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ON MINIMAL POSITIVE HARMONIC FUNCTIONS

by Kohur GOWRISANKARAN

Let Ω be a locally compact connected Hausdorff space with a countable base for its open sets. Let there be a harmonic system of Brelot on Ω satisfying the axioms 1, 2 and 3. Let there be a potential > 0 of this system on Ω. We shall consider here the poles on any compactification of Ω corresponding to minimal harmonic functions, and give some applications. Let S^+ and H^+ be respectively the positive superharmonic functions and harmonic functions on Ω. Let Λ be a compact base (in the T-topology) of S^+, and Δ_1 the set of minimal harmonic functions contained in Λ.

1. Minimal harmonic functions and their poles.

Let $\overline{\Omega}$ be a compact metrizable space containing Ω as a dense open subspace. Let Γ be the boundary of Ω, viz. $\Gamma = \overline{\Omega} - \Omega$. Let $w > 0$ be a fixed harmonic function on Ω, and μ_w its canonical measure on Λ (carried by Δ_1). Corresponding to each $w \in H^+$, $w > 0$, let Σ_w be the class of all lower bounded w-hyperharmonic functions on Ω. The following minimum principle is proved easily by standard methods.

LEMMA 1. - *For any* $v \in \Sigma_w$, *the condition*

$$\liminf_{x \to z} v(x) \geqslant 0 , \quad \textit{for all} \quad z \in \Gamma \quad \textit{implies that} \quad v \geqslant 0 .$$

This shows that the set of traces of all filters of neighbourhoods of the points of Γ is associated to Σ_w [1]. Let us consider the associated Dirichlet problem. Let us denote, corresponding to any extended real valued function f on Γ, the upper (resp. lower) solution of this problem by $\overline{\Gamma}_f^w$ (resp. $\underline{\Gamma}_f^w$). Any of these functions, if finite, will be a w-harmonic function on Ω. A resolutive function for this problem will be called Γ^w-resolutive, and the solution denoted by Γ_f^w. Corresponding to each $x \in \Omega$, let $\Gamma_{x,w}$ be the Daniell measure associated to this problem.

LEMMA 2. - *For any* $A \subset \Gamma$, *if* φ_A *is the characteristic function of* A, *then*

$$w \, \overline{\Gamma}_{\varphi_A}^w = \inf_{i \in I} R_w^{V_i \cap \Omega} ,$$

where $\{V_i\}_{i\in I}$ *is the family of all open sets in* $\overline{\Omega}$ *containing* A .

Proof. – Consider any such open set V_i . Then, $\frac{1}{w} R_w^{V_i\cap\Omega}$ is in Σ_w , and further the lim inf of this function majorises φ_A on Γ , and hence we get

$$w\, \overline{\Gamma}^w_{\varphi_A} \leqslant \inf_{i\in I} R_w^{V_i\cap\Omega} .$$

On the other hand, if v belongs to the upper class corresponding to φ_A and if $\varepsilon > 0$, then we can find an open set V such that

$$V \supset A \quad \text{and} \quad v > 1 - \varepsilon \quad \text{on} \quad V \cap \Omega .$$

Hence,

$$\frac{wv}{1-\varepsilon} \geqslant R_w^{V\cap\Omega} .$$

Now, by varying $\varepsilon > 0$, and then v , in the upper class defining $\overline{\Gamma}^w_{\varphi_A}$, we get the required opposite inequality. This proves the lemma.

COROLLARY. – *For any compact set* $K \subset \Gamma$, *we can find a sequence* $\{V_n\}$ *of open sets of* $\overline{\Omega}$ *such that* :

(i) $V_n \supset \overline{V}_{n+1}$ *for every* $n \geqslant 1$,
(ii) $\cap V_n = K$,
(iii) $\overline{\Gamma}^w_{\varphi_K} = \frac{1}{w} \lim_{n\to\infty} R_w^{V_n\cap\Omega}$.

