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STABILITY AND GENERICITY IN DYNAMICAL SYSTEMS

par Stephen SMALE

A general reference to this subject, with examples, written about the summer of 1967 is [7], (reported in a recent Bourbaki Seminar by C. Godbillon). Here I will try to emphasize developements since. An important source of much of this more recent work should appear in the immediate future [1].

For simplicity we restrict ourselves to a dynamical system defined by a diffeomorphism f of a compact manifold M into itself. This is the case of a discrete differentiable dynamical system with time represented by the number of times f is iterated or the n in f^n. Most of the results discussed here are valid also in the case of a dynamical system defined by a 1st order ordinary differential equation.

The space of all dynamical systems will be denoted by $\mathrm{Dyn}(M)$, topologized by putting the C^r uniform topology on the corresponding diffeomorphism f, $1 \leq r \leq \infty$. The study of the dynamical system is the study of the orbits $O(x) = \{f^n(x) \mid n \in Z\}$ of f especially from the global point of view. Thus a natural equivalence relation is topological conjugacy, i.e., $f, g \in \mathrm{Dyn}(M)$ are <u>topologically conjugate</u> if there is a homeomorphism $h : M \to M$ such that $fh = hg$. Clearly such an h sends the orbits of f onto the orbits of g.

More than 10 years ago I posed the problem of finding a dense open set U (or at least a Baire set) of $\mathrm{Dyn}(M)$ such that the elements of U could somehow be described qualitatively by discrete numerical and algebraic invariants. Since this problem has been often quoted since, I would like to take this opportunity to revise the problem in the light of what has been learned in these 10 years.

The problem posed in this way is too simple, too rough and too centralized. I believe now that the main problems of dynamical systems can't be unified so elegantly. The above problem however can be split apart so that it makes good sense and in my opinion gives some perspective to the subject. This goes as follows.

One should search for a sequence of subsets U_i of $\mathrm{Dyn}(M)$, $U_1 \subset U_2 \subset \ldots \subset U_k \subset \mathrm{Dyn}(M)$, k not too large, U_i open (or at least say a Baire subset of an open set) with U_k dense in $\mathrm{Dyn}(M)$. One main feature of the U_i is that as i increases, U_i includes a substantially bigger class of dynamical system, but as i decreases one has a deeper understanding and the elements of U_i have greater regularity (or stability properties). So U_1 should consist of the simplest best-behaved non-trivial class of dynamical systems, and U_k cannot be expected to have very strong stability properties at all.

To give a better idea of what I am saying, I will give a schema of the U_i now which to some extent illustrates our state of knowledge of dynamical systems. (There will always be some arbitrariness in the exact choice of the U_i.) We will first state briefly the choice of the U_i's and the rest of the talk will give some justification for our choice, defining the necessary terms as we proceed. In each of the following U_i there is a large class of examples not in the preceeding U_{i-1}. The reader may consult the literature cited for many of these.

$U_1 = \{f \in \mathrm{Dyn}(M) \mid f$ satisfies Axiom A, $\Omega(f)$ is finite and f satisfies the transversality condition$\}$

$U_2 = \{f \in \mathrm{Dyn}(M) \mid f$ satisfies Axiom A and the strong transversality condition$\}$

$U_3 = \{f \in \mathrm{Dyn}(M) \mid f$ satisfies Axiom A and the no cycle condition$\}$

$U_4 = \{f \in \mathrm{Dyn}(M) \mid f$ satisfies the main known generic properties$\}$

We have used the unifying language of Axiom A which can be stated as follows : $f \in \mathrm{Dyn}(M)$ satisfies <u>Axiom A</u> if the non-wandering set $\Omega = \Omega(f)$ has a hyperbolic structure and the periodic points of f are dense in Ω . We recall that Ω is the closed invariant set of $x \in M$ such that for any neighborhood U of x there is some $n > 0$ with $f^n(U) \cap U \neq \emptyset$. A <u>hyperbolic structure</u> on Ω is a continuous splitting of the tangent bundle $T_\Omega(M)$ of M restricted to Ω , $T_\Omega(M) = E^u + E^s$, invariant under the derivative, Df , such that Df is contracting on E^s and $D(f^{-1})$ is contracting on E^u . Finally $Df : E^s \to E^s$ is said to be <u>contracting</u> if given a Riemannian metric on M , there is $c > 0$, μ , $0 < \mu < 1$ with $\|Df^m(x)\ (v)\| \leq c\,\mu^m \|v\|$ for all $v \in E^s$.

