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Kleinian groups (a survey)

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KLEINIAN GROUPS
[A Survey]

by William James HARVEY

The study of discontinuous groups in the plane was begun in the 1880's by Schottky, Poincaré and Klein, although the main area of activity lay in specific subclasses. A good impression of the subject before 1960 can be gleaned from the treatise of Fatou [1].

Developments since then have been the introduction of quasi-conformal mappings by Ahlfors and Bers, which caused the modern revival in the general theory, and an amalgam of techniques due to Maskit in the topology of planar domains and coverings linked with methods of construction which are in direct descent from some work of Klein and Koebe. Recently the original idea of Poincaré to use the extended action on hyperbolic 3-space has been implemented by Marden using results of Waldhausen. Following the realisation by Greenberg of the significance of the assumption that fundamental polyhedra have finitely many faces, attention has naturally focussed on groups which satisfy this condition.

Among topics which go unmentioned for lack of space we list the study of automorphic forms and cohomology, and the structure of deformation spaces. Details and references for these will be found in [10,20].

§ 1. Generalities

We consider two closely related actions of the group $G_{\mathbb{C}} = SL_2(\mathbb{C})/\{\pm I\}$. The first is as the group of Möbius transformations of $\mathbb{P}_1\mathbb{C} = \mathbb{C} \cup \{\infty\}$:

$$(1.1) \qquad z \longmapsto T(z) = \frac{az + b}{cz + d} .$$

Here the subgroup of real matrices $G_{\mathbb{R}}$ preserves the upper and lower half-planes $\mathcal{U}$ and $\mathcal{L}$.

$G_{\mathbb{C}}$ also acts on the symmetric space $X \cong G_{\mathbb{C}} / SU_2(\mathbb{C})$, which is the 3-dimensional hyperbolic space $\mathcal{H} = \{(x , y , t) : t > 0\}$. The link with (1.1) is most neatly established by representing $\mathbb{R}^3$ as the set of quaternions

[1] Vol. II of Fonctions Algébriques by Appell and Goursat, Gauthier-Villars, 1930.

$\{q = x + y\underline{i} + t\underline{j} + 0.\underline{k}\}$ and writing

$$(1.2) \qquad T(q) = (aq + b)(cq + d)^{-1} .$$

It follows that $G_{\mathbb{C}}$ preserves the subspace $\mathcal{H}$ and the boundary $\partial\mathcal{H}$, which is naturally identifiable with $\mathbb{C} \cup \{\infty\}$; moreover (1.2) extends (1.1).

We shall need the familiar classification of elements of $G_{\mathbb{C}}$ as <u>elliptic</u>, <u>parabolic</u> or <u>loxodromic</u> (hyperbolic if in $G_{\mathbb{R}}$) according as they are conjugate to a rotation, a translation or a dilation.

Let Γ be a discrete subgroup of $G_{\mathbb{C}}$. Then Γ acts properly discontinuously on $\mathcal{H}$, as was proved by Poincaré, but the action on $\mathbb{P}_1\mathbb{C}$ may not be discontinuous. We denote by $\Lambda(\Gamma)$ the <u>limit</u> set of Γ , which is the set of accumulation points of Γ-orbits (any single Γ-orbit suffices to determine $\Lambda(\Gamma)$ unless Γ is cyclic), and write $\Omega(\Gamma)$ for the complementary set in $\mathbb{P}_1\mathbb{C}$, referred to as the <u>discontinuity region</u> or ordinary set of Γ .

DEFINITION.- A <u>Kleinian</u> group is a group Γ with $\Omega(\Gamma) \neq \emptyset$.

It follows easily for Kleinian groups that Λ is either finite (0 , 1 or 2 points), in which case the group is termed <u>elementary</u>, or a nowhere dense perfect set.

For discrete groups Γ one can form the quotient space $\mathcal{M}(\Gamma) = \mathcal{H}/\Gamma$, and if Γ is Kleinian there is an associated 3-manifold with boundary, $\mathcal{M}(\Gamma) = [\mathcal{H} \cup \Omega(\Gamma)]/\Gamma$. The boundary $\partial\mathcal{M}(\Gamma) = \Omega/\Gamma$ carries a complex structure and we consider it as a disjoint union of Riemann surfaces

$$(1.3) \qquad \Omega(\Gamma)/\Gamma = S_1 \cup S_2 \cup \dots \quad .$$

The connected components of Ω fall into corresponding Γ-conjugacy classes $\{\Omega_i\}$ covering the various S_i . If we denote by Γ_i the stabiliser in Γ of a given component $\Omega_i \subseteq \Omega$ then of course $\Omega_i/\Gamma_i = S_i$. Such a subgroup is termed a <u>component subgroup</u> of Γ .

An important class of Kleinian groups is those which possess a component $\Omega_0 \subseteq \Omega(\Gamma)$ which is preserved by all elements of Γ , i.e. the stabiliser Γ_0 is Γ. These groups are called <u>function groups</u> and Ω_0 is called an <u>invariant component</u>.

