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The combinatorial formula of Gabrielov, Gelfand and Losik for the first Pontrjagin class

THE COMBINATORIAL FORMULA OF GABRIELOV, GELFAND AND LOSIK

FOR THE FIRST PONTRJAGIN CLASS

by Robert MacPHERSON

The problem addressed here is to find a formula for the Pontrjagin classes of a polyhedral manifold X in terms of the combinatorial structure of X . The exis-tance of these classes was first established by Thom [18] by a nonconstructive argu-ment. One motivation for wanting an explicit formula is the hope of extending the deep results on signatures of elliptic operators which relate to the curvature for-mula for the Pontrjagin classes to the framework of difference operators on polyhe-dra [16]. Another motivation is the question of whether there exists a purely local formula for the Pontrjagin numbers[1] i.e. a formula using only the set of stars of vertices of X , not how they are put together. It is known that no topological inva-riants other than the Pontrjagin numbers or the Euler characteristic can be given by a purely local formula [10].

Combinatorial formulas exist for the other characteristic classes of X . For the Euler class, Euler's formula $\Sigma(-1)^i f_i$ where f_i is the number of i dimen-sional faces is a solution (see also [1]). The Poincaré dual of the total Stiefel-Whitney class in $H_i(X ; \mathbb{Z}/2\mathbb{Z})$ is the sum of all the i simplices in the barycen-tric subdivision ([17], [19], [2], [8]). Also there are other approaches to the Pontrjagin class than the one described here ([9], [11], [12], [13], [14]).

We report here on the remarkable formula of Gabrielov, Gelfand, and Losik for the first Pontrjagin class of a simplicial manifold [4-7], [3]. This is the first general, explicitly computable formula for a Pontrjagin class.


Let X be a simplicial complex and let σ be a simplex in X . The link of σ , $L\sigma$, is the subcomplex of M consisting of simplices σ' such that

1) $\sigma' \cap \sigma = \emptyset$,

2) for some σ'' , $\sigma' \subset \sigma'' \supset \sigma$.

The cone over the link of σ , $C L\sigma$, has a natural structure as a simplicial com-plex. A <u>flattening</u> of X at σ is a homeomorphism ψ from $C L\sigma$ to a neighborhood of the origin in $\mathbb{R}^k$ for some k which is linear on each simplex and which takes

[1]   See note 1, p. 497-18.

the cone point to the origin. The integer k is called the codimension of σ .
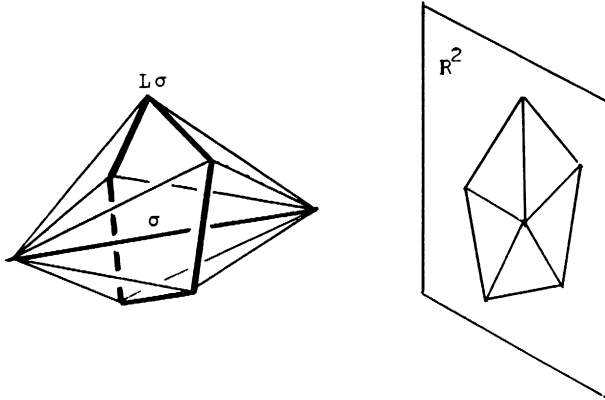
A flattening of σ .

DEFINITION.- A <u>simplicial manifold</u> is a simplicial complex which admits a flattening at each simplex.

(Note : there exist in dimension 4 and higher simplicial complexes that are piecewise linear manifolds but are not simplicial manifolds. However some barycentric subdivision of them will always be a simplicial manifold.)

For any simplicial manifold X we will construct a cell complex Γ over X

$$\Gamma$$
$$\downarrow \pi$$
$$X$$

and a cellular 4-cochain $\widetilde{P}$ in Γ so that

1) Γ and $\widetilde{P}$ are locally determined : for any open set U ⊂ X , $\pi^{-1}(U)$ and $\widetilde{P}$ restricted to $\pi^{-1}(U)$ depend only on the combinatorial structure in U .

2) $\pi^* : H^4(X) \to H^4(\Gamma)$ is injective.

3) $\widetilde{P} = \pi^* P_1(X)$ where $P_1(X)$ is the first Pontrjagin class of X .

The situation is analogous to the differentiable case. If X is a smooth manifold, the Pontrjagin class is represented by a natural differential form $\widetilde{P}$ on certain bundles over X , for example on the bundle of two jets of exponential maps. To get a representative differential form on X one must choose a section s and take $s^* \widetilde{P}$. There is a fundamental difference, however. In the smooth case there is no canonical section - one must choose an additional structure such as a connection in order to induce one. In the simplicial case a rational homological section (see § 3) can be chosen canonically. This provides an affirmative answer to the question of the existance of a purely local formula in this case.

§ 1. The hypersimplicial complex

In order to define Γ , we must first construct two other cell complexes over X . In this section we construct the hypersimplicial complex Δ ; in the next, the configuration complex K . Γ will the be a subcomplex of the fiber product of K and Δ over X .

For any finite set A , denote by $|A|$ the number of elements of A .

DEFINITION.- Let Z and A be disjoint sets and let n be an integer such that $|A| \le n \le |Z \cup A| - 2$. The hypersimplex $\Delta_n(Z,A)$ is the set of all functions h : Z ∪ A → **R** so that
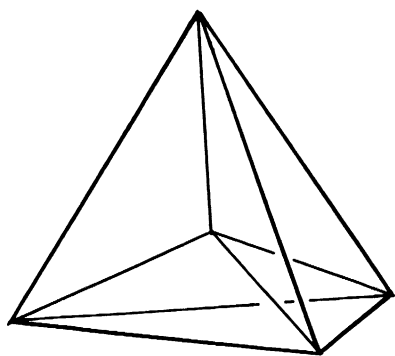
1) $0 \le h(e) \le 1$ if e ∈ Z

2) $h(e) = 1$ if e ∈ A

3) $\sum_{e \in Z \cup A} h(e) = n + 1$.

