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ON THE FINITE SIMPLE GROUPS

[according to Aschbacher and Gorenstein]

by Zvonimir JANKO

We begin with the Aschbacher's classification of the finite simple thin groups because this is a nice model for the future general classification.

DEFINITION 1.- Let X be a finite group and T be a non identity 2-subgroup of X. Then $N_X(T)$ is called a 2-local subgroup of X.

DEFINITION 2.- A finite group X of even order is of characteristic 2 type if $C_M(O_2(M)) \subseteq O_2(M)$ for every 2-local subgroup M of X.

DEFINITION 3.- A finite group X of even order is thin if for each 2-local subgroup M of X and each odd prime p, the Sylow p-subgroups of M are cyclic.

DEFINITION 4.- Let X be a finite group of even order. Then $m_{2,p}(X)$ = 2-local p-rank of $X = \max\{m_p(A) \mid A$ ranging over elementary abelian p-subgroups of X which lie in a 2-local subgroup of $X\}$, where p is an odd prime and $m_p(A)$ denotes the minimal number of generators of A.

DEFINITION 5.- Let X be a finite group of even order. Then $e(X) = \max\{m_{2,p}(X) \mid p$ ranging over all odd primes $p\}$.

We shall now discuss the following remarkable :

THEOREM 1 (Aschbacher).- Let G be a non-abelian finite simple thin group. Then G is isomorphic to the one of the following simple groups : $L_2(q)$, $L_3(p)$, $p = 1 + 2^a 3^b$, $U_3(p)$, $p = -1 + 2^a 3^b$, p an odd prime, $b = 0$ or 1, $Sz(2^n)$, $U_3(2^n)$, $L_3(4)$, ${}^2F_4(2)'$, ${}^3D_4(2)$, M_{11} and J_1.

In the first part of the proof of the theorem Aschbacher shows that a minimal counter-example to the theorem is a finite simple thin group G of characteristic 2 type. I have shown the same result several years ago but I could not go any further. It seems likely that in the near future the problem of determining the finite simple groups will be reduced to the determination of the characteristic 2 type groups. The principal model for the investigation of the characteristic 2 type groups is the Thompson's work on N-groups. He subdivides his arguments into the cases $e(G) = 1$, $e(G) = 2$ and $e(G) \geq 3$. This subdivision arrises from the difference in the uniqueness theorems obtained in the three cases.

One expects that in the general characteristic 2 type classification this same subdivision would naturally occur for the same reason. Hence the thin group classification may be regarded as one step in the classification of the finite simple groups.

A second motivation for the Aschbacher's work is to supply another model for characteristic 2 type investigation. Thompson's work on N-groups has been extended by a number of authors to a classification of groups in which all 2-local subgroups are solvable. This work of Aschbacher suggests that Thompson's techniques can be successfully extended to situations in which 2-local subgroupes are not solvable.

Most of the techniques in the Aschbacher's work are extensions of Thompson's ideas. That is certain uniqueness theorems are established which make it possible to carry on weak closure arguments on elementary abelian normal subgroups of the maximal 2-local subgroups of G . In addition certain ideas os Sims and Glauberman are exploited which make possible "pushing up theorems". The author also appeals to my classfication of thin groups with solvable 2-local subgroups to produce a nonsolvable 2-local subgroup of G . It is helpful to have Gorenstein and Harada's classification of groups of sectional 2-rank 4 (where every 2-subgroup is generated by at most 4 generators), and a theorem of Harada which bounds the sectional 2-rank in certain situations. The work of Timmesfeld on TI-sets is also helpful.

We shall now formulate the new result of Gorenstein and Lyons about simple groups G of characteristic 2 type with $e(G) \geq 4$. For that we need some more definitions.

DEFINITION 6.- Let X be a finite group of even order. Then we set

$\beta_k(X) = \{\text{odd primes } p \mid m_{2,p}(X) \geq k\}$, where k is a natural number ;

$\varepsilon_p(X)$ = the set of all elementary abelian p-subgroups of X , p an odd prime ;

$\varepsilon_{k,p}(X) = \{A \in \varepsilon_p(X) \text{ with } m_p(A) = k\}$;

$\beta_{max}(X ; p) = \{B \mid B \in \varepsilon_p(X) , m_p(B) = m_{2,p}(X) ,$ B lies in a 2-local subgroup of $X\}$ = the set of elementary abelian p-subgroups of X of maximal rank subject to lying in a 2-local subgroup of X ;

For $D \in \varepsilon_p(X)$, we set $\Delta_X(D) = \bigcap_{d \in D^{\#}} O_{p'}(C_X(d))$.

If $P \in Syl_p(X)$, p a prime, and k is a natural number, we set $\Gamma_{P,k}(X) = \langle N_X(Q) \mid Q \subseteq P , m_p(Q) \geq k \rangle$, where $Syl_p(X)$ denotes the set of all Sylow p-subgroups of X .

When there is no ambiguity, we write $\varepsilon_k(X)$, $\beta_{max}(p)$, and Δ_D for $\varepsilon_{k,p}(X)$, $\beta_{max}(X ; p)$, and $\Delta_X(D)$, respectivaly.

