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RECENT ADVANCES IN ENVELOPING ALGEBRAS OF SEMI-SIMPLE LIE-ALGEBRAS

[a report on work of
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by Walter BORHO

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie-algebra, G its adjoint group, and $U(\mathfrak{g})$ its universal associative enveloping algebra. The enveloping algebra $U(\mathfrak{g})$ may be thought of as a "non-commutative version" of the symmetric algebra $S(\mathfrak{g})$, the deviation from commutativity being given by the Lie-bracket on $\mathfrak{g}$ (cf. 2.1 for a more precise statement). The particular interest in $U(\mathfrak{g})$ arises from the study of representations of the group G : To a certain extent the representation theory of G is determined by that of its Lie-algebra $\mathfrak{g}$, and representations of $\mathfrak{g}$ (or $\mathfrak{g}$-modules) are nothing else but representations of $U(\mathfrak{g})$ (or $U(\mathfrak{g})$-modules).

One would like to know all irreducible representations of $\mathfrak{g}$. The finite-dimensional ones have been known for a long time (E. Cartan), but a complete classification of the general infinite-dimensional irreducible representations seems to be out of reach at present. However, a rough classification may be obtained by looking at the kernels in $U(\mathfrak{g})$. The kernel of a representation of $U(\mathfrak{g})$ is an ideal ("ideal" means "two-sided ideal"). The kernel of an irreducible representation (in other words, the annihilator of a simple module) is called a <u>primitive ideal</u>. For instance, all maximal ideals are primitive.

Let $\underline{\underline{X}}$ denote the set of all primitive ideals of $U(\mathfrak{g})$. This set is the correct generalization to non-commutative algebra of the "maximal spectrum" considered in commutative algebra. In particular, one can define a topology on $\underline{\underline{X}}$, the so-called Jacobson-topology, in the analoguous way (see [11], § 1).

<u>Main Problem</u> : <u>Give a complete description of</u> $\underline{\underline{X}}$, <u>as a set and as a topological space</u> !

The recent advances to be reported on here are all concerned with this main problem. The <u>methods</u> employed for studying this problem split into three groups, corresponding to the following three fields :

I. <u>non-commutative algebra</u>

II. <u>representation theory</u>

III. <u>algebraic geometry</u>

I. The first type of method uses non-commutative localization techniques and explicit calculations in $U(\mathfrak{g})$ to get hold of the ideals (even in terms of generators). This was begun by Dixmier, and in [1], and was developed to a certain state of technical perfection by Joseph.

II. The aim of the representation theorists was to find a particular well-understood class of simple modules, such that each primitive ideal occurs already as an annihilator of a module in this class. Here the breakthrough was made by Duflo (after preliminary work of Dixmier and Conze-Duflo). This opened the way to translate (see [12]) certain results of Jantzen on the classification of those modules into results on the classification of primitive ideals.

III. The philosophy of the third group of methods is to study the non-commutative algebra $U(\mathfrak{g})$ by comparing it with several closely related <u>commutative</u> algebras : namely a) its centre, b) the associated graded algebra, c) the ring of polynomial functions on $\mathfrak{g}$ (or $\mathfrak{g}^*$), and d) the symmetric algebra of a Cartan-subalgebra. This amounts to comparing the space $\underline{\underline{X}}$ with certain (more or less well-known) geometrical objects, for instance G-orbits in $\mathfrak{g}^*$ (or $\mathfrak{g}$). This point of view was introduced by Dixmier, and further developed in [2], [3], [12].

There are various relationships between the methods I, II, III, and the only adequate way to study $U(\mathfrak{g})$ seems to be a combination of all three. However, in the space available for this exposé, it is not possible to adequately discuss all three. Therefor non-commutative algebra is only touched upon here (§ 1), and representation theory is a bit neglected (§§ 4, 5), in favour of a more thorough discussion of the geometrical aspects (§§ 2, 3, 4, 6). Unfortunately, a report on the geometrical description of the orbit space $\mathfrak{g}^*/G$ as a "complex" of algebraic varieties (using Dixmier's notion of "sheets" of orbits of the same dimension), although essential for the proof of theorem 6.7, would lead us too far away from enveloping algebras and had to be omitted. Only a very few proofs have been indicated, and those very sketchily. As a kind of compensation, each section includes an open problem or a conjecture at the end. This seemed to me to be a proper way to characterize the present state of this quickly developing theory.

1. Structure of residue class algebras

1.1. A primitive ideal J of $U(\mathfrak{g})$ is called induced, if there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a finite dimensional simple $\mathfrak{p}$-module E such that J is the annihilator ideal of the $\mathfrak{g}$-module induced from E, i.e.

$$J = \operatorname{Ann} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E .$$

It is very useful to find out whether a given primitive ideal J is induced, because then we know a lot about the structure of its residue class algebra $U(\mathfrak{g})/J$.

1.2. For a finite dimensional vector space F with basis $x_1,\ldots,x_n$ denote by $\operatorname{Diff} F$ the algebra of "differential operators" on the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$, generated by the partial derivations $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and (the multiplications with) $x_1,\ldots,x_n$. This is a non-commutative algebra without zero-divisors.

THEOREM (Conze [4]).- Let J, $\mathfrak{p}$, E be as in 1.1. Then there is an embedding of algebras $i : U(\mathfrak{g})/J \hookrightarrow \operatorname{End} E \otimes \operatorname{Diff} \mathfrak{g}/\mathfrak{p}$.

COROLLARY.- If $\dim E = 1$, then $U(\mathfrak{g})/J$ has no zero-divisors.

1.3. The Gelfand-Kirillov-dimension (GK-dimension) $\operatorname{Dim} A$ of a finitely generated algebra A is defined as follows : Fix a finite set of generators, denote for each $n \in \mathbb{N}$ by A_n the linear span of all monomials of degree $\leq n$ in these generators, and take the infimum of all real numbers d such that $\dim A_n \leq n^d$ for large n. This infimum d is easily seen to be independent of the choice of generators, and hence is an invariant of A, denoted $\operatorname{Dim} A$, the GK-dimension of A.

Examples : For commutative A, this is just the Krull-dimension. For finite-dimensional A, $\operatorname{Dim} A = 0$. For $\operatorname{Diff} F$ as in 1.2, $\operatorname{Dim} \operatorname{Diff} F = 2 \dim F$.