If $|A| = n + 1$, then $\Delta_n(A)$ denotes the function which is identically one on A .

The hypersimplex $\Delta_n(Z,A)$ is said to be of type p , q where p = $|Z \cup A| - 2 - n$ and $q = n - |A|$.

The following facts about hypersimplices will be useful :

1) The hypersimplex $\Delta_n(Z,A)$, of type p , q , is a convex polyhedron in Eucli-

yab ——————————————— zab

$\Delta_2(yz, ab)$

type 0 , 0

xab

yab   zab

$\Delta_2(xyz, ab)$

type 1 , 0

xya   xza

yza

$\Delta_2(xyz, a)$

type 0 , 1

wab

zab

xab   yab

$\Delta_2(wxyz, ab)$

type 2 , 0

wxa   wza

wya

xya   xza   yza

$\Delta_2(wxyz, a)$

type 1 , 1

wxz   wyz

wxy

xyz

$\Delta_2(wxyz, \emptyset)$

type 0 , 2
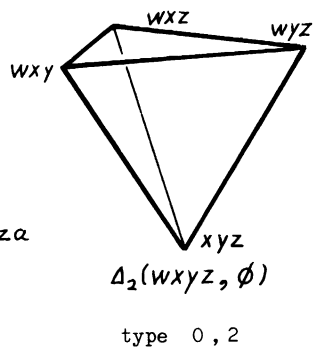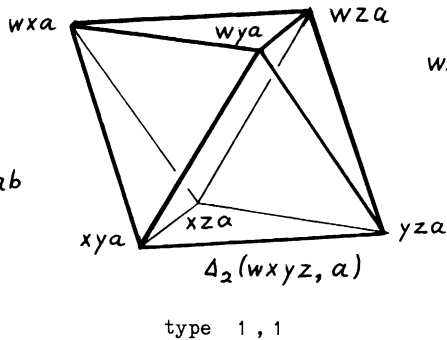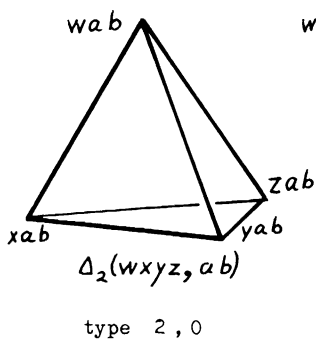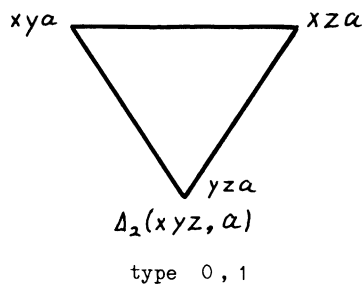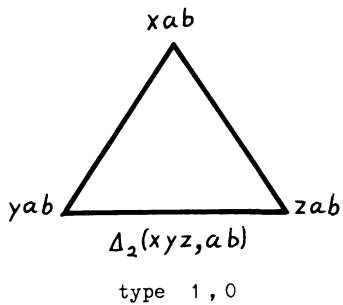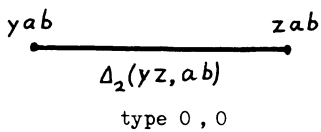
types 3 , 0 and 0 , 3
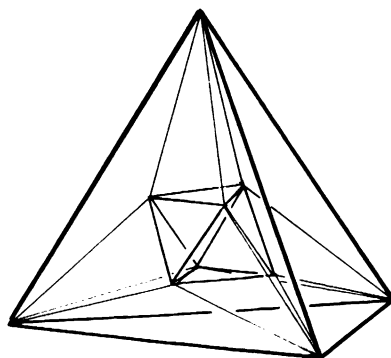
types 2 , 1 and 1 , 2

Hypersimplices

Vertices $\Delta_2(S)$ are labeled with the set S . The four dimensional figures are projected into R^3 .

dean $p + q + 1$ space. (Relations 2) and 3) of the definition define the space ; relation 1) gives the polyhedron.)

2)   The geometry of $\Delta_n(Z,A)$ depends only on p and q . This is clear from the fact that $\Delta_n(Z,A)$ is canonically isomorphic to $\Delta_{n-|A|}(Z,\emptyset)$.

3) The vertices of $\Delta_n(Z,A)$ are the $\Delta_n(S)$ where $A \subset S \subset Z \cup A$. ($\Delta_n(S)$ is contained in $\Delta_n(Z,A)$ by extension by zero.)

4)   The faces of positive dimension of $\Delta_n(Z,A)$ are the $\Delta_n(Z',A')$ where $A \subset A'$ and $Z' \cup A' \subset Z \cup A$. (Again $\Delta_n(Z',A')$ is contained in $\Delta_n(Z,A)$ by extension by zero.)

5)   An orientation of $\Delta_n(Z,A)$ is a parity class of orderings of Z . This is because $\Delta_n(Z,A)$ is subspace of the simplex obtained by replacing condition 1 with

1') $0 \leq h(e)$ if $e \in Z$

and Z corresponds to the vertices of this simplex.

From now on we consider $\Delta_n(Z,A)$ to be a cell complex whose cells are its faces as a convex set as described in 3) and 4) above.

For any simplicial complex Y , denote by $\underline{v}Y$ the set of vertices of Y .

DEFINITION.- The hypersimplicial complex Δ , a subcomplex of $X \times \Delta_n(\underline{v}X , \emptyset)$, is the union of the cells

$$\sigma \times \Delta_n(\underline{v}L\sigma , \underline{v}\sigma)$$

where σ is a simplex of X and n is the dimension of X .

A cell of Δ then is of the form $\sigma \times \Delta_n(Z,A)$ where $Z \cup A$ is a subset of the vertices of the star of σ and A contains the vertices of σ .

§ 2. The configuration complex

In this section, we construct a cell complex over X out of its flattenings. We think of a flattening $\psi : CL\sigma \to R^k$ as being given by the images of $\underline{v}L\sigma$, the vertices of the link of σ .