DEFINITION 7.- A non-abelian finite 2-group T is of symplectic type if it possesses no noncyclic characteristic abelian subgroup. Such groups T have been classified by P. Hall. It turns out that T is either of maximal class (and so dihedral, semidihedral or generalized quaternion) or the central product of an extraspecial 2-group with a cyclic group or with a 2-group of maximal class.

DEFINITION 8.- A finite group X of even order is said to be a K-group (K for known !) if the composition factors of X are known simple groups.

DEFINITION 9.- Let X be a finite group of even order. We formulate two conditions on an odd prime p :

$(\Gamma_3)_p$ If $P \in Syl_p(X)$, then $\Gamma_{P,3}(X)$ is contained in a 2-local subgroup of X .

$(\Delta O)_p$ For every $B \in \beta_{max}(p)$ and every $D \in \varepsilon_2(B)$, $|\Delta_D|$ is odd if $p \geq 5$; while if $p = 3$, then $|[\Delta_D , B]|$ is odd.

THEOREM 2 (Gorenstein and Lyons).- Let G be a finite simple group of characteristic 2 type in which all proper subgroups are K-groups and $e(G) \geq 4$. Assume the following conditions hold for some integer $m \geq 4$:

(a) For all $q \in \beta_{m+1}(G)$, $(\Gamma_3)_q$ holds ;

(b) For all $q \in \beta_m(G)$, either $(\Gamma_3)_q$ or $(\Delta O)_q$ holds ; and

(c) For some $q \in \beta_m(G)$ (with $m_{2,q}(G) = m$), $(\Delta O)_q$ holds.

Then either (I) or (II) holds :

(I) $m_{2,3}(G) = m$, $(\Delta O)_3$ holds, and there exists a maximal 2-local subgroup M of G such that $O_2(M)$ is of symplectic type with M containing an element of $\beta_{max}(3)$ (equivalently, $m_3(M) = m_{2,3}(G)$) ; or

(II) There exists a prime p with $m_{2,p}(G) = m$, an element $B \in \beta_{max}(p)$, and $x \in B^{\#}$ such that $C = C_G(x)$ has the following properties :

(i) C has a quasisimple normal subgroup L (i.e. $L' = L$ and $L/Z(L)$ is a nonabelian simple group) such that $C_C(L)$ has cyclic Sylow p-subgroups ;

(ii) Either (1) or (2) holds :

(1) $p = 3$ and L is either a sporadic simple group or of Lie type over $GF(2)$; or

(2) L is of Lie type over $GF(2^n)$ with $p \mid 2^n - 1$ if L is of untwisted and $p \mid 2^{2n} - 1$ if L is of twisted type ;

(iii) B induces inner or diagonal automorphisms on L ;

(iv) $e(L) \leq m$;

(v) B normalizes but does not centralize a 2-subgroup of L ;

(vi) $O_{p'}(C)$ is of odd order ; and

(viii) $\langle x \rangle$ is not weakly closed in a Sylow p-subgroup of C with respect to G .

In order to improve Theorem 2, one would have to consider the cases that some (or all) conditions (a), (b) or (c) fail. But in this case one hopes to get a contradiction because one can show that in those cases G possesses "an almost strongly p-embedded 2-local subgroup" for some odd prime p . Namely, for the past year Aschbacher was working on the case that the simple group G is of characteristic 2 type, $m_{2,3}(G) \geq 4$ and G has "a strongly 3-embedded 2-local subgroup M ". He has completed the proof that no such group exists and Gorenstein hopes that he will be able to generalize this Aschbacher's work in order to force the conditions (a), (b) and (c) in Theorem 2.

If and when this is done, it will remain to consider the groups which occur in the conclusion (I) or (II) of Theorem 2 in order to complete the classification of the simple groups of characteristic 2 type with $e(G) \geq 4$. The groups which occur in (I) are almost done as the following two results indicate :

THEOREM 3 (Aschbacher).- Let G be a nonabelian finite simple group which possesses a maximal 2-local subgroup M such that $C_M(O_2(M)) \subseteq O_2(M)$ and $O_2(M)$ is of symplectic type but not extraspecial. Then G is isomorphic to the one of the following groups : $L_2(2^n \pm 1)$, for some n , $U_3(3)$ or the Higman-Sims sporadic group.

THEOREM 4 (Timmesfeld).- Let G be a nonabelian finite simple group which possesses a maximal 2-local subgroup M such that $C_M(O_2(M)) \subseteq O_2(M)$ and $Q = O_2(M)$ is an extraspecial 2-group of order 2^{2n+1} (n is the width of Q). Set $\bar{M} = M/Q$. Then one of the following holds :

(1) $n \leq 6$.

(2) $G \simeq U_{n+2}(2)$.

(3) $n = 2m$ and $\bar{M} \simeq \Sigma_3 \times \Omega^{\pm}(2m, 2)$ or $\Sigma_3 \times O^{\pm}(2m, 2)$, where Σ_3 denotes the symmetric group in 3 letters.