§ 2. Examples

(i) The elementary Kleinian groups consist of :

(a) the finite groups of symmetries of the regular solids,

(b) the infinite cyclic and dihedral groups (and finite extensions),

(c) the doubly periodic groups of translations (and finite extensions).

If $\Gamma = \langle T \rangle$, with T loxodromic, then $\mathfrak{M}(\Gamma)$ is a solid torus, while if T is parabolic then $\mathfrak{M}(\Gamma)$ is homeomorphic to $\{0 < |z| \leq 1\} \times (0,1)$ with boundary a twice-punctured sphere.

If $\Gamma = \langle z \mapsto z + w_1 , z \mapsto z + w_2 \rangle$, then $\mathfrak{M}(\Gamma)$ is homemorphic to $\{0 < |z| \leq 1\} \times S^1$ and $\partial\mathfrak{M}(\Gamma)$ is a torus.

(ii) A <u>Fuchsian group</u> is a Kleinian group which is conjugate in $G_{\mathbf{C}}$ to a subgroup of $G_{\mathbf{R}}$. The theory of these groups is quite highly developed and is a frequent source of motivation and a useful tool in the general theory. If Γ is a (finitely generated) Fuchsian group of the first kind preserving $\mathcal{U}$ (and $\mathcal{L}$) then $\mathcal{U}/\Gamma$ is a Riemann surface of finite topological type. In fact $\mathcal{U}/\Gamma$ is conformally a compact surface with a finite number of branch points and deleted points corresponding to the conjugacy classes of elliptic and parabolic elements of Γ , and on fixing a Poincaré area measure in $\mathcal{U}$, usually $d\mu = (2\,\mathrm{Im}\, z)^{-2}\, dx\, dy$, one can speak of the <u>Poincaré area</u> of $S = \mathcal{U}/\Gamma$ as $\mu(S) = \iint_D d\mu$, for D some suitable Borel-measurable fundamental set for Γ . By the Gauss-Bonnet theorem, one has

$$\mu(S) = -2\pi\chi(S)$$

if there is no torsion and S is compact.

For Fuchsian groups Γ the 3-manifold $\mathfrak{M}(\Gamma)$ is homeomorphic to the product of $S = \mathcal{U}/\Gamma$ with a closed interval, and the boundary is the disjoint union of S and its mirror image $\mathcal{L}/\Gamma$.

(iii) A <u>Schottky group</u> is a group generated by a finite set of loxodromic elements $T_1,\ldots,T_n$ for which there is a collection $\gamma_1 , \gamma_1' ; \ldots ; \gamma_n , \gamma_n'$ of mutually exterior Jordan curves (closed) in $\mathbb{P}_1\mathbf{C}$ such that T_i maps the exterior of γ_i onto the interior of γ_i' for $i = 1,\ldots,n$. The corresponding manifold is a handle body with boundary a single compact surface of genus n . It is easy to verify that Schottky groups are free. Conversely by a theorem of Maskit any free purely loxodromic Kleinian group is Schottky.

§ 3. Cusps, geometric finiteness and the structure of $\mathcal{M}(\Gamma)$

Let $d(\, , \,)$ denote hyperbolic distance in $\mathcal{H}$. For any discrete group $\Gamma \subseteq G_{\mathbf{C}}$ and base point $\sigma \in \mathcal{H}$ we define the Dirichlet polyhedron $\mathcal{D}(\Gamma, \sigma)$ by

$$\mathcal{D}(\Gamma, \sigma) = \bigcap \{h_\gamma, \gamma \in \Gamma\} ,$$

where h_γ is the half space $\{q \in \mathcal{H} : d(q, \sigma) \leq d(q, \gamma\sigma)\}$. It is easily verified that $\mathcal{D}$ is a fundamental set for Γ in $\mathcal{H}$ and if Γ is Kleinian then the Euclidean closure of $\mathcal{D}$ intersects $\partial\mathcal{H}$ in a fundamental set for Γ in $\Omega(\Gamma)$.

The naïve approach to a structure theorem for the manifolds $\mathcal{M}(\Gamma)$ using identification of congruent faces of a suitable fundamental set relies on there being finitely many faces essentially. This need not be true for finitely generated Kleinian (or merely discrete) groups (see § 4 for "degenerate" groups).

DEFINITION.- A discrete subgroup of $G_{\mathbf{C}}$ is called geometrically finite if it has a finite sided fundamental polyhedron in $\mathcal{H}$.

Note.- For such groups the polyhedron $\mathcal{D}(\Gamma, \sigma)$ is finite-sided for all points σ in $\mathcal{H}$.

To describe $\mathcal{M}(\Gamma)$ accurately we must first discuss the possible boundary points which do not lie in $\Omega(\Gamma)$. These correspond to the conjugacy classes of maximal parabolic subgroups of Γ and are usually referred to as the cusps of Γ.