1.4. COROLLARY.- In the theorem, we have $\operatorname{Dim} U(\mathfrak{g})/J \leq 2 \dim \mathfrak{g}/\mathfrak{p}$.

Proof. $\operatorname{Dim} U(\mathfrak{g})/J \leq \operatorname{Dim} \operatorname{End} E \otimes \operatorname{Diff} \mathfrak{g}/\mathfrak{p} = \operatorname{Dim} \operatorname{End} E + \operatorname{Dim} \operatorname{Diff} \mathfrak{g}/\mathfrak{p} = 2 \dim \mathfrak{g}/\mathfrak{p}$. In fact $\operatorname{Dim} U(\mathfrak{g})/J = 2 \dim \mathfrak{g}/\mathfrak{p}$ holds. For the reason see 2.5.

1.5. If I, J are primitive ideals with $I \subset J$, then $\operatorname{Dim} U(\mathfrak{g})/I = \operatorname{Dim} U(\mathfrak{g})/J$ implies $I = J$ [18]. This nice property is one reason why GK-dimension is very useful in classification of primitive ideals.

1.6. The Goldie-rank $\operatorname{rk} A$ of an algebra A is defined as the maximal number n

such that A contains a direct sum of n nonzero left ideals.

Examples : In 1.2, we have $\mathrm{rk}\,\mathrm{End}\,E = \dim E$, but $\mathrm{rk}\,\mathrm{Diff}\,F = 1$.

If $A = U(\mathfrak{g})/J$ for a primitive ideal J , then the Goldie-rank has two important interpretations (which are immediate consequences of general ring-theoretical theorems of Goldie resp. Faith-Utumi).

1) There exists a ring of quotients of A isomorphic to a full ring of $n \times n$-matrices over some skew field, where $n = \mathrm{rk}\,A$. (A "ring of quotients" of A is an overring consisting of elements $s^{-1}r$ with $s , r \in A$, s not a zero-divisor.)

2) The maximal order of a nilpotent element in A is exactly $\mathrm{rk}\,A$. (The order of a nilpotent element b is the minimal m such that $b^m = 0$.)

1.7. COROLLARY.- For E , J as in the theorem, we have $\mathrm{rk}\,U(\mathfrak{g})/J \leq \dim E$.

This is immediate from the Faith-Utumi theorem.

1.8. In this Corollary, equality does not hold in general. But it does in "almost all" cases, that is, under certain restrictions on E (excepting those E with weights on certain "exceptional hyperplanes"), as was shown by Conze and Duflo [5]. Utilizing their results (and under the same restrictions), Joseph [15] can prove the following much more precise structure theorem :

The embedding i in Conze's theorem extends to an isomorphism

$$Q(U(\mathfrak{g})/J) \xrightarrow{\sim} \mathrm{End}\,E \otimes Q(\mathrm{Diff}\,\mathfrak{g}/\mathfrak{p}) ,$$

where $Q(\ldots)$ denotes the maximal ring of quotients (1.6). ($Q(\mathrm{Diff}\,\mathfrak{g}/\mathfrak{p})$ exists and is a skew field.) The proof depends on complicated non-commutative algebra developed in [17].

1.9. The adjoint group G of $\mathfrak{g}$ acts on $U(\mathfrak{g})$ and on $U(\mathfrak{g})/J$ locally finitely. This action of G is completely reducible by Weyl's theorem ; each irreducible representation of G occurs with a certain multiplicity. These multiplicities of $U(\mathfrak{g})/J$ are computed by Conze and Duflo [5] for "almost all" induced primitive ideals J (restrictions as in 1.8). They provide a powerful tool for the classification of induced ideals (cf. 6.8).

1.10. It was discovered only recently that there exist also non-induced primitive ideals (in case $\mathfrak{g} = \mathfrak{sp}_4$, Conze-Dixmier). Joseph [14] constructs for each simple

$\mathfrak{g}$ except $\mathfrak{sl}_n$ a particular non-induced ideal, which has Goldie-rank 1 and the least possible GK-dimension > 0 (and is characterized by these properties). In [12] we show that various further non-induced primitive ideals exist (in general), and that they always occur in infinite families.

Problem.- Compute the GK-dimensions, Goldie-ranks, and multiplicities of (the residue class algebras of) these non-induced ideals !

The Goldie-ranks are known only in case $\mathfrak{g} = \mathfrak{sp}_4$ [16].

2. Associating to each primitive ideal a cone in $\mathfrak{g}$

2.1. We may consider the symmetric algebra $S(\mathfrak{g})$ as the associated graded algebra of $U(\mathfrak{g})$ (Theorem of Poincaré-Birkhoff-Witt). More precisely, $U(\mathfrak{g})$ has a natural filtration by finite dimensional subspaces

$$U_0(\mathfrak{g}) = \mathbb{C} \ , \quad U_1(\mathfrak{g}) = \mathbb{C} + \mathfrak{g} \ , \quad U_2(\mathfrak{g}) = \mathbb{C} + \mathfrak{g} + \mathfrak{g} \cdot \mathfrak{g} \ , \ \ldots$$

such that for each $n \in \mathbb{N}$, $U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is canonically isomorphic to the n-th symmetric power $S^n(\mathfrak{g})$. Denote by gr_n the canonical linear map $U_n(\mathfrak{g}) \to S^n(\mathfrak{g})$. For each subspace $V \subset U(\mathfrak{g})$ define $gr\, V \subset S(\mathfrak{g})$ by

$$gr\, V := \sum_n gr_n(V \cap U_n(\mathfrak{g})) \ .$$

If I is an ideal of $U(\mathfrak{g})$, then $gr\, I$ is an ideal of $S(\mathfrak{g})$, the "associated graded ideal" of I.

Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the Killing-form, we consider $S(\mathfrak{g})$ as the ring of polynomial functions on $\mathfrak{g}$.

DEFINITION.- To each ideal I of $U(\mathfrak{g})$, we associate the variety

$$\underline{\underline{K}}\, I := \{x \in \mathfrak{g} \mid f(x) = 0 \ \text{ for all } \ f \in gr\, I\} \ .$$

Since $gr\, I$ is a homogeneous ideal, $\underline{\underline{K}}\, I$ is a closed cone in $\mathfrak{g}$. Therefor $\underline{\underline{K}}\, I$ is called the "associated cone" of I.