Let $A \subset \Gamma$ and $\{V_i\}_{i\in I}$ be the open sets as in the above.

LEMMA. – *Each* $V_i \cap \Omega$ *being an open subset of* Ω , *the set* $\mathcal{E}_i$ *of all points of* Δ_1 , *where* $V_i \cap \Omega$ *is not thin, is a Borel subset* [4]. *Let*

$$\mathcal{E}_A = \bigcap_{i\in I} \mathcal{E}_i .$$

It is clear then that $\mathcal{E}_A \subset \mathcal{E}_B$ *for* $A \subset B \subset \Gamma$ *and that* $\mathcal{E}_\Gamma = \Delta_1$.

LEMMA 3. – *For every compact set* $K \subset \Gamma$, *the set* $\mathcal{E}_K$ *is Borel, and further*

$$w\, \overline{\Gamma}^w_{\varphi_K} = \int_{\mathcal{E}_K} h\, \mu_w(dh) .$$

Proof. – We can find a sequence $\{V_n\}$ of open sets of $\overline{\Omega}$ such that $V_n \supset \overline{V}_{n+1}$ for every $n \geqslant 1$ and $\cap V_n = K$. It is easy to verify that

$$\mathcal{E}_K = \bigcap_{n=1}^{\infty} \mathcal{E}_n \quad ,$$

where $\mathcal{E}_n$ is the set of all points of Δ_1 where $V_n \cap \Omega$ is not thin. Hence, $\mathcal{E}_K$ is a Borel subset of Δ_1 . Further,

$$w\, \overline{\Gamma}^{w}_{\varphi_K} = \lim_{n\to\infty} R_w^{V_n \cap \Omega} \quad ,$$

and this limit (being a decreasing limit) is nothing but

$$\lim \int_{\mathcal{E}_n} h\, \mu_w(dh) \quad ,$$

the greatest harmonic minorant of $R_w^{V_n \cap \Omega}$ [4], and we deduce the required result. This proves the lemma.

COROLLARY 1. - <u>Let</u> C <u>and</u> D <u>be compact subsets of</u> Γ <u>satisfying</u> :

(i) $C \subset D$,
(ii) $\overline{\Gamma}^{w}_{\varphi_D} \neq 0$.

<u>Then</u>,

$$w_D\, \overline{\Gamma}^{w_D}_{\varphi_C} = w\, \overline{\Gamma}^{w}_{\varphi_C} \quad , \qquad \underline{\text{where}} \quad w_D = w\, \overline{\Gamma}^{w}_{\varphi_D} \quad .$$

By the above lemma, the canonical measure μ_{w_D} of w_D is nothing but μ_w restricted to $\mathcal{E}_D$, and hence we get

$$w_D\, \overline{\Gamma}^{w_D}_{\varphi_C} = \int_{\mathcal{E}_C} h\, \mu_{w_D}(dh) = \int_{\mathcal{E}_C \cap \mathcal{E}_D} h\, \mu_w(dh) = \int_{\mathcal{E}_C} h\, \mu_w(dh) = w\, \overline{\Gamma}^{w}_{\varphi_C} \quad .$$

COROLLARY 2. - <u>If</u> u_1 , $u_2 \in H^+$, <u>then, for any compact set</u> $K \subset \Gamma$, <u>we have</u>

$$(u_1 + u_2)\, \overline{\Gamma}^{u_1+u_2}_{\varphi_K} = u_1\, \overline{\Gamma}^{u_1}_{\varphi_K} + u_2\, \overline{\Gamma}^{u_2}_{\varphi_K} \quad .$$

THEOREM 1. - <u>The following four statements are equivalent</u> :

($\mathcal{R}_w^{\Gamma}$ (1)) <u>Every finite continuous function on</u> Γ <u>is</u> Γ^w-<u>resolutive</u> ;

($\mathcal{R}_w^{\Gamma}$ (2)) <u>The characteristic functions of compact subsets of</u> Γ <u>are</u> Γ^w-<u>resolutive</u> ;