Let $P \in \Lambda(\Gamma)$ be a parabolic fixed point of Γ and denote by M_P its Γ-stabiliser. Then M_P is either (a) free abelian of rank 2 or (b) infinite cyclic modulo torsion.

DEFINITION.- A horosphere at P is an open Euclidean ball in $\mathcal{H}$ tangent at P to $\partial\mathcal{H}$.

It is possible to find horospheres $\{\mathcal{B}_P\}$ for all parabolic fixed points of Γ, such that $\mathcal{B}_P$ is precisely invariant under M_P, i.e.,

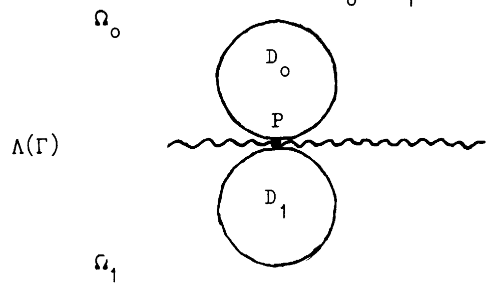
(i) $\quad T(\mathcal{B}_P^-) = \mathcal{B}_P^-$ $\quad$ for all $T \in M_P$;

(ii) $\quad$ if $T(\mathcal{B}_P^-) \cap \mathcal{B}_P \neq \emptyset$ $\quad$ with $T \in \Gamma$, $\quad$ then $T \in M_P$.

In case (a) above, one sees that $\mathcal{B}_P / M_P \cong \{0 < |z| < 1\} \times S^1$ (cf. example (i) (c) in § 2), and one refers to it as a solid cusp torus $\mathcal{T}_P$ for Γ at P. The collection $\{\mathcal{T}_P\}$ for a given group can be chosen to be mutually disjoint.

In case (b) the situation is more complicated. Certainly one has (cf. example (i) (b)) $\mathcal{B}_P / M_P \cong \{0 < |z| < 1\} \times (0,1)$ but one needs more information about the nature of $\Omega(\Gamma)$ near P. We distinguish those points P which are cusps (i.e. punctures) of some component surface S_o of Ω/Γ. Let Ω_o be a component of

$\Omega(\Gamma)$ corresponding to S_o and P. Then Ω_o contains a <u>horocyclic disc</u> D with $P \in \bar{D}$ and D precisely invariant under M_P, so that $(\partial D \setminus P)/M_P$ is a loop in S_o retractible to the puncture $\xi_o(P) \in S_o$. For some points P it is possible to have two associated punctures ξ_o, ξ_1 in $\Omega(\Gamma)/\Gamma$, either on the same surface S_o or on distinct ones S_o, S_1. In this case we say that ξ_o, ξ_1 are <u>paired</u> : there is then a pair of tangent horocyclic discs D_o, D_1 at P as pictured below



A simple example is the classical modular group $G_{\mathbb{Z}}$.

For points P representing two paired cusps (one might call P a bicuspidal point) there is an associated <u>solid cusp cylinder</u> $C_P \subseteq \mathcal{M}(\Gamma)$, conjugate to a set of the form

$$\{(x, y, t) \in \mathcal{H} \mid y^2 + t^2 > n^2\} / \langle T \rangle$$

where $T(z) = z + 1$. Here too the collection $\{\mathcal{C}_P\}$ for all P belonging to given Γ can be chosen disjoint.

Marden has given the following structure theorem for $\mathcal{M}(\Gamma)$.

THEOREM 1 [11].- Γ is geometrically finite if and only if there are a finite set of disjoint solid cusp tori and cylinders in $\mathcal{M}(\Gamma)$ such that the complement is compact.

The proof is similar to the classical one for Fuchsian groups.

<u>Remark</u>.- A further important consequence of the geometric finiteness assumption for Kleinian Γ is a result due to Ahlfors [3] that the limit set $\Lambda(\Gamma)$ has zero Lebesgue measure. It has been conjectured that this holds for <u>all</u> finitely generated Kleinian groups - counter examples are known [1] for infinitely generated groups.

§ 4. Quasi-conformal mappings and deformations of Kleinian groups

The powerful analytical methods developed by Ahlfors and Bers in the context of Teichmüller theory (see for example Gramain, Sém. Bourbaki, exposé 426 (1973)) make it possible to generate a large class of Kleinian groups by conjugation of a given one with suitable homeomorphisms of $\mathbb{P}_1\mathbf{C}$.

Recall that a homeomorphism f defined on a region $D \subseteq \mathbb{P}_1\mathbf{C}$ is called K-quasi-conformal on D (K finite) if it has generalised derivatives f_z , $f_{\bar z}$ which are locally in $L^2(D)$ and satisfy

$$|f_{\bar z}(z)| \leq \frac{K-1}{K+1} \; |f_z(z)|$$

almost everywhere in D . Notice that a 1-q-c mapping on D is conformal.