2.2. THEOREM.- Let I be an ideal of $U(\mathfrak{g})$.

a) The closed cone $\underline{\underline{K}}\, I \subset \mathfrak{g}$ is stable under the action of the adjoint group G.

b) The GK-dimension of I is given by $\mathrm{Dim}\, U(\mathfrak{g})/I = \dim \underline{\underline{K}}\, I$.

c) For I primitive, $\underline{\underline{K}}\, I$ is contained in the cone of nilpotent elements in $\mathfrak{g}$.

2.3. As is well-known, the number of G-orbits of nilpotent elements in $\mathfrak{g}$ is finite (Dynkin). Their dimensions are even. Hence we conclude from 2.2 :

COROLLARY [18].- For each primitive ideal J, the associated cone is a finite union of (nilpotent) G-orbits. The GK-Dimension of J is equal to the dimension of some nilpotent orbit. In particular, the GK-dimension is even.

2.4. Let us sketch a proof of theorem 2.2c). As a primitive ideal, I contains a maximal ideal p of the center $Z(\mathfrak{g}) = U(\mathfrak{g})^G$ of $U(\mathfrak{g})$. Hence $\operatorname{gr} J \supset \operatorname{gr} p$. It is easy to see that $\operatorname{gr} Z(\mathfrak{g})$ is just the ring $S(\mathfrak{g})^G$ of invariant functions on $\mathfrak{g}$, and that $\operatorname{gr} p$ is the maximal ideal of $S(\mathfrak{g})^G$ consisting of the functions vanishing at 0 . These latter functions generate in $S(\mathfrak{g})$ the prime ideal $\underline{\underline{I}}(\underline{\underline{N}})$ defining the cone $\underline{\underline{N}} \subset \mathfrak{g}$ of all nilpotent elements (B. Kostant). Now $\operatorname{gr} J \supset S(\mathfrak{g}) \operatorname{gr} p = \underline{\underline{I}}(\underline{\underline{N}})$ implies $\underline{\underline{K}}\, J \subset \underline{\underline{N}}$.

2.5. Now we consider the associated cone of an induced primitive ideal, say $J = \operatorname{Ann} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ as in 1.1. Let $\mathfrak{n}$ denote the nilradical of the parabolic subalgebra $\mathfrak{p}$, and $G\mathfrak{n}$ the G-invariant cone generated by $\mathfrak{n}$.

THEOREM [2].- The associated cone $\underline{\underline{K}}\, J$ of J contains $G\mathfrak{n}$.

The cone $G\mathfrak{n}$ is closed and irreducible, and hence it contains a unique dense G-orbit. This orbit has been studied by Richardson, who in particular determined the dimension : $\dim G\mathfrak{n} = 2 \dim \mathfrak{n} = 2 \dim \mathfrak{g}/\mathfrak{p}$. We conclude from 1.4, 2.2b) and the theorem, that

$$\operatorname{Dim} U(\mathfrak{g})/J = \dim \underline{\underline{K}}\, J = \dim G\mathfrak{n} = 2 \dim \mathfrak{g}/\mathfrak{p} .$$

In particular, $G\mathfrak{n}$ is an irreducible component of maximal dimension of $\underline{\underline{K}}\, J$.

CONJECTURE : $G\mathfrak{n}$ is equal to $\underline{\underline{K}}\, J$.

2.6. Here is a special case where the conjecture can be proved : Keep the notation J , $\mathfrak{p}$, $\mathfrak{n}$, E , but assume now $\dim E = 1$. Furthermore, assume that the stabilizer-group $G_x \subset G$ of an element x generating the dense orbit in $G\mathfrak{n}$ is connected. Assume finally that $G\mathfrak{n} = \overline{Gx}$ is a normal variety.

THEOREM.- These assumptions imply that the associated graded ideal $\operatorname{gr} J$ is prime. Consequently, tne associated cone is irreducible. So $\underline{\underline{K}}\, J = G\mathfrak{n}$ is true in this case. (Cf. [3], [13], [19].)

Remarks.- 1) If $\mathfrak{g} = \mathfrak{sl}_n$, then G_x is always connected.

2) It is conjectured by algebraic geometers that the closure of any G-orbit in $\mathfrak{g}$ should be a normal variety. Special cases have been settled (Kostant, Hesselink) ; in general, this seems to be a difficult problem.

2.7. <u>Example</u>.- Let us look at the special case where $\mathfrak{p}$ is a Borel-subalgebra. Then $G\mathfrak{n} = \underline{\underline{N}}$ is already the whole cone $\underline{\underline{N}}$ of nilpotent elements in $\mathfrak{g}$. This is known to be normal (Kostant). But in this case, $\underline{\underline{K}}\,J = G\mathfrak{n}$ is already clear from $G\mathfrak{n} \subset \underline{\underline{K}}\,J \subset \underline{\underline{N}}$ (2.5 and 2.2c)).

For later application, let us note an even more precise consequence for this special case : <u>We have</u> $J = U(\mathfrak{g})p$, <u>where</u> p <u>denotes the maximal ideal of the centre contained in</u> J , as in 2.4. In fact, the argument in 2.4 gave $\mathrm{gr}\ J \supset \mathrm{gr}\ U(\mathfrak{g})p \supset \underline{\underline{I}}(\underline{\underline{N}})$, and 2.5 gives $\underline{\underline{I}}(\underline{\underline{N}}) \supset \mathrm{gr}\ J$. Hence these inclusions are equalities, and $\mathrm{gr}\ J = \mathrm{gr}\ U(\mathfrak{g})p$ implies $J = U(\mathfrak{g})p$.

2.8. For an arbitrary primitive ideal J , a lower estimate for $\underline{\underline{K}}\,J$ is still known [13], which is very useful for actual computations of GK-dimensions.

*(Using the notation of §§ 4, 5, this estimate may be stated as follows. For each root α , let x_α denote a root-vector. For each subset B' of the base, set $x_{B'} := \Sigma\ x_\alpha$, $\alpha \in B'$. Let $J = J_\lambda$ be associated to the weight λ (§ 4). Then a sufficient condition for $x_{B'} \in \underline{\underline{K}}\,J$ is $\mathbb{N}B' \cap R^+_\lambda = \emptyset$ (5.1).) *

2.9. Conjecture 2.5 is a special case of the following important open problem. Denote by $\underline{\underline{N}}/G$ the set of G-orbits in the cone $\underline{\underline{N}}$ of all nilpotent elements in $\mathfrak{g}$. This is a finite set (Dynkin).