The set of flattenings $\psi : CL\sigma \to R^k$ forms a topological space on which the general linear group $GL(k)$ acts by composition. The quotient or orbit space of this action is called the configuration space $\Sigma(\sigma)$.

A flattening of X at a codimension k simplex σ is c-fold degenerate if exactly c of the k element subsets of $\underline{v}L\sigma$ are mapped into a linearly depen-

dent set in R^k . $\Sigma^c(\sigma)$ represents the subspace of the configuration space $\Sigma(\sigma)$ consisting of orbits containing c-fold degenerate flatte-nings. We call $\Sigma_1^c(\sigma)$,..., $\Sigma_j^c(\sigma)$,... the connected components of $\Sigma^c(\sigma)$; there are finitely many of them because $\Sigma^c(\sigma)$ is a semialgebraic set. It can be shown that every connected component of $\Sigma^1(\sigma)$ lies in the closure of exactely two con-nected components of $\Sigma^0(\sigma)$. (The determination of the set $\{\Sigma_j^c(\sigma)\}$ in terms of the triangulation of $I\sigma$ is the only part of the construction that is not purely com-binatorial i.e. programmable on a computer.)

For each σ of codimension ≤ 4 , we construct a cell complex $K(\sigma)$.

If σ is of codimension 4 , $K(\sigma)$ is a 0-complex with a point for each connected component of $\Sigma^0(\sigma)$. Denote by $K\Sigma_j^0(\sigma)$ the point for $\Sigma_j^0(\sigma)$.

If σ is of codimension 3 , $K(\sigma)$ is a graph. For each $\Sigma_j^0(\sigma)$ it has a vertex $K\Sigma_j^0(\sigma)$ and for each $\Sigma_j^1(\sigma)$ it has an edge $K\Sigma_j^1(\sigma)$ connecting the two vertices representing components of $\Sigma^0(\sigma)$ whose closures contain $\Sigma_j^1(\sigma)$.

If σ is of codimension 2 , $K(\sigma)$ is a 2-complex. The 1-skeleton is cons-tructed just as for codimension 3 . It is easily seen that each $\Sigma_j^2(\sigma)$ is in the closure of four components of $\Sigma^0(\sigma)$ and four of $\Sigma^1(\sigma)$ whose corresponding ver-tices and edges form a square. Fill this in wich a disk $K\Sigma_j^2(\sigma)$. The following figure illustrates the situation. Near each cell of $K(\sigma)$ a flattening is drawn whose orbit is in the corresponding component of $\Sigma(\sigma)$.
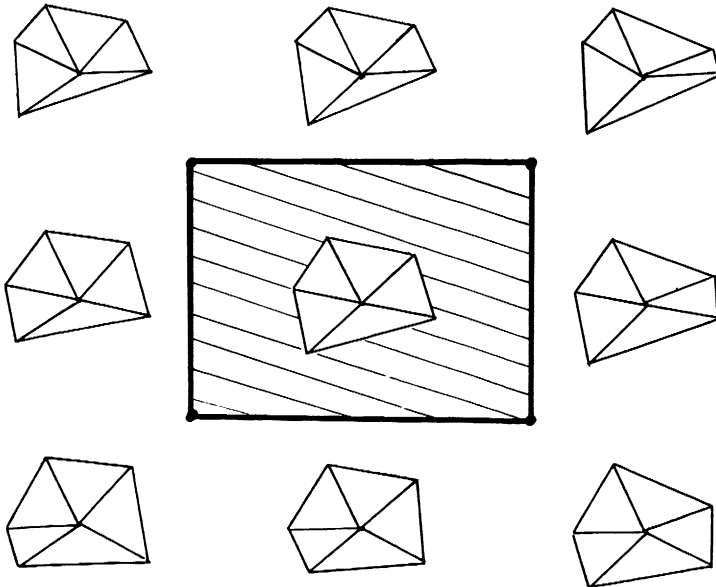
If σ has codimension 1 or 0 , $K(\sigma)$ is a point.

Suppose σ' , of codimension k' is contained in σ . Then a map $a_{\sigma'\sigma} : \Sigma(\sigma') \to \Sigma(\sigma)$ is defined to take the orbit of ψ' to the orbit of ψ in the following diagram :

$$\begin{array}{ccc}
\underline{v}L\sigma' \xrightarrow{\psi'} & R^{k'} \longrightarrow & R^{k'}/\,\text{span}(\psi'\,\underline{v}L\sigma' \cap \underline{v}\sigma) \\
\cup & & \downarrow \cong \\
\underline{v}L\sigma \xrightarrow{\hspace{3cm}\psi} & R^{k} &
\end{array}$$

where the isomorphism $\cong$ is chosen arbitrarily. This map takes any $\Sigma^{c'}_{j}(\sigma')$ to some $\Sigma^{c}_{j}(\sigma)$ for $c \le c'$. If $k' \le 4$, a cellular map $\bar{a}_{\sigma'\sigma} : K(\sigma') \to K(\sigma)$ exists taking $K\Sigma^{c'}_{j}(\sigma')$ to $K\Sigma^{c}_{j}(\sigma)$ whenever $a_{\sigma'\sigma}(\Sigma^{c'}_{j}(\sigma')) \subset \Sigma^{c}_{j}(\sigma)$.

For any σ in X , the dual cell denoted by $D\sigma$ is defined to be the union of all simplices in the barycentric subdivision of X whose intersection with σ is the barycenter of σ . The dual cells form a regular cell decomposition of X called the dual cell decomposition ; we call X with this cellulation X^{D} . The dimension of $D\sigma$ is the codimension of σ ; if $\sigma' \subset \sigma$ then $D\sigma \subset D\sigma'$.

DEFINITION.- The <u>configuration complex</u> K is the cell complex constructed as follows : we take the disjoint union $\quad \cup \, D\sigma \times K(\sigma)$ over all simplices σ of codimension ≤ 4 and for each $\sigma' \subset \sigma$ we perform the identifications

$$(ix \times y) \sim (x \times \bar{a}_{\sigma'\sigma}y)$$

where $i : D\sigma \subset D\sigma'$ is the inclusion.