The fundamental result which is used in Kleinian group theory is the existence theorem below.

THEOREM 2 [4].- For each measurable complex function μ in $L^{\infty}(\mathbf{C})$ with $\|\mu\|_{\infty} = \frac{K-1}{K+1} < 1$, there is a unique K-q-c homeomorphism w^{μ} of $\mathbb{P}_1\mathbf{C}$ fixing 0 , 1 and ∞ and satisfying the Beltrami equation

$$(4.1) \qquad w_{\bar z} = \mu(z)\, w_z \; .$$

Moreover the rule $\mu \longmapsto w^{\mu}$ is holomorphic.

Remark.- There is an explicit estimate of w^{μ} as $\mu \to 0$ which is of great importance for the analytic study of deformations.

To apply this we observe that if Γ is Kleinian and w^{μ} is quasi-conformal on $\Omega(\Gamma)$, then the group

$$(4.2) \qquad \Gamma^{\mu} = w^{\mu} \circ \Gamma \circ (w^{\mu})^{-1}$$

is discontinuous on $w^{\mu}(\Omega)$. The deformed group Γ^{μ} is therefore Kleinian if and only if it is contained in $G_{\mathbf{C}}$, and this occurs precisely when

$$(4.3) \qquad \mu(Tz) \cdot \frac{\overline{T'(z)}}{T'(z)} = \mu(z)$$

for all $T \in \Gamma$ and $z \in \Omega$, since $w^{\mu} \circ T$ must be a solution to (4.1).

DEFINITION.- A function $\mu \in L^{\infty}$ with support in $D \subseteq \Omega(\Gamma)$ and $\|\mu\| < 1$ satisfying (4.3) is called a Beltrami coefficient for Γ on D . The corresponding group Γ^{μ} is called a q-c deformation of Γ .

An important class of groups arises in this way if we take for $\Gamma \subseteq G_{\mathbb{R}}$ a Fuchsian group of the first kind (with finite area). The space of Beltrami coefficients for Γ contains a subspace arising from the automorphic forms of weight 4 for Γ : if $B_4(\Gamma, \mathcal{U})$ denotes this space of cusp forms on $\mathcal{U}$ (with finite L^∞ norm : $\|\varphi\| = \sup|4y^2\,\varphi(z)|$) then the rule

$$\varphi \longmapsto \mu = \begin{cases} (4y^2)\,\overline{\varphi(z)} & \text{for } z \in \mathcal{U}, \\ 0 & \text{for } z \in \mathcal{L} \cup \mathbb{R} \end{cases}$$

determines a family of q-c deformations of Γ with support in $\mathcal{U}$. The corresponding Kleinian groups represent two homeomorphic surfaces of which one is always conformally equivalent to $\mathcal{L}/\Gamma$, while the other is (usually) different. A key result due to Bers [5] states that <u>any</u> pair of homeomorphic surfaces can be represented in this way. Such groups are called <u>quasi-Fuchsian</u>.

§ 5. <u>Schwarzian derivatives</u>, <u>Teichmüller space and degenerate groups</u>

Although we shall not discuss Teichmüller space here, it should be noted that the two theories have developed in parallel and there are important interactions of which we must describe one in order to introduce the mysterious class of groups known as <u>degenerate</u>.

Consider for a fixed Fuchsian group Γ the space $B(\Gamma) = B_4(\Gamma, \mathcal{L})$ of cusp forms on $\mathcal{L}$. For $\varphi \in B(\Gamma)$, we denote by w_φ the unique solution to the <u>Schwarzian differential</u> equation

$$(5.1) \qquad \{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \varphi$$

which satisfies $w_\varphi(z) = \frac{1}{z+i} + O(|z+i|)$ near $-i$, recalling that such solutions arise as quotients of solutions to the linear D.E .

$$(5.2) \qquad 2u'' + \varphi(z).u = 0 .$$

A direct computation shows that for $T \in \Gamma$, $w_\varphi \circ T$ is a solution to (5.1) and so $w_\varphi \circ T = \hat{T} \circ w_\varphi$, determining a deformation

$$\theta_\varphi : \Gamma \to G_{\mathbb{C}}$$

of Γ by $\theta_\varphi(T) = \hat{T}$.

<u>Note</u>.- If w_φ extends to a Γ-compatible q-c mapping of $\mathbb{P}_1$, then $\theta_\varphi(\Gamma)$ is of course quasi-Fuchsian. The set of all such φ is denoted $\mathcal{T}(\Gamma)$. It is the <u>Bers embedding</u> of Teichmüller space, a bounded domain in $B(\Gamma)$ which lies inside the ball

$$\|\varphi\| \leq 6$$

(see [7] and references there).

We are interested in the boundary of $\mathcal{T}(\Gamma)$ for the following reason.

THEOREM 3 [7].- Every point φ in $\partial\mathcal{T}(\Gamma)$ determines a univalent function w_φ, and the associated deformation Γ_φ is a Kleinian group with a simply connected invariant component which contains $w_\varphi(\mathcal{L})$.