<u>Problem</u>.- <u>Is the associated cone of a primitive ideal always irreducible</u> ? <u>If yes, attach to each primitive ideal</u> $J \in \underline{\underline{X}}$ <u>the dense orbit in</u> $\underline{\underline{K}}\,J$. <u>Is this map</u> $\underline{\underline{X}} \to \underline{\underline{N}}/G$ <u>onto</u> ?

3. Associating to each primitive ideal a central character

3.1. Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$, and $\underline{\underline{Z}}$ the space of maximal ideals of $Z(\mathfrak{g})$. Let ℓ be the rank of $\mathfrak{g}$. It is well known that $Z(\mathfrak{g})$ has ℓ algebraically independent generators ("Casimir-elements" ; see 4.3 for the reason). Chosing these generators for coordinates, we may think of the maximal spectrum $\underline{\underline{Z}}$ as a complex ℓ-space $\mathbb{C}^\ell$, endowed with the Zariski-topology.

3.2. Every simple $\mathfrak{g}$-module M admits a central character, that is to say $Z(\mathfrak{g})$ acts by scalars on M , or equivalently, the annihilator ideal $\mathrm{Ann}\ M$ in $U(\mathfrak{g})$ contains a maximal ideal of $Z(\mathfrak{g})$ (Raïs). Hence we obtain a natural map $\pi : \underline{\underline{X}} \to \underline{\underline{Z}}$ of the primitive spectrum $\underline{\underline{X}}$ of $U(\mathfrak{g})$ into the maximal spectrum of $Z(\mathfrak{g})$ by sending J onto $J \cap Z(\mathfrak{g})$. Let us denote by $\underline{\underline{X}}_p$ the fibre $\pi^{-1}(p)$ over a point $p \in \underline{\underline{Z}}$. This set is ordered by inclusion.

THEOREM.- a) *The map* π *is surjective, continuous, open, and closed* [12].

b) *All the fibres of* π *are finite* (Dixmier).

c) *Each fibre* $\underline{\underline{X}}_p$ $(p \in \underline{\underline{Z}})$ *has a unique maximal element* (Dixmier).

d) *Each* $\underline{\underline{X}}_p$ *has a unique minimal element, namely* $U(\mathfrak{g})p$ (Dixmier-Duflo).

Thus we have a very nice projection of $\underline{\underline{X}}$ onto a complex ℓ-space.

3.3. Let $\underline{\underline{X}}^{max}$ denote the set of all maximal ideals of $U(\mathfrak{g})$. This is a subspace of $\underline{\underline{X}}$. Let $\underline{\underline{X}}^{min}$ denote the set of minimal primitive ideals of $U(\mathfrak{g})$. Combining parts a), c), d) of the theorem, we can say :

COROLLARY.- *Restriction of* π *gives homeomorphisms* $\underline{\underline{X}}^{max} \xrightarrow{\sim} \underline{\underline{Z}}$, $\underline{\underline{X}}^{min} \xrightarrow{\sim} \underline{\underline{Z}}$.

3.4. Comments on the proof of theorem 3.2 : Parts a) and c) are rather elementary ; they mainly depend on the fact that $Z(\mathfrak{g})$ is an isotypical component for the (completely reducible) adjoint action of G on $U(\mathfrak{g})$. Part b) seems to be very deep ; no purely algebraic proof is known so far ; the known proofs depend on the theory of principal series representations. For part d) we shall sketch a simple proof below (4.4).

3.5. The next sections (§§ 4, 5) will be concerned with detailed results on the cardinality of the fibres $\underline{\underline{X}}_p$. Here we present only a rough statement about the fibres of cardinality one or two. Let us denote by $\underline{\underline{Z}}^m$ the set of points $p \in \underline{\underline{Z}}$ such that the fibre over p has cardinality $\geq m$.

THEOREM.- a) $\underline{\underline{Z}}^2$ *is a locally finite union of algebraic hypersurfaces* (Dixmier).

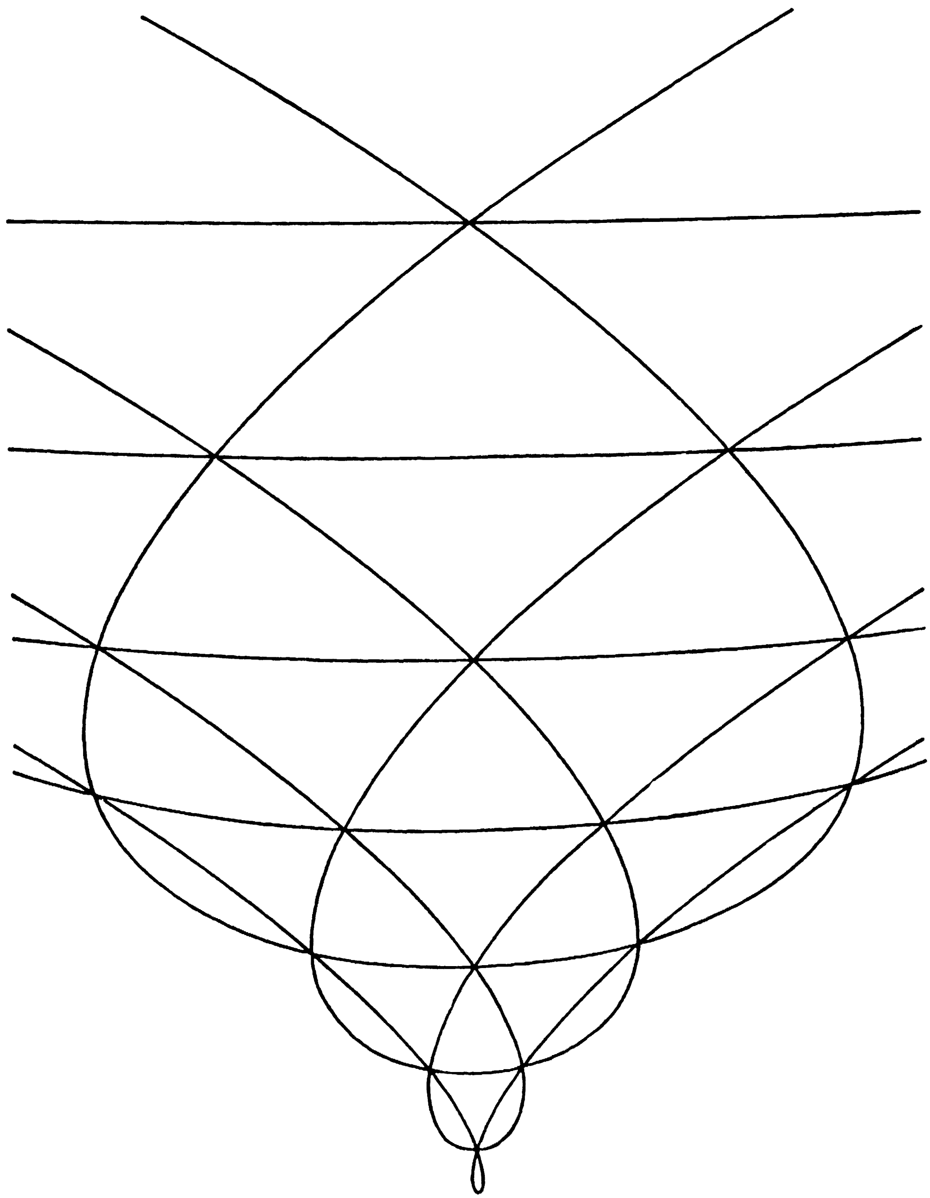
b) $\underline{\underline{Z}}^3$ *is contained in the set of singularities of* $\underline{\underline{Z}}^2$ [2].