A <u>generic property</u> is a property that is true for some Baire set in $\mathrm{Dyn}(M)$.

A basic notion for the study of dynamical systems is the notion of stable manifold. Given $f \in \mathrm{Dyn}(M)$ and some fixed metric on M , we say that $x \sim_s y$ if $d(f^m(x)\,,\,f^m(y)) \to 0$ as $m \to \infty$. This is an equivalence relation ; the equivalence class of x is denoted by $W^s(x)$ and called the <u>stable manifold</u> of x . The following theorem is a consequence of the work of a number of mathematicians, see especially [1], [5].

THEOREM 1.- <u>If</u> $f \in \mathrm{Dyn}(M)$ <u>satisfies Axiom A, than for each</u> $x \in M$, $W^s(x)$ <u>is a smoothly, injectively immersed open cell.</u>

It is an outstanding question as to whether the conclusion of Theorem 1 is a generic property.

The <u>unstable manifolds</u> of f are the stable manifolds of f^{-1} and are denoted by $W^u(x)$.

With this behind us consider the dynamical systems in U_1 . That Ω is finite implies that Ω consists of the periodic points of f , and Axiom A amounts to saying that if $x \in \Omega$ has period m , then $Df^m(x) : T_x \to T_x$ has no eigenvalues of absolute value 1 . The transversality condition of U_1 means that if $x \in M$, then $W^s(x)$ and $W^u(x)$ meet transversally at x . (Stated in this manner, this transversality condition coincides with the strong transversality condition of U_2 .)

If $f \in U_1$, then f has indeed very strong stability properties. Say that $f \in \mathrm{Dyn}(M)$ is <u>structurally stable</u> if it possesses a neighborhood of diffeomorphisms, each topologically conjugate to f .

THEOREM 2 (<u>Palis-Smale</u> [1]).- <u>If</u> $f \in \mathrm{Dyn}(M)$ <u>with</u> $\Omega(f)$ <u>finite</u>, <u>then</u> f <u>is structurally stable if and only if</u> $f \in U_1$.

It was known for sometime via gradient dynamical systems that for any M , U_1 is not empty and more recently U_1 was shown to be open [6].

Structural Stability is a very strong regularity property (now known to be not generic) and largely via the preceeding theorem, U_1 can be considered to consist of very well understood dynamical systems of relatively simple character. On the other hand, the proof of this theorem is not altogether simple because of the fact that structurally stable is such a strong (and subtle) property.

To define the remaining terms used in decribing U_2 , say that for $f \in \mathrm{Dyn}(M)$ satisfying Axiom A, f has the <u>strong transversality property</u> if for any $x \in M$, $W^s(x)$ and $W^u(x)$ meet transversally at x .

It has been conjectured that a necessary and sufficient condition for $f \in \mathrm{Dyn}(M)$ to be structurally stable is that $f \in U_2$. It is known that there exists $f \in U_2$, $f \notin U_1$ which is structurally stable. Among the rather complicated example of $f \notin U_1$, some have the property that a neighborhood doesn't even intersect U_2 because of lacking the strong transversality property. Also for f satisfying Axiom A, it can be seen that the strong transversality property is necessary for f to be structurally stable. Via this route it was first found that structurally stable was not a generic property. Proving that U_2 = structurally stable dynamical systems would cement U_2 into our hierarchy.

We pass to U_3 . To understand the no cycle property, we recall the spectral decomposition theorem which states that if f satisfies Axiom A, then $\Omega(f)$ can be written canonically as the finite union of closed invariant disjoint subsets $\Omega_1,\ldots,\Omega_k$ on each of which f has a dense orbit.

