} \times [-1,1]_m$ (subscripts here denoting coordinates), supported arbitrarily near the face $x_{i(1)} \times [-1,1]_m$, such that the $i(1)$ coordinate of $\theta_1(x_{i(1)},x_m)$ lies in $(-1,1)$. Let $\psi_1 = \theta_1 \times$ "identity" : $I^\infty \longrightarrow I^\infty$, that is ψ_1 leaves fixed all but the $i(1)$-th and m-th coordinates of any point in I^∞, and these coordinates are changed by applying θ_1. Next, let $\psi_2 : I^\infty \longrightarrow I^\infty$ be a homeomorphism defined just as ψ_1 was defined, to make the first ± 1 coordinate of $\psi_1(x)$ (which is in the $i(2)$-th slot, say) to lie in $(-1,1)$, at the expense of making the $m(2)$-th coordinate to be ± 1, for some arbitrarily large $m(2)$. Continuing this way define ψ_3, ψ_4, etc. Observe that <u>each</u> ψ_j <u>can be constructed to be arbitrarily close to</u> $\operatorname{id}(I^\infty)$. On account of this, $h = \lim_{n\to\infty} \psi_n \cdots \psi_2 \psi_1$ can be made a homeomorphism because it is a limit of homeomorphisms, each of which can be constructed arbitrarily close to the preceding one, and this enables one to keep distinct points of I^∞ distinct in the limit. This establishes the homogeneity of I^∞.

APPENDIX 2. THE MILLER-WEST THEOREM

Here we sketch the proof of the Miller-West Theorem ([Mi] and [We]), mentioned in § 2. Actually, all that is needed for this article is Miller's half, so we focus on it (see Note 1 below). (Alternatively, one could use the Chapman-West infinite mapping cylinder construction to prove Miller's Theorem, as explained in [Ch_2, XIII], at least in the locally compact case.)

MILLER'S THEOREM.- Suppose X is a (separable complete metric) ANR .
(1) If X is locally compact, there is a proper fine homotopy equivalence (i.e. cell-like map) $f : M \longrightarrow X \times (0,1)$ from some I^∞-manifold M onto $X \times (0,1)$.
(2) In general, there is a fine homotopy equivalence $f : M \longrightarrow X$ from some ℓ_2-manifold onto X .

Note 1.- In the locally compact case, Statement (1) suffices as input for Toruńczyk's Characterization Theorem, hence for all of the corollaries, because one can deduce West's half of the Miller-West Theorem using the Approximation Theorem, as follows. Suppose X is a locally compact ANR . Combining the Approximation Theorem with Miller's Theorem, one obtains that $X \times (0,1) \times I^\infty$ is a I^∞-manifold. Hence so is $X \times D^2 \times I^\infty$ ($\approx X \times I^\infty$) , it being locally homeomorphic to $X \times (0,1) \times [0,1) \times I^\infty$ which is locally homeomorphic to $X \times (0,1) \times I^\infty$. Thus X is a cell-like image of the I^∞-manifold $X \times I^\infty$.

Note 2.- As suggested in the footnote in § 2, in the non-locally-compact case Statement (2) allows one to deduce Toruńczyk's ANR Theorem as a corollary to his Characterization Theorem.

Note 3.- In [Mi] Miller remained entirely in the finite dimensional setting, but the above versions of his theorem are obvious extensions, requiring no change in method of proof. The following proof is a simple reinterpretation of Miller's argument.

Proof of Miller's Theorem. The proofs of Statements (1) and (2) are virtually identical. For ease of exposition we focus on the case where X is compact.

We suppose X is embedded in I^∞ as a negligible compactum, and we show that $X \times (0,1)$ has a I^∞-manifold mapping cylinder neighborhood M in $I^\infty \times (0,1)$. The mapping cylinder retraction of M to $X \times (0,1)$ then serves as the desired cell-like map f .