c) *Suppose that the Coxeter-diagram of* $\mathfrak{g}$ *has no double or triple lines. Then* $\underline{\underline{Z}}^3$ *coincides with the set of singularities of* $\underline{\underline{Z}}^2$ [2].

3.6. The theorem is illustrated by the diagram below for the case $\mathfrak{g} = \mathfrak{sl}_3$. Here the rank is 2 , so $\underline{\underline{Z}}$ is a complex plane, and the exceptional hyperplanes forming $\underline{\underline{Z}}^2$ are curves. The diagram shows the real points of these curves. In this case, $\underline{\underline{Z}}^3$ is the set of intersection points of the curves. For more diagrams, see [2].

3.7. In view of theorem 3.5 b), c) and of further results of Conze and Duflo [5] on non-primitive ideals, it is natural to ask the following.

Problem.- *Let* $p \in \underline{\underline{Z}}^2$. *To which extent does the local geometric structure of* $\underline{\underline{Z}}^2$ *at the point* p *determine the structure of the fibre* $\underline{\underline{X}}_p$, *or even the complete ideal structure of* $U(\mathfrak{g})/U(\mathfrak{g})p$?



Exceptional subset in $\underline{\underline{Z}}$ for $\mathfrak{g} = \mathfrak{sl}_3$:

$$-x = t^2 + nt + n^2 \quad , \quad y = (2t + n)(t - n)(t + 2n) \; , \quad t \in \mathbf{C} \; , \; n = 1 , 2 , 3 , \ldots .$$

4. Associating to each weight a primitive ideal

4.1. Fix a Cartan-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a Borel-subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Denote by R the root-system (in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$), by $B \subset R$ the base of R corresponding to $\mathfrak{b}$, by R_+ the set of positive roots, and by W the Weyl-group.

4.2. For each $\lambda \in \mathfrak{h}^*$ denote by E_λ a $\mathfrak{b}$-module of dimension 1 and weight λ. (E_λ is spanned by a vector e such that $he = \lambda(h)e$ for all $h \in \mathfrak{h}$.) An element of any $\mathfrak{g}$-module is called a $\mathfrak{b}$-eigenvector or a "highest weight vector" of weight λ, if it spans a $\mathfrak{b}$-submodule isomorphic to E_λ. Any $\mathfrak{g}$-module generated by a $\mathfrak{b}$-eigenvector of weight λ is a homomorphic image of a certain "universal" such module, the so-called "Verma-module", denoted $M(\lambda)$. This $M(\lambda)$ is nothing else but the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} E_\lambda$ (1.1). There exists a beautiful theory of these modules, developed by Bernstein, Gelfand, Verma, and others, and reported already in this seminar [8] by Dixmier.

4.3. The module $M(\lambda)$ admits a central character. Setting $\varphi(\lambda) := Z(\mathfrak{g}) \cap \operatorname{Ann} M(\lambda)$, we obtain a map $\varphi : \mathfrak{h}^* \to \underline{\underline{Z}}$. This map has a nice description. Consider the "translated action" of the Weyl-group defined by $w.\lambda = w(\lambda + \rho) - \rho$ for $w \in W$, $\lambda \in \mathfrak{h}^*$, where ρ is half the sum of positive roots. This induces an action of W on the ring $S(\mathfrak{h})$ of regular functions on $\mathfrak{h}^*$. Denote $S(\mathfrak{h})^{W.}$ the ring of invariants. This ring gives the structure of an algebraic variety to the set of orbits $\mathfrak{h}^*/W. = \{W.\lambda \mid \lambda \in \mathfrak{h}^*\}$ of that action in $\mathfrak{h}^*$. Now there is a natural isomorphism $Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^{W.}$, due to Harish-Chandra [8]. This provides an isomorphism $\mathfrak{h}^*/W. \xrightarrow{\sim} \underline{\underline{Z}}$. Now <u>our map $\varphi : \mathfrak{h}^* \to \underline{\underline{Z}}$ is obtained by composing this isomorphism and the canonical map $\mathfrak{h}^* \to \mathfrak{h}^*/W.$</u>.

(In passing, we note that the Harish-Chandra-isomorphism gives the reason why $Z(\mathfrak{g})$ is a polynomial ring (3.1) : Namely, $S(\mathfrak{h})^{W.}$ is known to be a polynomial ring, since W is generated by reflections.)

4.4. We note in particular that each maximal ideal $p \in \underline{\underline{Z}}$ is equal to $\varphi(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. <u>In the sequel we always denote the fibre $\underline{\underline{X}}_p = \underline{\underline{X}}_{\varphi(\lambda)}$ by $\underline{\underline{X}}_\lambda$</u>. Let us insert now the reason why p generates a primitive ideal and hence is the smallest element of $\underline{\underline{X}}_p$ (3.2d)). By the theory of Verma-modules, $M(w.\lambda)$ is irreducible for some $w \in W$. Since $\varphi(w.\lambda) = \varphi(\lambda) = p$, we may assume $M(\lambda)$ irreducible. The ideal $J := \operatorname{Ann} M(\lambda)$ is therefor primitive, and is induced in the sense of 1.1 from a Borel-subalgebra. We have shown in 2.7 that this implies : $U(\mathfrak{g})p = J$ is primitive.