K projects cellularly to the 4-skeleton of X^{D} .

§ 3. Homological sections

First we define our fundamental object $\Gamma \xrightarrow{\pi} X$. The hypersimplicial complex Δ projects cellularly to X and the configuration complex K projects cellularly to X^{D} . The fiber product $K \times_{X} \Delta$ has an obvious cell decomposition that projects cellularly to X' , the common refinement of X and X^{D} . A cell in X' has the form $\sigma \cap D\sigma'$ where $\sigma' \subset \sigma$. A cell in $K \times_{X} \Delta$ will have the form

$$(\sigma \cap D\sigma') \times K\Sigma^{c}_{j}(\sigma') \times \Delta_{n}(Z,A)$$

where $\Delta_{n}(Z,A)$ is a face of $\Delta_{n}(\underline{v}L\sigma , \underline{v}\sigma)$. We denote this cell by $\gamma(\sigma' , c , j , \sigma , Z , A)$.

DEFINITION.- Γ is the subcomplex of $K \times_X \Delta$ consisting of those cells $\gamma(\sigma', c, j, \sigma, Z, A)$ such that if $\Delta_n(S)$ is a vertex of $\Delta_n(Z, A)$ and $\psi : C L \sigma' \to R^k$ is a flattening whose orbit is in $\Sigma_j^c(\sigma')$, then the k-tuple $\psi(\underline{v} L \sigma' \cap S)$ is a linearly independent set of vectors in R^k.

We note that if $c = 0$, $\gamma(\sigma', c, j, \sigma, Z, A)$ will always be in Γ.

Denote by $C_i^D(X)$ the X^D cellular i-chains,

by $C_D^i(X)$ the X^D cellular i-cochains,

by $C_i'(X)$ the X' cellular i-chains,

and by $r : C_i^D(X) \to C_i'(X)$ the refinement map.

DEFINITION.- If R is a ring, an R-homological j-section of $\Gamma \xrightarrow{\pi} X$ is a homomorphism $s : C_i^D(X, R) \to C_i(\Gamma, R)$ for each $i \leq j$ so that

1) $\partial s = s \partial$ (s is a chain map),

2) $\pi_* s = r$,

3) support $s(c) \subset \pi^{-1}$ support c.

A homological j-section shares many of the properties of an ordinary section. We may define $s^* C^i(\Gamma) \to C^i(X)$ by the formula $s^* k(c) = k s(c)$ for $k \in C^i(\Gamma)$ and any $c \in C_i^D(X)$. This induces a map s^* on cohomology so that $s^* \pi^*$ is the identity. In particular if a homological section exists, π^* is an injection.

PROPOSITION 1.- $\pi : \Gamma \to X$ admits an integral homological 4-section.

This follows from the

Lemma.- Let σ be a simplex of X of codimension ≤ 4 and let $b D \sigma$ denote the boundary of $D \sigma$, i.e. the union of its proper faces. Then any integral or rational homological section s of $\pi^{-1} b D \sigma \xrightarrow{\pi} b D \sigma$ extends to the same on $\pi^{-1} D \sigma \xrightarrow{\pi} D \sigma$.

The proof, which we omit, uses that $K(\sigma)$ is 1-connected for σ of codimension 3, (see Note 2, p.497-18) and that $K(\sigma)$ is 2-connected for σ of codimension 2, which can be seen by direct analysis since configurations in R^2 are so simple.

Note that Γ is locally determined in the sense that $\pi^{-1} D \sigma$ depends only on the geometry of the star of σ. A homological section is called locally determined if its value on a chain with support in $D \sigma$ is given by a canonical procedure in

terms of the star of σ .

PROPÓSITION 2.- Γ has a locally determined rational 4 section $\tilde{s}$.

Proof. We may define $\tilde{s}$ in the following canonical if ad hoc manner. We proceed by induction on the dimension k of the dual cell $D\sigma$. For $k = 0$, there is a unique homological section since π is a homeomorphism. For $k > 0$, assume $\tilde{s}$ is already defined as a rational section of $\pi^{-1} b D\sigma \xrightarrow{\pi} b D\sigma$. Then S , the set of rational extensions to $\pi^{-1} D\sigma \xrightarrow{\pi} D\sigma$, is nonempty. Choose a generator $[D\sigma]$ of the integral chains $C_i^D(D\sigma , \mathbb{Z})$. For $s \in S$, let $j(s)$ be the smallest integer j' so that the rational chain $s([D\sigma])$ can be written with coefficients whose denominators are all divisors of j' . Let j be the minimum of $j(s)$ for all s and define S_j to be the set of $s \in S$ so that $j(s) = j$. Define the norm of s to be the sum of the absolute values of all the coefficients of $s([D\sigma])$. Let m be the minimum of the norms of s for $s \in S_j$ and define $S_j^m \subset S_j$ to be the set of elements with norm m .

Clearly, S_j^m is independent of the choice of $[D\sigma]$. Because Γ is a finite cell complex, S_j^m is a nonempty, finite set. Now define $\tilde{s}$ on $D\sigma$ to be the average of the elements of S_j^m

$$\tilde{s} = \frac{1}{|S_j^m|} \sum_{s \in S_j^m} s .$$

§ 4. The formula

We define a cellular 4 cochain $\tilde{P}$ on Γ . For each 4-cell $\gamma(\sigma' , c , j , \sigma , Z , A)$ and for each orientation $[\gamma]$ of that 4-cell, we must give a number $\tilde{P}([\gamma])$.

$\tilde{P}([\gamma]) = 0$ unless $\sigma' = \sigma$, $c = 0$, and $\Delta_n(Z,A)$ is of type $2,1$ or $1,2$.