The construction makes repeated use of the following

PROPOSITION.- Given any $\varepsilon > 0$, there is a surjective homotopy $f_t : I^\infty \longrightarrow I^\infty$, $0 \le t \le 1$, such that
(1) $f_0 = id(I^\infty)$ and each f_t , $0 \le t < 1$, is a homeomorphism (but not f_1),
(2) each f_t is ε-close to $id(I^\infty)$, and is the identity on $X \cup (I^\infty - N_\varepsilon(X))$, and

(3) $f_1^{-1}(X)$ is a neighborhood of X and f_1 is a homeomorphism over $I^\infty - X$, that is, $f_1| : f_1^{-1}(I^\infty - X) \longrightarrow I^\infty - X$ is a homeomorphism.

Proof of Proposition. Without loss X lies in a face $F = F \times 1$ of I^∞, where we write $I^\infty = F \times [-1,1]$. Since some neighborhood U of X in F can be retracted to X by a small homotopy of U in F starting at $id(U)$ and fixed on X, we can apply the negligible homotopy-isotopy principle in the ambient manifold I^∞ to find a homotopy $g_t : I^\infty \longrightarrow I^\infty$, $0 \leq t \leq 1$, satisfying (1) and (2) of the Proposition and (3') : $g_1^{-1}(X)$ is a neighborhood of X in F (such a $\{g_t\}$ can be gotten by taking an infinite stack of smaller and smaller ambient isotopies, provided by the Negligible Homotopy-Isotopy Theorem in § 4). Let $\pi_t : I^\infty \longrightarrow I^\infty$, $0 \leq t \leq 1$, be a homotopy satisfying (1) and (2) of the Proposition and (3") : $\pi_1^{-1}(g_1^{-1}(X)) = g_1^{-1}(X) \times [1 - \delta , 1]$ for some small $\delta > 0$ (such a $\{\pi_t\}$ can be gotten simply by crushing $g_1^{-1}(X) \times [1 - \delta , 1]$ to $g_1^{-1}(X) \times 1$, and damping this crush to the identity elsewhere in $F \times [-1,1] \approx I^\infty$). Then the homotopy $f_t = g_t\pi_t$ serves for the Proposition. □

The proof of the Theorem is divided into two steps.

Step 1. Here we use a construction reminiscent of the construction of a Urysohn function separating disjoint closed subsets of a normal space, to find a surjection $F : I^\infty \times (0,1) \longrightarrow I^\infty \times (0,1)$ having the following properties :

(1) F is level-preserving, that is, $F(I^\infty \times t) = I^\infty \times t$ for each $t \in (0,1)$,

(2) F is the identity on $X \times (0,1)$, and F is a homeomorphism over $(I^\infty - X) \times (0,1)$ (i.e., $F| : F^{-1}((I^\infty - X) \times (0,1)) \longrightarrow (I^\infty - X) \times (0,1)$ is a homeomorphism), and

(3) for $s,t \in (0,1)$, if $s < t$, then $\pi(F^{-1}(X \times s)) \subset \text{int}\, \pi(F^{-1}(X \times t))$, where $\pi : I^\infty \times (0,1) \longrightarrow I^\infty$ is the projection.

This last condition says that $F^{-1}(X \times (0,1))$ looks like a sort of inverted pyramid neighborhood of $X \times (0,1)$ in $I^\infty \times (0,1)$ (see Figure A2 below).

Given $\varepsilon > 0$ and $a \in (\varepsilon,1)$, we define first an auxiliary level preserving surjection $F[a,\varepsilon] : I^\infty \times (0,1) \longrightarrow I^\infty \times (0,1)$. Let $f_t : I^\infty \longrightarrow I^\infty$, $0 \leq t \leq 1$, be a homotopy provided by the Proposition for the given ε. Define $F[a,\varepsilon]$ by

$$F[a,\varepsilon] \mid I^\infty \times t = \begin{cases} \text{identity} & \text{if } t \leq a - \varepsilon \\ f_{(t-(a-\varepsilon))/\varepsilon} & \text{if } a - \varepsilon \leq t \leq a \\ f_1 & \text{if } a \leq t . \end{cases}$$

The definition is consistent, and $F[a,\varepsilon]$ is ε-close to $id(I^\infty \times (0,1))$ and satisfies properties (1) and (2) of the desired map F and in addition (3') : $F[a,\varepsilon]^{-1}(X \times (0,1)) = X \times (0,1) \cup f_1^{-1}(X) \times [a,1)$. Now given a sequence of positive numbers $\varepsilon_1,\varepsilon_2,\ldots$, define a sequence of maps $\{F_i : I^\infty \times (0,1) \longrightarrow I^\infty \times (0,1)\}$ thus :