4.5. One of the elementary properties of a Verma-module $M(\lambda)$ is that it has a unique largest submodule, and hence a unique simple quotient, denoted by $L(\lambda)$. To each weight $\lambda \in \mathfrak{h}^*$, we associate the primitive ideal $J_\lambda := \operatorname{Ann} L(\lambda)$. Together with the maps introduced above, this new map $\lambda \mapsto J_\lambda$ makes up a commutative diagram :

$$\begin{tikzcd} \mathfrak{h}^* \arrow[r] \arrow[d] \arrow[dr, dashed, "\varphi"] & \underline{\underline{X}} \arrow[d, "\pi"] \\ \mathfrak{h}^*/W. \arrow[r] & \underline{\underline{Z}} \end{tikzcd}$$

4.6. THEOREM (Duflo [9]).- <u>The map</u> $\mathfrak{h}^* \to \underline{\underline{X}}$, $\lambda \mapsto J_\lambda$ <u>is surjective</u>.

In view of the diagram above, the following description of the fibres $\underline{\underline{X}}_\lambda$ of $\underline{\underline{X}}$ follows readily :

4.7. COROLLARY.- $\underline{\underline{X}}_\lambda = \{J_{w.\lambda} \mid w \in W\}$ <u>for each</u> $\lambda \in \mathfrak{h}^*$.

Comments : This theorem is one of the deepest facts known about enveloping algebras of semi-simple Lie-algebras. The proof is based upon the relations between the theory of Verma-modules and the theory of principal series representations revealed by Duflo and Conze [5], [9], [6], 9.6. (A nice report on the principal series of complex semi-simple groups, as developed by Zelobenko, Wallach, Varadarajan, et al., originating from the work of Gelfand, Naimark, Harish-Chandra, is presented in [10].)

The theorem implies, of course, the finiteness of the fibres (3.2b)). It even gives a common upper bound for the cardinality : $\# \underline{\underline{X}}_\lambda \leq \# W$. This bound may be improved as follows.

4.8. Let $\mathbb{Z}R$ denote the lattice generated by the roots. For $\lambda \in \mathfrak{h}^*$, let $W_\lambda \subset W$ denote the subgroup of W fixing the lattice $\lambda + \mathbb{Z}R$, and $W_\lambda^o \subset W_\lambda$ the subgroup of W fixing $\lambda + \rho$. Both are generated by reflections.

THEOREM (Duflo [9]).- $\# \underline{\underline{X}}_\lambda \leq \# \{w \in W_\lambda \mid w^{-1} \in W_\lambda^o w W_\lambda^o\}$.

4.9. COROLLARY.- <u>If</u> $W_\lambda^o = 1$, <u>then the cardinality of</u> $\underline{\underline{X}}_\lambda$ <u>is at most the number of involutions in</u> W_λ , <u>that is</u> $\# \underline{\underline{X}}_\lambda \leq \# \{w \in W_\lambda \mid w^2 = 1\}$.

4.10. It is now easy to determine the "exceptional" subset $\underline{\underline{Z}}^2$ of $\underline{\underline{Z}}$ (3.5) : It is given by $\underline{\underline{Z}}^2 = \varphi(\{\lambda \in \mathfrak{h}^* \mid W_\lambda \neq W_\lambda^o\})$, an image of a family of hyperplanes.

4.11. CONJECTURE : <u>The structure of a fibre</u> $\underline{\underline{X}}_\lambda$ (as an abstract ordered set) <u>is uniquely determined by the pair</u> $(W_\lambda , W_\lambda^o)$ <u>alone</u> (considered as an abstract pair of reflection groups, one containing the other), <u>even independently of</u> $\mathfrak{g}$.

This is closely related to problem 3.6. It was suggested by various examples, and by the "translation principle" to be reported on now.

5. <u>Relating primitive ideals of different central characters by a translation principle</u>

5.1. We introduce a few technical abbreviations, which are useful to express frequently-occuring positivity and integrality conditions for weights. For a root α , let $H_\alpha \in \mathfrak{h}$ denote the dual root. For each weight $\lambda \in \mathfrak{h}^*$, set

$$R_\lambda := \{\alpha \in R \mid (\lambda+\rho)(H_\alpha) \in \mathbb{Z}\} \quad \text{and} \quad R_\lambda^+ := \{\alpha \in R \mid (\lambda+\rho)(H_\alpha) \in \mathbb{N}\setminus 0\} .$$

Note that R_λ is a root-system : the one with Weyl-group W_λ (4.8).

5.2. <u>Example</u> : E. Cartan's classification of all finite dimensional simple $\mathfrak{g}$-modules may be summarized as follows : Up to isomorphism, they are just the modules $L(\nu)$ for $\nu \in \mathfrak{h}^*$ such that $R_\nu^+ = R_+$. (In classical language, these weights ν are called "dominant integral" weights.)

5.3. THEOREM [12].- <u>Let</u> λ , $\nu \in \mathfrak{h}^*$ <u>be such that</u> $R_\nu^+ = R_+$ <u>and</u> $R_\lambda^+ = R_{\lambda+\nu}^+$. <u>Then there exists an isomorphism</u> $T_\lambda^{\lambda+\nu} : \underline{\underline{X}}_\lambda \xrightarrow{\sim} \underline{\underline{X}}_{\lambda+\nu}$ <u>of ordered sets</u>.