For $\gamma(\sigma , 0 , j , \sigma , Z , A)$ where $\Delta_n(Z,A)$ is of type $2,1$, let the orbit of $\psi : CL(\sigma) \to R^k$ be in $\Sigma_j^0(\sigma)$ and call θ the composition

$$Z \subset \underline{v} L\sigma \xrightarrow{\psi} R^k \longrightarrow R^k / \text{span } \psi(\underline{v} L\sigma \cap A) = V \quad ;$$

V will be two dimensional. Pick arbitrarily an orientation $\mho$ of V , $\mho \in \Lambda^2 V$. Pick an ordering $<$ of Z that gives the orientation $[\gamma]$. Let i be the number of ordered pairs z_1 , z_2 , of elements of Z so that $z_1 < z_2$ and $\theta z_1 \wedge \theta z_2$ agrees in sign with $\mho$. Then

$$\tilde{P}([\gamma]) = -\frac{(-1)^i}{48} \quad .$$

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For $\gamma(\sigma, 0, j, \sigma, Z, A)$ where $\Delta_n(Z,A)$ is of type $1, 2$, define $\theta : Z \rightarrow V$ analogously ; now V is three dimensional. Let i be the number of ordered triples z_1, z_2, z_3 of elements of Z so that $z_1 < z_2 < z_3$ and $\theta z_1 \wedge \theta z_2 \wedge \theta z_3$ agrees in sign with σ. Then

$$\widetilde{P}([\gamma]) = \frac{(-1)^i}{48} .$$

Note that $\widetilde{P}([\gamma])$ is independent of the choice of ψ and σ, and changes sign when $[\gamma]$ does, i.e. when the ordering of Z is subjected to an odd permutation.

THEOREM.- $\widetilde{P}$ represents $\pi^* P_1(X)$, the pull up of the first Pontrjagin class of X.

A proof of this theorem is sketched in $\S 5 - \S 8$.

COROLLARY 1. $P_1(X) = s^* \widetilde{P}$ for any homological 4-section s of Γ.

We may choose s to be an integral homological 4-section and obtain a formula with 48 in the denominator. Or we may choose $\widetilde{s}$ of Proposition 2 and obtain a canonical formula without control on the denominators.

Suppose that X is oriented and that for each simplex σ of codimension 4 generators $[\sigma] \in C_{n-4}(\sigma ; \mathbb{Z})$ and $[D\sigma] \in C_4^D(D\sigma, \mathbb{Z})$ are chosen so that $[\sigma]$ and $[D\sigma]$ have intersection multiplicity $+ 1$.

COROLLARY 2.- The Poincaré dual class to $P_1(X)$ is given by

$$\sum_\sigma \widetilde{P}(s([D\sigma])) \cdot [\sigma]$$

where s is any homological section.

If in corollary 2 s is chosen to be $\widetilde{s}$, we have a purely local formula : the coefficient of $[\sigma]$ depends only on the combinatorial structure of $L\sigma$.

Remarks.- 1) With more work, one can have a formula with at most 12 in the denominators.

2) The cells of X need not be simplices. A simple cell is a convex polyhedron in Euclidean space whose faces meet mutually transversely. (A solid cube is one, a solid octahedron isn't.) The formula holds for a manifold decomposed into simple cells so that the inclusions preserve the affine structure ; it is just slightly

harder to define Γ .

§ 5. The idea of the proof

If M is a smooth n-manifold embedded in Euclidean space R^N , it has a Gauss map
$g : M \rightarrow G$, where G is the Grassman variety of n-planes in R^N , which associa-
tes to each point x on M the tangent space to M at x . There is an invariant
differential form $\bar{P}$ on G so that $P_1(M) = g^*\bar{P}$. So the Pontrjagin class is a
particular measure of how $g(x)$ changes as x moves in M , i.e. of the curvature
of M .

Now suppose X were embedded in Euclidean space in such a way that it could be
approximated by a smooth manifold M .
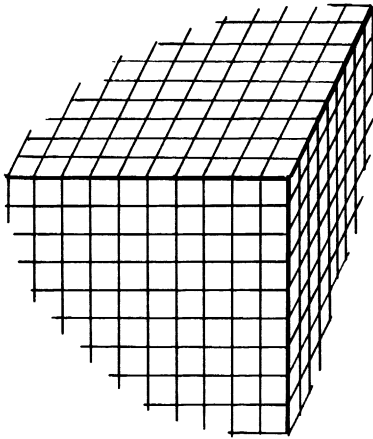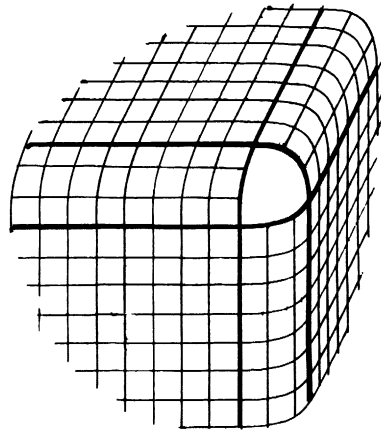
$$X \qquad\qquad\qquad\qquad M$$

As the approximation becomes better, the Gauss map for M becomes discontinuous.
Intuitively in the limit it will be constant along the n-simplices, and will take
a k dimensional set of values at a codimension k simplex σ . Motivated by this,
we will construct a homological Gauss map which assigns to $D\sigma$ a k dimensional
set of "virtual" tangent planes. But first we need a digression on Grassmannians.

§ 6. Structure of the Grassmannian

In this section, we show that a point in a Grassman manifold is determined by a
projective configuration and a point in a hypersimplex.

Let Z and A be disjoint finite sets and let $R^{Z \cup A}$ denote the $|Z \cup A|$

dimensional vector space generated by $Z \cup A$. The Grassman manifold $G_n(Z,A)$ is the space of $n+1$ dimensional planes in $\mathbb{R}^{Z \cup A}$ which contain $\mathbb{R}^A$. Note that $G_n(Z,A)$ identifies canonically with $G_{n-|A|}(Z, \emptyset)$.