$F_1 = F[1/2,\varepsilon_1]$

$F_2 = F[3/4,\varepsilon_2]F[1/2,\varepsilon_1]F[1/4,\varepsilon_2]$

$F_3 = F[7/8,\varepsilon_3]F[3/4,\varepsilon_2]F[5/8,\varepsilon_3]F[1/2,\varepsilon_1]F[3/8,\varepsilon_3]F[1/4,\varepsilon_2]F[1/8,\varepsilon_3]$

etc.

Each F_i satisfies properties (1) and (2) of the desired map F, and the sets $F_i^{-1}(X \times (0,1))$ look like this :

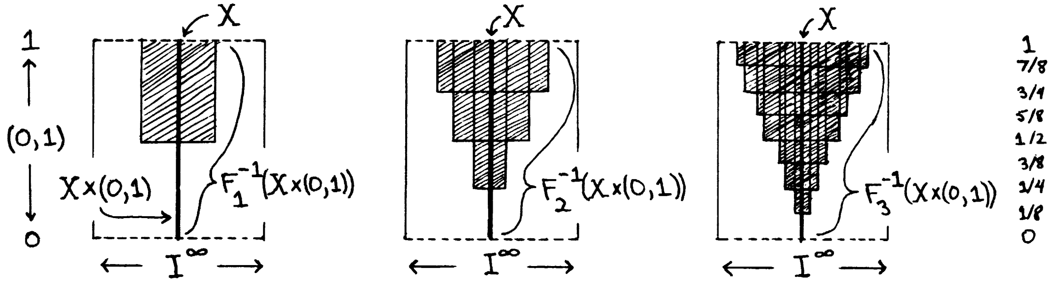

Figure A1

The desired map F of Step 1 is defined as $F = \lim_{i\to\infty} F_i$ for some sequence $\varepsilon_1,\varepsilon_2,\ldots$ which converges to 0 sufficiently rapidly.

Step 2. Let $\lambda : I^\infty \longrightarrow [0,1]$ be defined by $\lambda(z) = \sup\{t \in [0,1] \mid F(z,t) \notin X \times t\}$. By property (3) of F, λ is continuous. The set $L = \{(z,t) \in I^\infty \times (0,1) \mid t \geq \lambda^2(z)\}$ is a I^∞-manifold closed mapping cylinder neighborhood of $F^{-1}(X \times (0,1))$ in $I^\infty \times (0,1)$, where the mapping cylinder retraction is just vertical upward projection (see Figure A2). Consequently, the set $M = F(L)$ is the desired I^∞-manifold closed mapping cylinder neighborhood of $X \times (0,1)$ in $I^\infty \times (0,1)$.

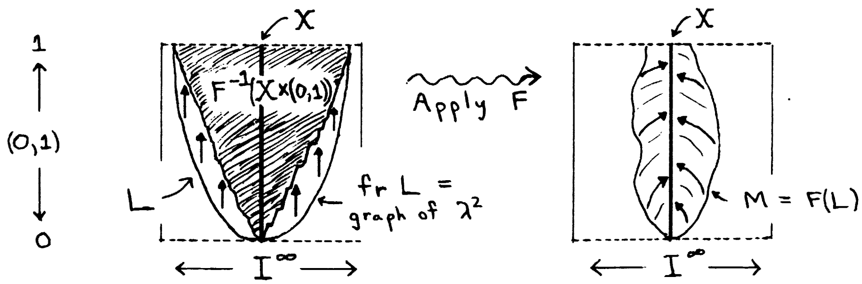

Figure A2

For the general locally compact case the construction of M is the same, except that one replaces I^∞ by $I^\infty \times R^1$ (in order to be able to embed X as a closed subset), and one must replace the fixed ε's by arbitrarily small majorant maps $\varepsilon : I^\infty \times R^1 \longrightarrow (0,1)$.