Moreover, this $T_\lambda^{\lambda+\nu}$, the "<u>translation from</u> λ <u>to</u> $\lambda+\nu$ " is given by

$$(*) \qquad T_\lambda^{\lambda+\nu} J_{w.\lambda} = \operatorname{Ann} M(\lambda+\nu) + \operatorname{Ann} L(w.\lambda) \otimes L(\nu) ,$$

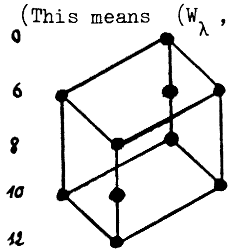
and satisfies

$$T_\lambda^{\lambda+\nu} J_{w.\lambda} = J_{w.(\lambda+\nu)} \qquad \text{for all } w \in W .$$

Finally, it preserves GK-dimensions :

$$\operatorname{Dim} U(\mathfrak{g})/T_\lambda^{\lambda+\nu} J = \operatorname{Dim} U(\mathfrak{g})/J \qquad \text{for all } J \in \underline{\underline{X}}_\lambda .$$

5.4. <u>Example</u> : Let λ be an integral point in the interior of some Weyl-chambre. (This means $(W_\lambda , W_\lambda^o) = (W , 1)$ in the notation of § 4.) The "translation principle" (5.3) says that for all such weights λ , the fibres $\underline{\underline{X}}_\lambda$ are isomorphic. The diagram shows, how they look like for $\mathfrak{g} = \mathfrak{sl}_4$ (points = ideals, lines = inclusions, numbers = GK-dimensions). For more diagrams, see [12], [13].



5.5. A more refined version [12] of the translation principle says : <u>We can deduce the structure of the fibres</u> $\underline{\underline{X}}_\mu$ <u>for</u> μ <u>on a wall from the structure of fibres</u> $\underline{\underline{X}}_\lambda$, for λ inside a Weyl-chambre.

To be a little more precise : If in theorem 5.3 the conditon $R_\lambda^+ = R_{\lambda+\nu}^+$ is weakened to $R_\lambda^+ \supset R_{\lambda+\nu}^+$, then the translation operator $T_\lambda^{\lambda+\nu}$ defined by formula (*) still sends a certain part of $\underline{\underline{X}}_\lambda$ order-isomorphically onto $\underline{\underline{X}}_{\lambda+\nu}$, but the other part is sent onto the single ideal $U(\mathfrak{g})$.

So under a translation onto a wall, a primitive ideal may "degenerate" to $U(\mathfrak{g})$.

5.6. Primarily, the translation principle provides only relative information, relating some fibres. But applied with some skill, we may use it to produce also absolute information on the structure of a fibre $\underline{\underline{X}}_\lambda$. Here we sketch one of the methods to do so. Another one will be indicated in 6.8.

Fix a weight λ inside a chambre. In the root system R_λ , the positive system $R_\lambda \cap R_+$ determines a base B_λ of R_λ . We can arrange $R_\lambda^+ = R_\lambda \cap R_+$ (and we do so). To each primitive ideal $J \in \underline{\underline{X}}_\lambda$, we now associate a subset τJ of B_λ as follows : τJ is the set of those $\alpha \in B_\lambda$, such that J "degenerates" if translated onto the wall corresponding to α . Thus we define a map $\tau : \underline{\underline{X}}_\lambda \to \underline{P}(B_\lambda)$ of $\underline{\underline{X}}_\lambda$ into the set $\underline{P}(B_\lambda)$ of subsets of B_λ .

THEOREM [12].- τ is an order-homomorphism of $\underline{\underline{X}}_\lambda$ onto $\underline{P}(B_\lambda)$.

In particular, we obtain a lower estimate for $\# \underline{\underline{X}}_\lambda$ ([12], cf. [9]) :

5.7. COROLLARY.- Let λ be as before. Then $\# \underline{\underline{X}}_\lambda \geq 2^{\operatorname{rank} R_\lambda}$.

5.8. In general, equality does not hold in this corollary : If R_λ is linearly closed in R , then equality holds in corollary 5.7 iff R_λ has no simple component of rank ≥ 3 [12]. In particular the " τ-invariant" alone almost never separates all elements of $\underline{\underline{X}}_\lambda$.

Example : For $R_\lambda = R$ of type A_3 , A_4 , A_5 , we have $\# \underline{\underline{X}}_\lambda = 10 , 26 , 76$ (> 8 , 16 , 32) respectively [12], [13].

The additional invariants used to prove these statements (and many other results) are GK-dimension and associated cones.

5.9. Problem.- Let λ be integral and not on a wall (i.e. as in 5.4). Here is a precise conjecture how $\underline{\underline{X}}_\lambda$ should look like in case $\mathfrak{g} = \mathfrak{sl}_n$: To begin with, problem 2.9 should have a positive answer for this case, i.e. the associated cone $\underline{\underline{K}}J$ should contain a dense orbit for each $J \in \underline{\underline{X}}_\lambda$. Denote this orbit by $\underline{\underline{OKJ}}$. Now recall the Jordan "normal form" of a nilpotent matrix, and note that the nilpotent orbits of $\mathfrak{sl}_n$ are in natural bijection to the partitions of n (given by the sizes of "Jordan-blocks"). On the other hand, note that the Weyl-group is here the symmetric group S_n, and recall that the classes of irreducible complex representations of S_n are also in bijection to the partitions of n (considered as "Young-diagrams"). Thus the orbit $\underline{\underline{OKJ}}$ corresponds to a certain irreducible representation of the Weyl-group, denoted $\underline{\underline{ROKJ}}$.

CONJECTURE (Jantzen).- <u>For each primitive ideal</u> $J \in \underline{\underline{X}}_\lambda$, <u>we should have</u> :

$$\dim \underline{\underline{ROKJ}} = \# \{ I \in \underline{\underline{X}} \mid \underline{\underline{K}}I = \underline{\underline{K}}J \} = \# \{ L(\mu) \mid J = \operatorname{Ann} L(\mu) \} .$$

<u>Example</u> : This is true for $\mathfrak{g} = \mathfrak{sl}_5$ [12] : There are 7 partitions of 5, hence 7 nilpotent orbits in $\mathfrak{g}$, of dimension 0, 8, 12, 14, 16, 18, 20. Such an orbit is dense in the associated cone $\underline{\underline{K}}J$ of 1 resp. 4, 5, 6, 5, 4, 1 different ideals $J \in \underline{\underline{X}}_\lambda$, and such an ideal J is the annihilator of 1 resp. 4, 5, 6, 5, 4, 1 different modules $L(\mu)$. We have $5! = 120 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2$ elements in the Weyl-group, and we have exactly $26 = 1+4+5+6+5+4+1$ elements in $\underline{\underline{X}}_\lambda$.