Define $W^s(\Omega_i) = \bigcup_{x\in\Omega_i} W^s(x)$ and $W^u(\Omega_i) = \bigcup_{x\in\Omega_i} W^u(x)$.

A <u>cycle</u> is a sequence of distinct $\Omega_1,\ldots,\Omega_k$, $k > 1$, with the property $W^s(\Omega_i) \cap W^u(\Omega_{i-1}) \neq \emptyset$, $i = 2,\ldots,k$, and $W^s(\Omega_1) \cap W^u(\Omega_k) \neq \emptyset$. Then f (supposed to satisfy Axiom A) has the <u>no cycle property</u> if there are no cycles. Dynamical systems in U_3 have the important regularity property known as Ω-stability which is defined as follows. First say that f , $g \in \mathrm{Dyn}(M)$ are Ω-<u>conjugate</u> or conjugate on Ω if there is a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $hf(x) = gh(x)$ for all $x \in \Omega(f)$. Then f is Ω-stable if it has a neighborhood N in $\mathrm{Dyn}(M)$ of diffeomorphisms which are Ω-conjugate to f. Clearly structurally stable implies Ω-stable.

THEOREM 3 (The Ω-stability Theorem).- If $f \in U_3$, then f is Ω-stable.

The converse of Theorem 3 is an open problem. More generally one can ask what stability properties of dynamical systems are valid outside of U_3. Can even some dynamical system not in U_3 be structurally stable ? Another version of these questions is : Does structural stability imply Axiom A ? or even does Ω-stability imply Axiom A ? My feeling is that the questions of this paragraph are very hard and important to settle.

Some other regularity properties are true of $f \in U_3$. For example one can define the "zeta function" $\zeta(f) = \sum_{n=1}^{\infty} N_n t^n$ where N_n is the number of fixed points of f^n. It is an open question whether $\zeta(f)$ having a positive radius of convergence is a generic property. But on the other hand, the following theorem was very recently proved by J. Guckenheimer [4].

THEOREM 4.- If $f \in U_3$, then $\zeta(f)$ not only has a positive radius of convergence, but it is a rational function whose zeros and poles are algebraic numbers.

R. Bowen [1], [2], [3] has studied dynamical systems satisfying Axiom A in the direction of ergodic theory and has obtained the following rather striking results.

THEOREM 5.- *Let* f *satisfy Axiom A and* Ω_i *be one of the subsets given by the spectral decomposition theorem. Then there exists an invariant ergodic measure* μ_f *on* Ω_i *, positive on open sets, zero on points (unless* Ω_i *is finite) which is the unique invariant normalized Borel measure on* Ω_i *maximizing entropy. The (measure theoretic) entropy coincides with the topological entropy and this entropy is equal to the log of the radius of convergence of the zeta function of* f .

Also Bowen gets good information on the distribution of periodic points in Ω_i and shows that $(f|_{\Omega_i}, \mu_f)$ is a K-automorphism in the "C-dense" case, a mild condition which is met for example in the case Ω_i is connected.

Let me emphasize again that indeed these last theorems cover situations which are very rich in examples and that I am not giving them here.

What happens outside of U_3 ? At the present time, there are a large number of examples outside of U_3 whose import is that one cannot obtain any dense open $U_k \subset \mathrm{Dyn}(M)$ with very strong regularity or stability properties. Some of them are as follows :

Abraham-Smale [1] show that Ω-stability and Axiom A are not generic properties. Shub has an example of an open set in $\mathrm{Dyn}(M)$ where $\Omega = M$, where Axiom A and Ω-stability fail. Newhouse [1] shows that if $r > 1$, with the C^r topology, even on S^2 , Axiom A and Ω-stability are not generic properties. The earlier examples, with r arbitrary in the range $1 \leq r \leq \infty$, where on higher dimensional manifolds. C. Simon has recently shown that the zeta function being rational is not a generic property.

Under perturbation, some features of these examples seem to be preserved. On the other hand the above examples and others need much study to get some understanding of the area beyond U_3 .

As far as I know, there has been not much progress in the finding of new generic properties (or study of U_4) since [7] was written.

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