($\mathcal{R}_w^\Gamma$ (3)) <u>For any two disjoint compact subsets</u> C <u>and</u> D <u>of</u> Γ ,

$$\bar{\Gamma}^w_{\varphi_{C\cup D}} = \bar{\Gamma}^w_{\varphi_C} + \bar{\Gamma}^w_{\varphi_D} \;;$$

($\mathcal{R}_w^\Gamma$ (4)) <u>For any two disjoint compact subsets</u> C <u>and</u> D <u>such that</u> $w_D = w\,\bar{\Gamma}^w_{\varphi_D} \neq 0$,

$$\bar{\Gamma}^{w_D}_{\varphi_C} \equiv 0 \quad .$$

<u>Proof</u>. - The equivalence of the first two statements can be proved by standard Baire class arguments. We shall now show that these are equivalent to the last two. If we assume ($\mathcal{R}_w^\Gamma$ (2)), then, from the additivity of the w-solutions, we find that ($\mathcal{R}_w^\Gamma$ (3)) is true. Let us now assume ($\mathcal{R}_w^\Gamma$ (3)), and show that ($\mathcal{R}_w^\Gamma$ (4)) is a consequence.

Let C and D be two disjoint compact sets such that $w_D \neq 0$. From ($\mathcal{R}_w^\Gamma$ (3)), we get that $w_{C\cup D} = w_C + w_D$. But, from the corollary 2, lemma 3, we deduce that

$$w_{C\cup D}\,\bar{\Gamma}^{w_{C\cup D}}_{\varphi_C} = w_C\,\bar{\Gamma}^{w_C}_{\varphi_C} + w_D\,\bar{\Gamma}^{w_D}_{\varphi_C} \quad ,$$

i. e.

$$w_C = w_C + w_D\,\bar{\Gamma}^{w_D}_{\varphi_C} \quad .$$

Since $w_D \neq 0$, we get that $\bar{\Gamma}^{w_D}_{\varphi_C} = 0$.

Finally, let us suppose that ($\mathcal{R}_w^\Gamma$ (4)) is true. We shall show that ($\mathcal{R}_w^\Gamma$ (2)) is true. Let $K \subset \Gamma$ be a compact set. There is nothing to prove if $\bar{\Gamma}^w_{\varphi_K} = 0$. Hence, let us assume that

$$w_K = w\,\bar{\Gamma}^w_{\varphi_K} \neq 0 \quad .$$

Let us first note that

$$w\,\Gamma^w_{\varphi_K} \geqslant w_K\,\underline{\Gamma}^{w_K}_{\varphi_K} \quad .$$

Now, $\Gamma - K$ is the limit of an increasing sequence of compact sets A_n . For each n , $\bar{\Gamma}^{w_K}_{\varphi_{A_n}} = 0$, and hence, $\bar{\Gamma}^{w_K}_{\varphi_{\Gamma-K}} = 0$. This implies that

$$\Gamma^{w_K}_{\varphi_K} = 1 \quad .$$

It follows that

$$\frac{w_K}{w} \leqslant \underline{\Gamma}^w_{\varphi_K} \leqslant \overline{\Gamma}^w_{\varphi_K} \leqslant \frac{w_K}{w} \quad .$$

We deduce that φ_K is Γ^w-resolutive, and moreover that

$$w \, \overline{\Gamma}^w_{\varphi_K} = w_K \quad .$$

This completes the proof of the theorem.

DEFINITION 1. - <u>The harmonic function</u> w <u>is said to satisfy the</u> resolutivity axiom relative to Γ (<u>or simply</u> $(\mathcal{R}^\Gamma_w)$), <u>if anyone of the above four (equivalent) properties holds good for</u> w .

The following lemma is proved easily.

LEMMA 4. - <u>Let</u> h <u>be an element in</u> Δ_1 . <u>Then,</u> $\overline{\Gamma}^h_{\varphi_A}$ <u>is identically</u> h <u>or zero, for any</u> $A \subset \Gamma$.