The proof involves only the Hurwitz theorem on limits of univalent functions and the properties of equation (5.1).

DEFINITION.- A Kleinian group is termed a b-<u>group</u> if it has a simply connected invariant component.

In analogy with the theory of moduli for elliptic curves there is a subset of $\partial\mathcal{T}(\Gamma)$ referred to as the set of <u>cusps</u> of $\mathcal{T}(\Gamma)$ distinguished by the following property.

DEFINITION.- A <u>cusp</u> of $\mathcal{T}(\Gamma)$ is a Kleinian group Γ_φ which contains a parabolic element $T_\varphi = \theta_\varphi(T)$ with T hyperbolic in Γ. Such an element T_φ is termed <u>accidental parabolic</u> for Γ_φ, because it does not (as it should) determine a puncture of $w_\varphi(\mathcal{L})/\Gamma_\varphi$.

<u>Note</u>.- The nomenclature suggests that there should be a discontinuous group acting on $\mathcal{T}(\Gamma)$ which might contain elements fixing a cusp group, and indeed there is such a group - the Teichmüller modular group.

DEFINITION.- A b-group is called <u>totally degenerate</u> if it contains no accidental parabolic elements.

The question of existence of degenerate groups is settled by the observation that the condition for an element T_φ to be accidental parabolic in Γ_φ may be expressed as

$$|\mathrm{Trace}\ \theta_\varphi(T)|^2 = 4 ,$$

and this determines an analytic set in $B(\Gamma)$ for each of the countably many hyperbolic elements $T \in \Gamma$. Thus most points of $\partial\mathcal{T}(\Gamma)$ are not cusps.

THEOREM 4 [7].- A totally degenerate group $\Gamma_\varphi \in \partial\mathcal{T}(\Gamma)$ has discontinuity set the single (invariant) component, $w_\varphi(\mathcal{L})$.

For if Ω_1 is another component, then either it is Γ_φ-invariant in which case w_φ extends to $\mathbb{P}_1\mathbb{C}$ and Γ_φ is quasi-Fuchsian contradicting the assumption $\Gamma_\varphi \in \partial\mathcal{C}(\Gamma)$, or Ω_1 is not invariant so the inverse image $\Gamma_1 = \theta_\varphi^{-1}$ stab(Ω_1) in Γ , having infinite index in Γ , contains a boundary hyperbolic element which must have parabolic image in Γ_φ , contradicting degeneracy.

Remarks.- 1) No explicit example of a degenerate group is known, although there are clearly an uncountable number representing each conformal type of Riemann surface.

2) Bers has conjectured that the set of cusps of $\mathcal{C}(\Gamma)$ is dense in the boundary.

§ 6. The finiteness theorem

A crucial rôle in providing a focal point of interest and new methods was played by Ahlfors in proving the following result.

THEOREM 5 [2].- Let Γ be a finitely generated Kleinian group. Then the space $\Omega(\Gamma)/\Gamma$ consists of a finite number of surfaces S_i , each a Riemann surface of finite area.

In order to outline the method of proof we need to recall the Eichler cohomology of Γ .

Let V_q denote the vector space of polynomials in $\mathbb{C}[X]$ of degree $\leq 2q-2$, $q = 2, 3, \ldots$. Then $G_{\mathbb{C}}$ acts on V_q by the rule

$$V \cdot \gamma = (V \circ \gamma)\,(\gamma')^{1-q} ,$$

(for $q = 2$ one finds this is isomorphic to the adjoint action on the Lie algebra of $G_{\mathbb{C}}$). Denote by $H^1(\Gamma, V_q)$ the first cohomology group of Γ with respect to this action. One easily verifies that if Γ has N generators then $\dim H^1(\Gamma, V_q) \leq (2q-1)(N-1)$. Ahlfors' result follows from the fact that there is an injective mapping β_q (anti-linear) from the space of cusp forms $B_{2q}(\Gamma, \Omega(\Gamma))$ for Γ into $H^1(\Gamma, V_q)$ for each $q \geq 2$; for it is well known that these spaces are finite dimensional if and only if every component S_i is of finite area and they are finite in number, since $B_{2q}(\Gamma, \Omega) \cong \bigoplus_i B_{2q}(\Gamma_i, \Omega_i)$ with (Γ_i, Ω_i) running through a system of non-Γ-conjugate component subgroups.