(4) $n = 10$ and $\bar{M} = \bar{M}_0\langle\bar{s}\rangle$, where $\bar{s}^2 = 1$ and $\bar{M}_0 \simeq L_6(2)$ or $U_6(2)$.

(5) $n = 11$ and $\bar{M}$ is isomorphic to the Conway group .2 .

(6) $\bar{M}$ is nonabelian simple group which possesses an involution $\bar{t}$ such that $\bar{N} = O_2(C_{\bar{M}}(\bar{t}))$ is extraspecial, $C_{\bar{M}}(\bar{N}) \subseteq \bar{N}$ and we have only the following possibilities :

(i) $n = 12$, $C_{\bar{M}}(\bar{t})/\bar{N} \simeq D_4(2)$ and $|\bar{N}| = 2^9$;

(ii) $n = 16$, $C_{\bar{M}}(\bar{t})/\bar{N} \simeq \Sigma_3 \times D_4(2)$ and $|\bar{N}| = 2^{17}$.

(iii) $n = 28$, $C_{\bar{M}}(\bar{t})/\bar{N} \simeq D_6(2)$ and $|\bar{N}| = 2^{33}$.

We mention also that Bierbrauer and Tran van Trung are working full time on the groups which appear in the conclusion (1) of Theorem 4 and they will be certainly finished within 2 years. Also, the last sporadic simple group which I have discovered in May 1975 appears within these groups with $n = 6$. Its order is 86 , 775 , 571 , 046 , 077 , 562 , 880 . Reifart is working on the case (4) of Theorem 4 and he is practically finished.

It will be much more difficult to determine the simple groups which appear in the conclusion (II) of Theorem 2. In fact this is so difficult that it seems that Theorem 2 part (II) will take another 20 years !

All that will in turn leave $e(G) \leq 3$. But what then ? Chermak seems to have made good progress in the $e(G) = 2$ case but nobody has yet seen his work.

Gorenstein and Lyons have completed the $e(G) = 3$ analysis in the special case that G has the 2-local 3-rank 1 . Much of this looks as though it will generalize with little work. The aim is to show that if $e(G) = 3$, then either the conclusion of Theorem 2 holds or $m_{2,3}(G) = 3$ and $m_{2,q}(G) \leq 2$ for all primes $q \geq 5$.

And this should be handled by methods analogous to those used in $e(G) = 2$ problem. The point is that it is simply not possible to construct a good signalizer functor for the prime 3 when $m_{2,3}(G) = 3$.

It is of interest to give finally the following generalization of the Theorem 4 :

THEOREM 5 (Stroth).- Let G be a nonabelian finite simple group which possesses an involution z such that $H = C_G(z)$ has the following properties :

(1) We have $C_H(Q) \subseteq Q$, where $Q = O_2(H)$.

(2) We have $Q = E \times F$, where E is a nonidentity elementary abelian group and F is extraspecial with $z \in F'$. Set $|F| = 2^{2n+1}$.

(3) We have $C_H(Q/Z(Q)) = Q$.

Then we have one of the following possibilities :

(i) $z^G \cap Q \subseteq Z(Q)$.

(ii) There is a subgroup R of $Z(Q)$ with $|Z(Q) : R| = 2$ and $C_G(R) \nsubseteq C_H(R) . O(C_G(R))$. Further G is not of characteristic 2 type .

(iii) We have $n = 2$ and $H/Q \simeq \Sigma_3 \times \Sigma_3$ or $\Sigma_3 \wr Z_2$.

(iv) We have $n = 3$ and $H/Q \simeq A_8$. Further, $Q/Z(Q)$ is the natural $SO^+(6, 2)$-module and $Z(Q) = Z(H) \times V$, where V is the natural $L_4(2)$-module.

(v) We have $n = 4$ and $H/Q \simeq O^-(6, 2)$. Further, $Q/Z(Q)$ is the natural $U_4(2)$-module and $[H', Z(Q)] = 1$ but $Z(Q) \nsubseteq Z(H)$.

(vi) We have $n = 4$ and $H/Q \simeq (\Omega^-(6, 2) \times Z_3)\langle x \rangle$, where $\Omega^-(6, 2)\langle x \rangle \simeq O^-(6, 2)$ and $Z_3\langle x \rangle \simeq \Sigma_3$. Again $Q/Z(Q)$ is the natural $U_4(2)$-module and $[H'', Z(Q)] = 1$. If $r \in H$ and $o(r) = 3$ with $rQ \in O_3(H/Q)$, then $|[r, Z(Q)]| = 4$.

(vii) We have $n = 4$ and $H/Q \simeq Sp_6(2)$. Further, $Q/Z(Q)$ is an irreducible $Sp_6(2)$-module and $Z(Q)/Z(H)$ is the natural $Sp_6(2)$-module.

(viii) We have $n = 2m > 2$ and $H/Q \simeq Sp_{2m}(2) \times \Sigma_3$. As an $Sp_{2m}(2)$-module we have $Q/Z(Q) \simeq V_1 \times V_2$, where $V_1 \simeq V_2$ is the natural $Sp_{2m}(2)$-module. Further, $[H'', Z(Q)] = 1$ but $Z(Q) \nsubseteq Z(H)$.