THEOREM 2. - <u>Let</u> $h \in \Delta_1$. <u>If</u> $K \subset \Gamma$ <u>is a compact set such that</u> $\overline{\Gamma}^h_{\varphi_K} = 1$, <u>then there is at least an element</u> $P \in K$ <u>such that</u>

$$\overline{\Gamma}^h_{\varphi_{\{P\}}} = 1 \quad .$$

<u>Proof</u>. - If there exists no such point in K , we can find an open neighbourhood V_Q (in $\overline{\Omega}$) of each point of K such that $R_h^{V_Q \cap \Omega}$ is a potential. Hence, we can choose a finite number of points $Q_1 , \dots , Q_m$ such that

$$V = \bigcup_{i=1}^{m} V_{Q_i}$$

covers K . We get immediately that $\overline{\Gamma}^h_{\varphi_K} = 0$ from the lemma 2 and the fact that

$$R_h^{V \cap \Omega} \leqslant \sum_{i=1}^{m} R_h^{V_{Q_i} \cap \Omega}$$

is a potential ; contradicting the assumption. The theorem is proved.

COROLLARY. - <u>Corresponding to each</u> $h \in \Delta_1$, <u>there is at least one point</u> $P \in \Gamma$ <u>such that</u>

$$\overline{\Gamma}^h_{\varphi_{\{P\}}} = 1 \quad .$$

DEFINITION 2. - Let $h \in \Delta_1$. Any point $P \in \Gamma$ such that $\bar{\Gamma}^h_{\varphi_{\{P\}}} = 1$ is called a pole corresponding to h. If the set of poles corresponding to a $h \in \Delta_1$ consists of a single point, then this point is called the unique pole of h on Γ.

The following proposition is an immediate consequence.

PROPOSITION 1. - Let $h \in \Delta_1$. A point P in Γ is a pole corresponding to h if $\Omega \cap V$ is not thin at h, for every neighbourhood V of P.

PROPOSITION 2. - Let p be the projection mapping from the set of all potentials in Λ with point support (p maps each such potential to its support in Ω). Let $Q \in \Gamma$ be a pole corresponding to $h \in \Delta_1$. The filter $p^{-1}(\mathfrak{F})$, where $\mathfrak{F}$ is the trace on Ω of the filter of all neighbourhoods of Q, is adherent to h.

Proof. - Let V be any open set of Ω. Then, it can be seen easily as in [2] (using a sequence of relatively compact open sets covering V) that R^V_u (for any positive harmonic function u) is represented as an integral with a measure supported by $p^{-1}(V)$ in the compact base Λ.

Let, now, W' be an open neighbourhood of Q, and let $W = \Omega \cap W'$. Since Q is a pole of h, $R^W_h \equiv h$. If $p^{-1}(W)$ does not have h as an adherent point, then R^W_h could be expressed as an integral over some subset of $H^+ \cap \Lambda$. This is impossible in view of the fact that h is minimal (and $R^W_h = h$). Hence, $p^{-1}(W)$ is adherent to h. This being true for each open neighbourhood W' of Q, the proposition is proved.

THEOREM 3. - There is a unique pole on Γ corresponding to a $h \in \Delta_1$, if anyone of the following two conditions is satisfied :

1° $(\mathcal{R}^\Gamma_h)$ is valid.
2° The fine filter $\mathfrak{F}_h$ is convergent.