We describe the map β_q for $q = 2$: let $\varphi \in B_4(\Gamma, \Omega)$. There is a function F_φ , continuous on $\mathbb{P}_1$, with

$$\frac{\partial F}{\partial \bar{z}} = \bar{\varphi}(z) \cdot \lambda(z)^{-2} \quad \text{on } \Omega \;, \quad (\lambda(z) = (2y)^{-1})$$

and $\frac{\partial F}{\partial \bar{z}} = 0$ on Λ, satisfying a growth condition $F(z) = O(|z|^2)$ at ∞. Such an F_φ can be written down explicitly. Now if $T \in \Gamma$ the expression

$$P_T(z) = F(T(z)).T'(z)^{-1} - F(z)$$

is a polynomial of degree ≤ 2, and $\{P_T\}$ is a 1-cocycle of Γ in V_2. To finish the proof for $q = 2$ one must show this rule is injective and this is carried out using a delicate estimate for $F_\varphi(z)$ as $z \to \Lambda$ and a smoothing operator which allows the use of Stokes' theorem to show $\iint_\Omega \bar{\varphi} \cdot \psi \cdot (\lambda(z))^{-2} \cdot dx\, dy = 0$ for all integrable holomorphic functions ψ on Ω.

A further development due to Bers [6] was the extension of this method to higher values of q. Using the Riemann-Roch theorem to write down dimensions of $B_{2q}(\Gamma_i, \Omega_i)$ and dividing by q, one obtains as $q \to \infty$ the formula for the area of S_i (up to multiple 2π). This implies the result

THEOREM 6.- The total area of Ω/Γ is at most $4\pi(N - 1)$, if Γ has N generators.

Remark.- More precise relations between N and Ω/Γ can be derived in certain cases. If Γ is torsion free then Ω/Γ has at most $2(N-1)$ components, while if Γ is loxodromic there are at most $N-1$.

In fact a purely topological argument (see [11]) shows that if S_i has genus g_i then $\Sigma g_i \leq N$ (the kernel of the surjection $H_1(\partial\mathfrak{M}(\Gamma)) \to H_1(\mathfrak{M}(\Gamma))$ has dimension $\frac{1}{2} \dim H_1(\partial\mathfrak{M})$). Therefore if Γ is loxodromic the number of components is at most $N/2$.

§ 7. Function groups and b-groups

The first step in classification of Kleinian groups is to study the component subgroups and these are function groups (§ 1). We shall sketch the classification of these, due to Maskit and, in the torsion free case, worked out by Marden using 3-manifolds.

It is natural to consider first the groups with two components.

THEOREM 7 [12].- A finitely generated group with two invariant components is quasi-Fuchsian.

By Ahlfors' theorem 5 each component Ω_1, Ω_2 of Γ with stabiliser Γ_1, resp. Γ_2, represents a finitely punctured surface. For elementary reasons there are no other components and each Ω_i is simply connected. The Riemann mappings $F_1 : \Omega_1 \to \mathcal{U}$, $F_2 : \Omega_2 \to \mathcal{L}$ determine type-preserving isomorphisms ψ_i, $i = 1, 2$, of Γ onto Fuchsian groups Γ_1', Γ_2' and by the Nielsen-Fenchel theorem [1] (see [ ]) there is a homeomorphism $f : \mathcal{U} \to \mathcal{U}$ which realises

$$\psi = \psi_2 \circ \psi_1^{-1} : \Gamma_1' \to \Gamma_2' \ ;$$

in fact f may be assumed piecewise linear (hence q-c). Therefore setting $f_1 = f \circ F_1$, and $\mu(z) = (f_1)_{\bar{z}} / (f_1)_z$ for $z \in \Omega_1$ and 0 elsewhere, one obtains by theorem 2 a global homeomorphism w^μ of $\mathbb{P}_1\mathbb{C}$ and a Kleinian group Γ^μ having two invariant components and possessing an automorphism θ induced by an orientation-reversing homeomorphism g of $\Omega(\Gamma^\mu)$:

$$g = w^\mu \circ F_2^{-1} \circ j \circ f_1 \circ (w^\mu)^{-1} \quad \text{on} \quad w^\mu(\Omega_1)$$, with j denoting complex conjugation.

It follows also that θ preserves the trace of each element of Γ^μ and this implies by a standard argument that θ extends to an automorphism of $G_{\mathbb{C}}$. Hence g is anti-Möbius and Γ^μ is Fuchsian.

THEOREM 8 [11].- If Γ has two invariant components then $\mathfrak{M}(\Gamma)$ is the product of a surface Ω_1/Γ with the interval $[0,1]$.

(For compact Ω_1/Γ this is a result of Waldhausen.)

The argument involves choosing a set $\{\alpha_j\}$ of simple loops in $S_1 = \Omega_1/\Gamma$, each intersecting only its successor (transversely and in one point). Using the Cylinder theorem [19] there is a corresponding set $\{\alpha_j'\}$ on S_2 and pairing cylinders C_j in $\mathfrak{M}(\Gamma)$ having the same intersection properties. Hence

$S_1 \setminus \{S_1 \cap \bigcup_j C_j\}$ is a disc, and similarly for S_2 with the result that

$\mathfrak{M}(\Gamma) \setminus \{\bigcup C_j\}$ is a 3-ball since it has boundary containing a sphere. The result now follows in the compact case. The non-compact case is more delicate, using theorem 1.