The most convenient way to think of a point in $G_n(Z,A)$ is as a $\underline{(Z, A, n)}$ $\underline{\text{vector configuration}}$ w. This is defined to be the choice of a vector $w(e)$ in a vector space V of dimension $n+1$ for every e in $Z \cup A$ such that V is the direct sum $\operatorname{span} w(Z) \oplus \operatorname{span} w(A)$ and the vectors $w(a)$, $a \in A$, are linearly independent. We consider two (Z, A, n) vector configurations $w : Z \cup A \to V$ and $w' : Z \cup A \to V'$ equivalent if there is a linear isomorphism $V \to V'$ taking $w(e)$ to $w'(e)$ for all $e \in Z \cup A$. Given a point p in $G_n(Z,A)$, the associated (Z, A, n) vector configuration is obtained by orthogonally projecting the unit vectors in $\mathbb{R}^{Z \cup A}$ to the plane V representing p.

Let R_+ be the positive real numbers and let $R_+^{Z \cup A}$ be the abelian group of functions $a : Z \cup A \to R_+$ whose group law is pointwize multiplication. $R_+^{Z \cup A}$ acts on $G_n(Z,A)$ by multiplying the vectors of the configurations :

$$aw(e) = a(e)w(e) .$$

The quotient space $G_n(Z,A)/R_+^{Z \cup A}$ is called the space of $\underline{(Z,A,n)}$ $\underline{\text{enhanced pro-}}$ $\underline{\text{jective configurations}}$ $P_n(Z,A)$; the quotient map is called τ. This space is T_0 but not T_1 or Hausdorff. One may identify a point in $P_n(Z,A)$ with an ordinary projective configuration in $\mathbb{R}P^n$ of points indexed by a set S where $A \subset S \subset Z \cup A$ enhanced by a lift to the double cover of $\mathbb{R}P^n$. Note that $P_n(Z,A)$ identifies canonically with $P_{n-|A|}(Z, \emptyset)$.

Our aim is to reconstruct $G_n(Z,A)$ from $P_n(Z,A)$ is as efficient a way as possible. To this end we define a map $\rho : G_n(Z,A) \to \Delta_n(Z,A)$ as follow : let $w : Z \cup A \to V$ be a point in $G_n(Z,A)$. Choose an invariant measure μ on V. For any $n+1$ element subset S of $Z \cup A$ denote by S^μ the measure of the parallelogram spanned by $w(S)$, i.e.

$$S^\mu = \mu(\{ \sum_{e \in S} \theta_e \, w(e) \mid 0 \le \theta_e \le 1 \}) .$$

Note that S^μ is the absolute value of the S Plücker coordinate. Now $\rho(w) = h$ where

$$h(e) = \frac{\displaystyle\sum_{S \text{ containing } e} S^\mu}{\displaystyle\sum_{\text{all } S} S^\mu}$$

for $e \in Z \cup A$. One may check that $h(e)$ is independent of μ and satisfies the

three conditions of the definition of $\Delta_n(Z,A)$.

If P is in $P_n(Z,A)$, let w be any lift to a (Z,A,n) vector configuration. Then $\Delta_n(Z,A,P)$ is defined to be that subset of the hypersimplex $\Delta_n(Z,A)$ which is the convex hull of the vertices $\Delta_n(S)$ for which $S^\mu > 0$.

Then $\Delta_n(Z,A,P)$ is itself a polyhedron possibly of smaller dimension. $\overset{\circ}{\Delta}_n(Z,A,P)$ is the interior of $\Delta_n(Z,A,P)$, that is $\Delta_n(Z,A,P)$ minus its proper faces. If $\Delta_n(Z,A,P) = \Delta_n(Z,A)$ then P is called <u>generic</u> ; the set of generic enhanced projective configurations is denoted $P_n^{gen}(Z,A)$.

PROPOSITION.- For any $P \in P_n(Z,A)$, ρ takes the $R_+^{Z \cup A}$ orbit given by P homeomorphically onto $\overset{\circ}{\Delta}_n(Z,A,P)$, and takes the closure of the orbit homeomorphically onto $\Delta_n(Z,A,P)$

COROLLARY.- $G_n(Z,A) \overset{\tau,\rho}{\subseteq} P_n(Z,A) \times \Delta_n(Z,A)$

is an embedding.

§ 7. <u>The homological Gauss map</u>

Embed X in $R^{\underline{v}X}$ by sending each vertex to the corresponding basis vector and extending linearly. In this section we construct a homological 4-map from Γ to $G_n(\underline{v}X , \emptyset)$, i.e. a chain map $C_i(\Gamma) \to C_i(G_n(\underline{v}X , \emptyset))$ for $i \le 4$. Then for any homological section s of Γ , $g \circ s$ will be our candidate for a Gauss map as in § 5 in the sense that if $V \subset R^{\underline{v}X}$ represents a point in the support of $g \circ s([D\sigma])$, then we consider $V \cap \{x_1, \ldots, x_{|\underline{v}X|} \mid \sum x_i = 1\}$ to be a "virtual" tangent plane to X at σ .

Consider a cell $\gamma = \gamma(\sigma', c, j, \sigma, Z, A)$ in Γ where $\Delta_n(Z,A)$ is of type p , q . The cell γ determines a connected component ξ_γ of $P_n^{gen}(Z,A) \cong P_p^{gen}(Z, \emptyset)$ in this way : let $\psi : C L(\sigma') \to R^k$ be a flattening whose orbit is in $\Sigma_j^c(\sigma')$, then

$$Z \subset \underline{v} L\sigma' \overset{\psi}{\longrightarrow} R^k \longrightarrow R^k / \text{span } \psi(\underline{v} L\sigma' \cap A)$$

gives a $(Z , \emptyset , p)$ vector configuration τ of which is generic, i.e. in $P_p^{gen}(Z , \emptyset)$, by the definition of Γ . All such choices of ψ lead to the same connected component ξ_γ of $P_p^{gen}(Z , \emptyset)$.