Proof. - Let the axiom $(\mathcal{R}^\Gamma_h)$ be valid. If P_1 and P_2 are two different poles corresponding to $h \in \Delta_1$, then

$$2 = \bar{\Gamma}^h_{\varphi_{\{P_1\}}} + \bar{\Gamma}^h_{\varphi_{\{P_2\}}} = \bar{\Gamma}^h_{\varphi_{\{P_1 \cup P_2\}}} \leq 1 \ .$$

This inequality is absurd, and hence there cannot be two different poles corresponding to h. Conversely, if P is the unique pole corresponding to h, then for any two disjoint compact subsets, C and D, evidently

$$\bar{\Gamma}^h_{\varphi_C} + \bar{\Gamma}^h_{\varphi_D} = \bar{\Gamma}^h_{\varphi_{C \cup D}} \quad .$$

Let us now suppose that $\mathfrak{F}_h$ is convergent to a P (evidently $P \in \Gamma$). Then, for any $Q \in \Gamma$, $Q \neq P$, there is a neighbourhood W' such that $W = W' \cap \Omega$ is thin at h . This implies that no point of Γ , other than P , can be a pole of h . Conversely, suppose P is the unique pole corresponding to h . Let W' be an open neighbourhood of P in Ω' , and let $W = W' \cap \Omega$. We can choose an open set V in $\bar{\Omega}$ such that $V \supset \Gamma - W'$ and $R_h^{V \cap \Omega}$ is a potential (theorem 2). Let

$$K = \Omega - (V \cup W') \quad ,$$

it is clear that K is compact. Hence,

$$W = \Omega - (K \cup \Omega \cap V)$$

is in $\mathfrak{F}_h$. This being true for all neighbourhoods of P , $\mathfrak{F}_h$ is convergent to P . The proof is complete.

DEFINITION 3. - <u>Let</u>

$$\Delta_1^\Gamma = \{h \in \Delta_1 : \text{there is a unique pole on } \Gamma \text{ corresponding to } h\} \quad ,$$

<u>and</u> Φ_Γ <u>the mapping</u> $\Delta_1^\Gamma \to \Gamma$ <u>which takes each</u> h <u>to its unique pole on</u> Γ .

LEMMA 5. - <u>Let</u> $K \subset \Gamma$ <u>be a compact set. Then, the set of all points of</u> Δ_1 <u>for which there exists at least one pole on</u> K <u>is a Borel subset of</u> Δ_1 .

<u>Proof</u>. - The lemma follows immediately by observing that the set in question here is nothing but $\mathcal{E}_K$ considered in the lemma 3.

The proof of the following theorem is exactly as in the chapter V, 6, and we omit it.

THEOREM 4. - <u>The set</u> Δ_1^Γ <u>is a Borel subset of</u> Δ_1 , <u>and</u> $\Phi^\Gamma : \Delta_1^\Gamma \to \Gamma$ <u>is a Borel mapping</u>.

We have the following corollary [7].

COROLLARY. - <u>The image by</u> Φ_Γ <u>of every Borel subset of</u> Δ_1^Γ <u>is universally measurable for all Borel measures on</u> Γ .

THEOREM 5. - <u>A necessary and sufficient condition in order that</u> $(\mathcal{R}_u^\Gamma)$ <u>be valid for a</u> $u \in H^+$ <u>is that</u>

$$\mu_u(\Delta_1 - \Delta_1^\Gamma) = 0 \quad .$$

<u>Proof.</u> – For any two compact sets, C and D, of Γ, if $u_C = u\,\overline{\Gamma}^u_{\varphi_C}$, then

$$u_C\,\overline{\Gamma}^{u_C}_{\varphi_D} = \int_{\mathcal{E}_D} h\,\mu_{u_C}(dh) = \int_{\mathcal{E}_D \cap \mathcal{E}_C} h\,\mu_u(dh) \quad .$$

Let $(\mathcal{R}^\Gamma_u)$ be valid. Then, for any two disjoint compact sets C and D,

$$\mu_u(\mathcal{E}_C \cap \mathcal{E}_D) = 0 \quad ,$$

i. e. the set of all points of Δ_1 which can have a pole simultaneously in C and D is of measure zero. Now (chap. V,), $\Delta_1 - \Delta_1^\Gamma$ can be expressed as a countable union of sets of the form $\mathcal{E}_C \cap \mathcal{E}_D$, where C and D are disjoint compact sets. We deduce that $\mu_u(\Delta_1 - \Delta_1^\Gamma) = 0$.