For degenerate b-groups the structure of $\mathfrak{M}(\Gamma)$ is still a mystery. Here is one reason why.

THEOREM 9 [9].- A degenerate b-group is not geometrically finite.

Proof. For simplicity assume Γ is a purely loxodromic group, degenerate and geometrically finite. By theorem 4, $\partial\mathfrak{M}(\Gamma) = \Omega/\Gamma$ is connected. Taking the double

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1 See A. Marden, Isomorphisms between Fuchsian groups, Lecture Notes in Math., vol. 505, p.56-71, Springer, 1974.

$\hat{\mathfrak{M}}$ of $\mathfrak{M}$ one finds that the Euler characteristics satisfy $\chi(\hat{\mathfrak{M}}) = 0 = 2\chi(\mathfrak{M}) - \chi(\partial\mathfrak{M})$. But $\chi(\partial\mathfrak{M}) = \chi(\mathfrak{M})$ here, so $\chi(\partial\mathfrak{M}) = 0$ which implies Ω/Γ is a torus ; this is impossible for a loxodromic group of this type.

If Γ is a b-group with no degenerate subgroups, the structure of $\mathfrak{M}(\Gamma)$ is analogous to that in theorem 8. If $\mathfrak{M}_0(\Gamma)$ denotes the result of removing all solid cusp cylinders in $\mathfrak{M}(\Gamma)$ which are disjoint from S_1 , then $\partial\mathfrak{M}_0(\Gamma)$ is not connected and it can again be shown that $\mathfrak{M}_0$ is a product $S_1 \times [0,1]$. The cusp cylinders here correspond to the conjugacy classes of primitive accidental parabolic elements of Γ . The various boundary surfaces S_i with $i \neq 1$ fit together using these linkings by paired cusps (see § 2) to make up a 2-complex K of the same homotopy type as S_1 , and one finds that

(7.1) $\quad$ Total area of $\Omega/\Gamma = 2$ (Area of S_1) .

When Γ contains degenerate subgroups some part of the complex K is missing, and the equality in (7.1) becomes a strict inequality.

In a recent series of papers Maskit has extended the above heuristic description to give a structure theorem for finitely generated function groups. It is appropriate to use the notion of a graph of groups in formulating his result (see [18] for the definition).

THEOREM 10 [14].- Associated to each function group Γ there is a graph of groups $(\mathcal{G}, K)$, whose vertex groups are either Fuchsian or elementary Kleinian groups and whose edge groups are elliptic or parabolic cyclic (or trivial), such that Γ is isomorphic to $\pi_1(\mathcal{G}, K)$.

Remark.- The vertex groups can be identified up to conjugacy in Γ as maximal subgroups H with a simply connected invariant component, having no accidental parabolic elements and containing all parabolic elements of Γ which have fixed point in $\Lambda(H)$.

The method of proof is to use the planarity of the covering $\Omega_1 \to S_1 = \Omega_1/\Gamma$ to find loops $\{w_j\}$ in S_1 which lift to loops in Ω_1 when raised to some power $\alpha_j \leq \infty$. These generate the defining group of the covering under normal closure in $\pi_1(S_1)$ and they give rise to the decomposition K of S_1 .

From theorem 10 one can then prove an isomorphism theorem : Γ and Γ' are geometrically isomorphic if and only if their graphs are isomorphic, and the classical notion of signature for describing the geometric type of a Fuchsian group extends naturally to the framework of the graph $(\mathcal{G}, K)$.

§ 8. Constructions

Combination methods formulated originally by Klein and used by Nielsen and Fenchel in Fuchsian group theory have been significantly developed by Maskit into a powerful tool for generating both examples and models for classification. The technique constitutes a geometrical realisation in the plane of familiar constructions in topology and group theory. Here are some characteristic examples.

(a) Free products. Given a Kleinian group Γ and two disjoint discs D_1 , D_2 in $\Omega(\Gamma)$ with a Möbius transformation $T : \mathbb{P}_1 \setminus D_1 \rightarrow D_2$, the group $\Gamma_1 = \langle \Gamma , T \rangle$ is Kleinian and $\Gamma_1 \cong \Gamma * \langle T \rangle$. This is the process of attaching a handle to a component of $\partial \mathcal{M}(\Gamma)$ or a connecting link between two components.

(b) Amalgamated products. Take horocyclic discs at two punctures in $\Omega(\Gamma)$, $\Omega(\Gamma')$, and assume things so arranged that the discs are complementary in $\mathbb{P}_1$ and the stabilising subgroups coincide ($= H$, say). Then $\Gamma_1 = \langle \Gamma , \Gamma' \rangle$ is Kleinian and isomorphic to $\Gamma *_H \Gamma'$.

(c) H.N.N. extensions. Take two horocyclic discs in $\Omega(\Gamma)$ and a transformation T , permuting the stabilising subgroups and taking the exterior of one disc onto the interior of the other. Then $\langle \Gamma , T \rangle$ is Kleinian and isomorphic to $\Gamma * \langle T \rangle$.