Now if $p = 0$ or $q = 0$, $P_p^{gen}(Z , \emptyset)$ is a discrete set, since there are no

moduli for $q + 2$ points in projective 0 space or $p + 2$ generic points in projective p space (the latter because such a system of $p + 2$ points can be used to coordinatize the space). So we can define a map $\bar{g}$ from $\overset{o}{\gamma}$, the interior of γ , to $G_n(\underline{v}X , \emptyset)$ by requiring commutativity of this diagram :

$$
\begin{array}{ccc}
\overset{o}{\gamma} & \overset{\bar{g}}{\longrightarrow} & G_n(\underline{v}X , \emptyset) \\
 & & \cup \\
 & & G_n(Z , A) \\
\Big\downarrow & & \cap \tau , \rho \\
\overset{o}{\Delta}_n(Z,A) & \longrightarrow & P_n(Z,A) \times \overset{o}{\Delta}_n(Z,A) \\
h & \longmapsto & \xi_\gamma \times h
\end{array}
$$

$\bar{g}$ defines a homomorphism $g = \bar{g}_* : C_i(\Gamma) \to C_i(G_n(\underline{v}X , \emptyset))$ where, if $C_i(G_n(\underline{v}X , \emptyset))$ is interpreted as i-currents on $G_n(\underline{v}X , \emptyset)$, to a chain in $C_i\Gamma$ we assign integration of an i-form over $\bar{g}$ of that chain. One may check that $\bar{g}$ extends continuously to the boundary of γ so $\bar{g}_*$ is a chain map defined on cells $\gamma \in \Gamma$ such that $p = q = 0$. Since we want a homological 4-map, it remains to consider types $(1,1)$, $(2,1)$ and $(1,2)$.
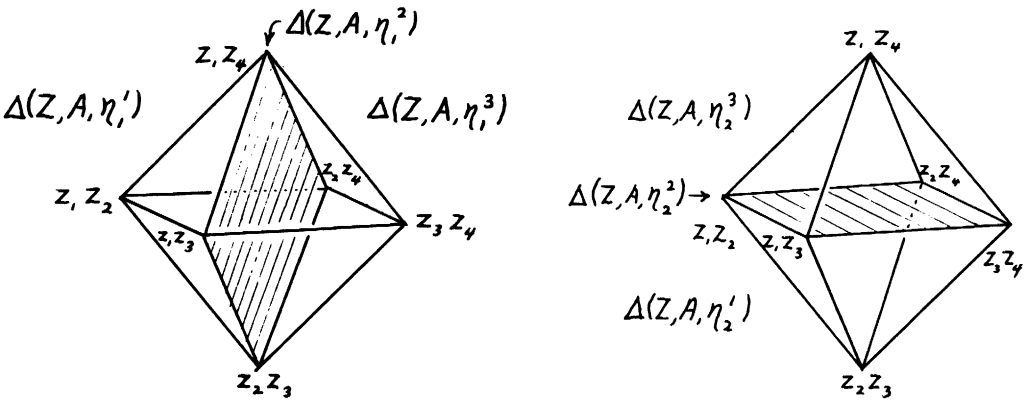
If $p = q = 1$, ξ_γ will be a connected component of generic enhanced configurations of four points on the projective line. These configurations have a continuous modulus, the cross-ratio, which we avoid by choosing modulus free configurations in the closure of ξ_γ . Suppose the elements of Z are labeled z_1 , z_2 , z_3 , z_4 in such a way that a configuration in ξ_γ is a lift of an ordinary projective configuration that looks like this
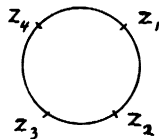
Here the circle represents the real projective line. Then there are unique elements η_i^k of the closure of ξ_γ that are lifts respectively of these

$$\eta_1^1 \qquad \eta_1^2 \qquad \eta_1^3 \qquad \eta_2^1 \qquad \eta_2^2 \qquad \eta_2^3$$
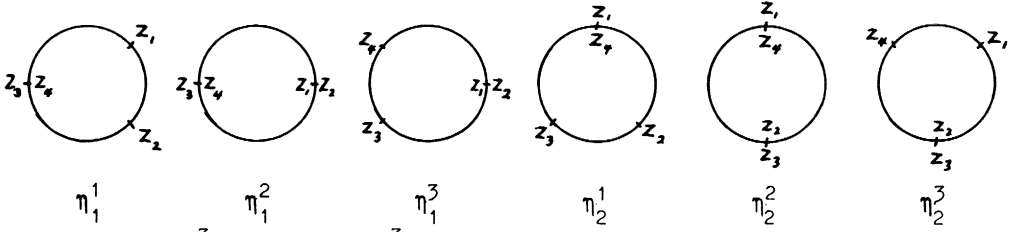
Now $\overset{\circ}{\Delta}_n(Z,A) = \bigcup\limits_{k=1}^{3} \overset{\circ}{\Delta}(Z,A,\eta_1^k) = \bigcup\limits_{k=1}^{3} \overset{\circ}{\Delta}(Z,A,\eta_2^k)$

We define two maps $\bar{g}_1$ and $\bar{g}_2$ on the interior of γ by requiring commutativity :

$$
\begin{array}{ccc}
\overset{\circ}{\gamma} & \xrightarrow{\quad \bar{g}_i \quad} & G_n(\underline{v}X,\emptyset) \\
\downarrow & & \cup \\
\overset{\circ}{\Delta}_n(Z,A) & & G_n(Z,A) \\
\cup & & \cap \\
\overset{\circ}{\Delta}_n(Z,A,\eta_i^k) & \xrightarrow{\qquad} & P_n(Z,A) \times \overset{\circ}{\Delta}_n(Z,A) \\
h & \longmapsto & \eta_i^k \times h
\end{array}
$$

Wishing to be canonical, we define g on γ to be $\frac{1}{2}(\bar{g}_{1*} + \bar{g}_{2*})$. Since both $\bar{g}_1$ and $\bar{g}_2$ extend our previous definitions continuously, g is still a chain map.