Conversely, if $\mu_u(\Delta_1 - \Delta_1^\Gamma) = 0$, then, for any two disjoint compact sets C and D, since $\mathcal{E}_C \cap \mathcal{E}_D \subset \Delta_1 - \Delta_1^\Gamma$, we get $\mu_u(\mathcal{E}_C \cap \mathcal{E}_D) = 0$. Hence $(\mathcal{R}^\Gamma_u\ (4))$ is true. The proof is complete.

THEOREM 6. – <u>Let</u> $(\mathcal{R}^\Gamma_u)$ <u>be true for</u> u. <u>Then</u> $\Gamma_{x,u}$, <u>which are Radon measures on</u> Γ, <u>are precisely the image by</u> Φ <u>of the measures</u> $\frac{h(x)}{u(x)}\,d\mu_u(h)$, <u>for every</u> $x \in \Omega$. <u>Hence, for all</u> Γ^u<u>-resolutive functions (or equivalently</u> $\Gamma_{x,u}$ <u>summable functions)</u> <u>on</u> Γ, <u>the</u> Γ^u<u>-solution is given by</u>

$$u(x)\;\Gamma^u_f(x) = \int (f \circ \Phi)(h)\,\frac{h(x)}{u(x)}\,\mu_u(dh) \quad .$$

<u>Proof.</u> – Let K be a compact set in Γ, and φ_K its characteristic function. Then,

$$\Gamma^u_{\varphi_K}(x) = \int \varphi_K(z)\;\Gamma_{x,u}(dz) \quad ,$$

for all $x \in \Omega$. But, from the last theorem and the lemma 3, we get

$$\Gamma^u_{\varphi_K}(x) = \int_{\mathcal{E}_K} \frac{h(x)}{u(x)}\,\mu_u(dh) = \int_{\mathcal{E}_K \cap \Delta_1^\Gamma} \frac{h(x)}{u(x)}\,\mu_u(dh)$$

$$= \int_{\Delta_1^\Gamma} (\varphi_K \circ \Phi)(h)\,\frac{h(x)}{u(x)}\,\mu_u(dh)$$

$$= \int \varphi_K(z)\;\Phi\Big[\frac{h(x)}{u(x)}\,\mu_u\Big]\,(dh) \quad .$$

This is true for all the compact sets $K \subset \Gamma$, and hence the two Radon measures $d\,\Gamma_{x,u}$ and $d\,\Phi\big(\frac{h(x)}{u(x)}\,\mu_u\big)$ are identical. This is again true whatever be the point

$x \in \Omega$. The rest of the proof follows by a standard theorem in the theory of Radon measures [7].

COROLLARY 1. - For any extended real valued function f on Γ,

$$\overline{\Gamma}^u_f(x) = \overline{\int} f(z)\ \Phi\left[\frac{h(x)}{u(x)}\mu_u\right](dz) \quad .$$

Proof. - We know that

$$\overline{\int} f(z)\ \Phi\left[\frac{h(x)}{u(x)}\mu_u\right](dz) \geqslant \overline{\Gamma}^u_f(x) \quad .$$

On the other hand, let $v \in \Sigma_u$ and satisfy

$$\liminf_{x\to z} v(x) \geqslant f(z) , \qquad \text{for all } z \in \Gamma \quad .$$

The function v in Ω continued by its lim inf on the boundary is lower semi-continuous, and this function $\psi \geqslant f$ on Γ. Hence, we get

$$v(x) = \overline{\Gamma}^u_\psi(x) = \int \psi(z)\ \Phi\left[\frac{h(x)}{u(x)}\mu_u\right](dz) \geqslant \overline{\int} f(z)\ \Phi\left[\frac{h(x)}{u(x)}\mu_u\right](dz) \quad .$$

This is true for all such v, and the required opposite inequality follows.