<u>Remark</u>.- Jantzen's conjecture would imply that Duflo's estimate 4.9 for $\# \underline{\underline{X}}_\lambda$ gives already the exact value in this case ($\mathfrak{g} = \mathfrak{sl}_n$, λ integral, not on a wall). In fact, it says that $\# \underline{\underline{X}}_\lambda$ should be the sum of the dimensions over all classes of irreducible complex representations of $W = S_n$, and this sum is known to be the number of involutions in W (Schur). For $\mathfrak{sl}_6$, $\# \underline{\underline{X}}_\lambda = \# \{ w \in W \mid w^2 = 1 \}$ (= 76) is true [13]. For $\mathfrak{g} \neq \mathfrak{sl}_n$, it is generally not true.

6. Associating primitive ideals to orbits in $\mathfrak{g}^*$

6.1. Even if we have completely determined all fibres $\underline{\underline{X}}_\lambda$ of $\underline{\underline{X}}$ we can not be satisfied, because this would describe $\underline{\underline{X}}$ only as an abstract set. We should rather like to describe the primitive spectrum $\underline{\underline{X}}$ as a topological space, or even better, parametrize it piece-wise by algebraic varieties. This is the ambitious aim of Dixmier's "orbit-method".

6.2. To explain the idea of this method, suppose for a moment that $\mathfrak{g}$ is a commutative Lie-algebra. Then the only irreducible representations of $\mathfrak{g}$ are those of dimension 1, and the only primitive ideals of $U(\mathfrak{g})$ (a polynomial ring now) are the maximal ideals. Associating to each linear form $f \in \mathfrak{g}^*$ the kernel of the corresponding character of $U(\mathfrak{g})$, we obtain a bijection $\mathfrak{g}^* \xrightarrow{\sim} \underline{\underline{X}}$. This is Hilbert's Nullstellensatz, the classical connection between geometry and <u>commutative</u> algebra.

6.3. Now let $\mathfrak{g}$ be a <u>non-commutative</u> Lie-algebra. Dixmier's idea is to describe $\underline{\underline{X}}$ also in this case by a similar map $\mathfrak{g}^* \to \underline{\underline{X}}$. Since now a linear form $f \in \mathfrak{g}^*$

need not be a representation of $\mathfrak{g}$, one first has to restrict it to suitable subalgebras, the "polarizations". A polarization of f is a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of codimension $= \frac{1}{2} \dim Gf$ (where G = adjoint algebraic group), such that f defines a representation of $\mathfrak{p}$. This representation induces a representation of $\mathfrak{g}$ and hence an annihilator ideal in $U(\mathfrak{g})$, denoted by $\tilde{I}(\mathfrak{p}, f)$. (Here we must take "twisted" induced representations, as explained in [6].) Although the induced representation need not be irreducible, it turns out that the induced ideal is primitive (at least for semi-simple or solvable $\mathfrak{g}$). Now two problems arise :

I. Does every linear form $f \in \mathfrak{g}^*$ admit a polarization $\mathfrak{p}$?

And if there exists more than one :

II. Is the induced ideal $\tilde{I}(\mathfrak{p}, f)$ independent of the choice of $\mathfrak{p}$?

6.4. Once we have answered these questions affirmatively, we have well-defined a map $\mathfrak{g}^* \to \underline{\underline{X}}$ by $f \mapsto \tilde{I}(\mathfrak{p}, f)$. It is easy to see then that this map is constant on G-orbits, and hence induces a map on the orbit-space $\mathfrak{g}^*/G$, denoted $\mathrm{Dix} : \mathfrak{g}^*/G \to \underline{\underline{X}}$.

6.5. For solvable $\mathfrak{g}$, the map $\mathrm{Dix} : \mathfrak{g}^*/G \to \underline{\underline{X}}$ is well-defined (Dixmier), surjective (Conze-Duflo-Vergne), and injective (Rentschler) [6], [11].

6.6. Unfortunately, for $\mathfrak{g}$ semi-simple questions I and II both have negative answers in general (see the bad situation in case $\mathfrak{g} = \mathfrak{so}_5$, described by Rentschler, and in [1]). The situation is better for $\mathfrak{g} = \mathfrak{sl}_n$, where the existence of polarizations was proved by Ozeki-Wakimoto [20].

6.7. THEOREM.- For $\mathfrak{g} = \mathfrak{sl}_n$, the map $\mathrm{Dix} : \mathfrak{g}^*/G \to \underline{\underline{X}}$ is well-defined [3] and injective [12].

6.8. Comments on the proofs. The proof of the well-definedness (the positive answer to question II) requires mainly a careful analysis of the multiplicities of the action of G in $U(\mathfrak{g})/\tilde{I}(f, \mathfrak{p})$ (cf. 1.9). One of the by-products of this analysis is theorem 2.6.

For the proof of the injectivity, we employ the translation principle (5.3) in the following way. Let $\lambda \in \mathfrak{h}^*$. Take a "dominant integral" weight ν such that all the translation operators $T_{\lambda}^{\lambda+m\nu}$ for $m \in \mathbb{N}$ are defined (cf. 5.3), and define a function $\tau_{\nu} : \underline{\underline{X}} \to \mathbb{N}$ by

$$\tau_{\nu} I := \mathrm{Dim}\, U(\mathfrak{g}) / \bigcap T_{\lambda}^{\lambda+m\nu} I \qquad \text{(intersection over } m \in \mathbb{N}) .$$

For induced ideals I , we are able to compute these numerical invariants. It turns out that this system of invariants contains enough information to reconstruct from

$\tilde{I}(\mathfrak{p}, f)$ the orbit Gf by combinatorial methods. This is the strategy of proof. However, the details are complicated.

6.9. Denote by $\underline{\underline{X}}^m \subset \underline{\underline{X}}$ the subspace of primitive ideals of Goldie-rank m (1.6). The image of $\mathfrak{g}^*/G$ in $\underline{\underline{X}}$ is contained in $\underline{\underline{X}}^1$ (this is immediate from Conze's theorem 1.2).

CONJECTURE (Dixmier).- <u>For</u> $\mathfrak{g} = \mathfrak{sl}_n$, <u>the map</u> Dix <u>is a bijection</u> $\mathfrak{g}^*/G \rightarrow \underline{\underline{X}}^1$.

This is true for $\mathfrak{sl}_3$ [7].

<u>Problem</u>.- <u>Find modifications of the orbit-method which apply</u>

1. <u>also to arbitrary semi-simple Lie-algebras</u> ;
2. <u>also to the parts</u> $\underline{\underline{X}}^m$ <u>of</u> $\underline{\underline{X}}$, <u>for</u> $m > 1$.

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