We observe that $g_1(\gamma)$ and $g_2(\gamma)$ together bound a very interesting 4-cell in $G_n(Z,A) \cong G_2(Z,\emptyset)$ which is itself 4-dimensional. This cell, which we call

N_γ , has as its interior $\tau^{-1}(\xi_\gamma)$. If we choose proper orientations $[N_\gamma]$ and $[\gamma]$ then $\partial[N_\gamma] = \bar{g}_{1*}[\gamma] - \bar{g}_{2*}[\gamma]$.

If $p = 2$ and $q = 1$, we proceed in a similar way taking the configurations determined from ξ_γ by the following list.

ξ_γ $\quad$ η_1^1 $\quad$ η_1^2 $\quad$ η_1^3 $\quad$ η_1^4 $\quad$ η_1^5
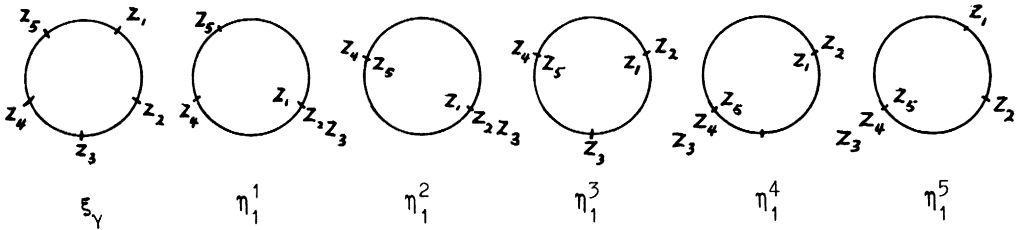
and η_i^k for $1 < i \leq 5$ is the same but with the indices of the z's cyclically permuted. Define $\bar{g}_i$ as before. But now the question of continuity becomes more interesting. If the cell with $p = q = 1$ described above is called γ' , then one easily verifies that $\bar{g}_1$ and $\bar{g}_4$ on γ extend continuously to $\bar{g}_2$ on γ' , and $\bar{g}_2$, $\bar{g}_3$ and $\bar{g}_5$ on γ extend to $\bar{g}_1$ on γ' . Therefore we are forced to define g on γ with correction terms

$$g([\gamma]) = \frac{1}{5} \sum_{i=1}^{5} \bar{g}_{i*}[\gamma] - \frac{1}{10} \sum_{\gamma'} [N_{\gamma'}]$$

where the second sum is over all five faces of type $1 , 1$ of γ ; $(-\frac{1}{10} = \frac{1}{2} - \frac{3}{5})$.

The story for type $2 , 1$ is entirely parallel : in fact there is a formal duality for Δ , G , and P reversing p and q .

PROPOSITION.- $g^*\bar{P} = \tilde{P}$ where $\bar{P}$ is the unique $O(|\underline{v}X|)$-invariant differential 4-form on $G_n(\underline{v}X , \emptyset)$ representing the Pontrjagin class of the tautological bundle η .

Essentially the reason for this is that $\bar{P}$ integrates to zero on all the types of 4-cells in $G_n(\underline{v}X , \emptyset)$ involved in the construction of g except for the N_γ where it integrates to $\frac{1}{24}$. (The Grassmannian of 2-planes in 4-space decomposes into 24 cells congruent to N_γ .) And $5 \cdot \frac{1}{10} \cdot \frac{1}{24} = \frac{1}{48}$.

The cell N_B has another distinguishing trait : all the other surfaces in $G_n(\underline{v}X , \emptyset)$ involved in the construction of g were orbits of the $R_+^{\underline{v}X}$ action

corresponding to projective configurations without continuous moduli. Thus we have
the following interpretation of the proof : the first Pontrjagin class is the obs-
truction to finding a homological Gauss map using only

τ^{-1} (the discrete part of $P_n(\underline{v}X, \emptyset)) \subset G_n(\underline{v}X, \emptyset)$.

§ 8. Microbundles and vector bundles

All that remains in order to prove the theorem is that $g \circ s$ deserves to be called
a Gauss map.

Let $S \subset \Gamma \times G_n(\underline{v}X, \emptyset)$ be the union of the sets $\gamma \times$ support $g([\gamma])$.

PROPOSITION.- On S , the tautological bundle η of $G_n(\underline{v}X, \emptyset)$ identifies as a
topological microbundle with $TX \oplus 1$, the tangent microbundle to X plus a tri-
vial line bundle.

Then since the Pontrjagin class extends to topological microbundles [15] and is
unchanged by $\oplus 1$, we have $g^*\overline{P} = \pi^*P_1(X)$ and we're done.

To establish the proposition, let Λ be the union of the following sets in
$\Gamma \times G_n(\underline{v}X, 0)$

$$\gamma(\sigma', c, j, \sigma, Z, A) \times [\tau \Sigma_j^c(\sigma') \times \overset{\circ}{\Delta}(\underline{v}L\sigma, \underline{v}\sigma, P_{\sigma', c, j})]$$

where $P_{\sigma', c, j}$ is any element of $\tau(\Sigma_j^c(\sigma'))$. Using the fact that $\Sigma_j^c(\sigma')$ is
connected, we can find a section λ of Λ over Γ . Now for each point in S we
have three subsets of $R^{\underline{vX}}$. The first is the fiber to $TX \oplus 1$: this embeds on
the TX factor by the inclusion of X and on the 1 factor by the line connec-
ting to the origin. The second is the fiber of η and the third is the fiber of
η pulled back by the section λ . Orthogonal projection takes the third homeomor-
phically onto both the first and the second. $^{\bullet}$

Note 1

This question has been answered affirmatively by Levitt and Rourke [20]. However their method does not give an explicit formula.

Note 2

That the space $K(\sigma)$ is connected for σ of codimension 3 follows from Theorem 1 of [21]. This eliminates the need for the condition [A] of [4-7] on the polyhedron X . I am greatful to R. Porter for pointing out the reference to me.

Note 3

Gabrielov has reportedly found a generalization of this work to the higher Pontrjagin classes.

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