In the non-locally-compact case one replaces I^∞ (or $I^\infty \times R^1$) by ℓ_2 (or by $\text{int } I^\infty$, if you wish), and one continues to use arbitrarily small majorant maps $\varepsilon : \ell_2 \longrightarrow (0,1)$. Also, Step 2 can be shortened a bit (although it is sound as is), simply taking as L the (non-closed) neighborhood $\pi(F^{-1}(X \times (0,1))) \times (0,1)$ of $X \times (0,1)$ in $\ell_2 \times (0,1)$. The ℓ_2-manifold neighborhoods L and M are now only open mapping cylinder neighborhoods, but still the retraction $M \longrightarrow X \times (0,1) \longrightarrow X$ is a fine homotopy equivalence, as desired. □

REFERENCES

The fundamental reference, for an introdution to the subject as well as all the basic theorems (locally compact, I^{∞}-manifold case), is [Ch_2].

[An_1] R. D. ANDERSON - Topological properties of the Hilbert cube and the infinite product of open intervals, Trans. Amer. Math. Soc. 126(1967), 200-216. Announced in Bull. Amer. Math. Soc. 72(1966), 515-519. See also [A-B].

[An_2] R. D. ANDERSON - On topological infinite deficiency, Michigan Math. J. 14(1967), 365-383.

[An_3] R. D. ANDERSON - Strongly negligible sets in Fréchet manifolds, Bull. Amer. Math. Soc. 75(1969), 64-67.

[An_4] R. D. ANDERSON - Homeomorphisms on infinite-dimensional manifolds, Actes, Congrès International des Mathématiciens, Nice 1970, Tome 2, 13-18.

[A-B] R. D. ANDERSON and R. H. BING - A completely elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines, Bull. Amer. Math. Soc. 74(1968), 771-792.

[Bi] R. H. BING - A homeomorphism between the 3-sphere and the sum of two solid horned spheres, Annals of Math. 56(1952), 354-362.

[Ch_1] T. A. CHAPMAN - Cell-like mappings of Hilbert cube manifolds : Solution of a handle problem, General Top. and Appl. 5(1975), 123-145.

[Ch_2] T. A. CHAPMAN - Lectures on Hilbert Cube Manifolds, Regional Conference Series in Mathematics n° 28, Amer. Math. Soc. 1976.

[C-S] D. W. CURTIS and R. M. SCHORI - 2^X and $C(X)$ are homeomorphic to the Hilbert cube, Bull. Amer. Math. Soc. 80(1974), 927-931.

[Ed] R. D. EDWARDS - The ANR Theorem, Chapter 14 in [Ch_2].

[Hat] A. E. HATCHER - Higher simple homotopy theory, Annals of Math. 102(1975), 101-137.

[Hav] W. E. HAVER - Mappings between ANR's that are fine homotopy equivalences, Pacific Jour. Math. 58(1975), 457-461.

[Hu] S.-T. HU - Theory of Retracts, Wayne State University Press, 1965.

[La] R. C. LACHER - Cell-like mappings, I, Pacific Jour. Math. 30(1969), 717-731.

[Mi] R. T. MILLER - Mapping cylinder neighborhoods of some ANR's, Annals of Math. 103(1976), 417-427. Announced in Bull. Amer. Math. Soc. 81(1975), 187-188.

[S-W] R. M. SCHORI and J. E. WEST - The hyperspace of the closed unit interval is a Hilbert cube, Trans. Amer. Math. Soc. 213(1975), 217-235. Announced in Bull. Amer. Math. Soc. 79(1973), 1286-1291.

[To$_1$] H. TORUŃCZYK - Absolute retracts as factors of normed linear spaces, Fund. Math. 86(1974), 53-67.

[To$_2$] H. TORUŃCZYK - On CE-images of the Hilbert cube and characterization of Q-manifolds, Fund. Math., to appear.

[To$_3$] H. TORUŃCZYK - Characterizing Hilbert space topology, Preprint 143, Institute of Mathematics, Polish Academy of Sciences, 1978.

[To$_4$] H. TORUŃCZYK - An Introduction to Infinite-Dimensional Topology, in preparation.

[We] J. E. WEST - Mapping Hilbert cube manifolds to ANR's : a solution of a conjecture of Borsuk, Annals of Math. 106(1977), 1-18. Announced in Bull. Amer. Math. Soc. 81(1975), 163-165.