Using (b) and (c) one can construct from Fuchsian groups models of all non-degenerate b-groups [12], and the method can be extended to encompass function groups too [14].

§ 9. Marden's isomorphism theorem

Using methods of Waldhausen [19], Marden has proved the following result on isomorphisms of Kleinian groups without torsion.

THEOREM 11.- Let Γ be geometrically finite and $\varphi : \Gamma \rightarrow \Gamma'$ an isomorphism onto a discrete group Γ' . Assume in addition that φ is induced by a q-c homeomorphism of $\Omega(\Gamma) \rightarrow \Omega(\Gamma')$. Then f lifts to a q-c map of $\mathcal{H} \rightarrow \mathcal{H}$ which induces φ and Γ' is geometrically finite.

We note that if f is in fact conformal on $\Omega(\Gamma)$ then it is the restriction of a Möbius transformation since $\Lambda(\Gamma)$ has zero area. The main tool in proving the theorem is the geometric realisation of isomorphisms between $\pi_1(\mathcal{M}(\Gamma))$ and $\pi_1(\mathcal{M}(\Gamma'))$ which respect the injections of $\pi_1(\partial \mathcal{M}(\Gamma))$ and $\pi_1(\partial \mathcal{M}(\Gamma'))$. This is possible because the manifolds $\mathcal{M}(\Gamma)$, $\mathcal{M}(\Gamma')$ are "Waldhausen" manifolds (see Gramain, Sém. Bourbaki, exposé 485 (Juin 1976), Proposition 9).

§ 10. Euler characteristics

By Maskit's decomposition theorem 10, one sees that the notion of Euler characteristic function (see for example [8]) extends to the class of finitely generated function groups and this can be used to derive an inequality for the total area of such groups which makes more precise the immediate estimate, due to Bers, which reads in usual notation

$$\text{Total Area of } \Omega/\Gamma \leq 2 \text{ (Area of } \Omega_1/\Gamma) .$$

One is naturally tempted to conjecture that all finitely generated Kleinian groups have finite Euler characteristic i.e., belong to Chiswell's class $FP(\mathbb{Q})$. This would follow if they were all constructible using Maskit's methods. Certainly the class of geometrically finite Kleinian groups are in $FP(\mathbb{Q})$ as it is known by results of Waldhausen that any such torsion-free group has a 3-manifold decomposition into a union of balls using a sequence of incompressible surfaces. Unfortunately there is no known a priori method of carrying this out.

There is an interesting example due to Jørgensen [1] of a discrete subgroup Γ of $G_{\mathbb{C}}$ (not Kleinian) in $FP(\mathbb{Q})$ which is a normal subgroup of infinite index in a group Γ^* . Here $\Gamma^* = \langle X,Y,T \ ; \ (XYX^{-1}Y^{-1})^n = 1 \quad TXT^{-1} = XY^{-1}, \ TYXT^{-1} = Y\rangle$ and $\Gamma = \langle X,Y \ ; \ (XYX^{-1}Y^{-1})^n = 1\rangle$. Note that Γ is isomorphic to a Fuchsian group so that $\chi(\Gamma) = -(n-1)/n$, whereas Γ^* has Euler characteristic 0 because χ/Γ^* is a compact 3-manifold. The group Γ is of course not geometrically finite.

§ 11. Comments

1) One would like to know more about discrete groups with finite volume in $G_{\mathbb{C}}$. Riley [17] has given a number of examples of knot and link groups admitting faithful representations into $G_{\mathbb{C}}$, and has deduced that the knot complement $R^3 \setminus k$ admits a structure of hyperbolic 3-space form usually with finite volume. The simplest examples are the figure-8 knot and the Boromean rings, which correspond to subgroups of $SL_2(\mathcal{O}_d)$ where $\mathcal{O}_d$ denotes the ring of integers in $\mathbb{Q}(\sqrt{-d})$, with d = 3 , 1 respectively. The knot k (or link) appears as the omitted central axis of the solid cusp torus (or tori) associated to the group.

2) Two major obstacles to progress in Kleinian groups are the shortage of explicit examples of uniformisations of Riemann surfaces (as opposed to families of examples which are plentiful) and the lack of information on degeneracy, which we have formu-

[1] Compact 3-manifolds of constant negative curvature fibering over the circle.

lated here only for b-groups. Known examples without invariant component exist (see [15], [1]) but no general pattern is known.

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Added postscript : A similar example to that of Jørgensen, (§ 10) but whithout torsion, has been discovered by Riley (oral communication) : inside the group $\mathcal{G}$ of a figure-8 knot embedded in $SL_2(O_3)$, the commutator subgroup $J = [\mathcal{G},\mathcal{G}]$ has infinite index and is free of rank 2 with a parabolic subgroup which is not cusped, so J is not geometrically finite.