COROLLARY 2. - A set $A \subset \Gamma$ is Γ^u-negligible (i. e. $\overline{\Gamma}^u_{\varphi_A} \equiv 0$), if

$$\mu^*_u[\Phi^{-1}(A)] = 0 \quad .$$

In particular, $\Gamma - \Phi_\Gamma(\Delta^\Gamma_1)$ is Γ^u-negligible.

Remark. - A real valued function f on Γ is Γ^u-resolutive (u as in the theorem), if $f \circ \Phi$ is μ_u-summable.

2. The case of "proportionality".

Let us now assume that the axiom of proportionality is valid in addition for the system on Ω, i. e. the potentials with the same point support are proportional to each other. In this case, Ω can be identified homeomorphically with the set of extreme potentials on Λ [2]. Let $\overline{\Omega}$ be the closure of Ω in Λ, Δ the boundary (i. e. Martin boundary). Then $\Delta \supset \Delta_1$, and Δ consists only of harmonic functions. We shall show that the axiom of resolutivity is valid in this case (for all $u > 0$).

THEOREM 7. - Let $u > 0$, $u \in H^+$, and $K \subset \Delta$ any compact set. Then, there exists a measure λ on Δ, supported by K, satisfying

$$u(x)\,\bar{\Delta}^{u}_{\varphi_K}(x) = \int h(x)\,\lambda(dh) \quad .$$

Proof. - The existence of the measure is proved by the method of Martin, and the adaptation of the proof is exactly as in [2]. We have only to note the fact that any open set V of Ω is the increasing union of a sequence of relatively compact open sets δ_n , and R_u^V is the limit of $R_u^{\delta_n}$.

THEOREM 8. - For every $h \in \Delta_1$, the fine filter $\mathfrak{F}_h$ is convergent to h .

Proof. - Let ω' be an open neighbourhood of $h \in \Delta_1$, and $\omega = \omega' \cap \Omega$. We have to show that $\Omega - \omega$ is thin at h . For this, it is enough to show that $\bar{\Delta}^{h}_{\varphi_K} = 0$, where $K = \Delta \cap \complement\omega'$.

Suppose $\bar{\Delta}^{h}_{\varphi_K} \neq 0$; then this function is $\equiv 1$ (lemma 2). Hence there exists at least one pole $h' \in K$ for the function h . Hence $\bar{\Delta}^{h}_{\varphi_{\{h'\}}} = 1$. But, by the last theorem, we get that

$$h(x)\,\bar{\Delta}^{h}_{\varphi_{\{h'\}}}(x) = \alpha h'(x) \;, \qquad \text{i. e.} \qquad \alpha h' \equiv h \quad .$$

This is impossible, since h and h' belong to the same base and are not proportional. The proof is complete.

COROLLARY 1. - Every $h \in \Delta_1$ has a unique pole (i. e. h) on Δ .

COROLLARY 2. - The axiom $(\mathcal{R}_u^\Delta)$ is valid for all $u \in H^+$, $u > 0$.

(Consequence of the theorem 5.)

Remark. - In this case (of proportionality), it is easily seen that the axiom $(\mathcal{R}_u)$ introduced in [2] is valid for all $u \in H^+$. In other words, we have shown here that the axiom D is not needed for proving the validity of this result.

3. A sufficient condition for the proportionality of potentials with same point support.

Let us now go back to the general case. We shall give a sufficient condition in order that the potentials with the same point support are proportional.

THEOREM 9. – Let $K \subset \Omega$ be any compact set, and $h \in \Delta_1$. Then $h' = h - R_h^K$ is a minimal harmonic function for a unique connected component of $\Omega - K$.

Proof.

Case 1 : K polar. Then $R_h^K = 0$. Let u be a positive harmonic function on $\Omega - K$ (which is connected) satisfying $0 < u < h$. Then, $\frac{u}{h}$ is a bounded h-harmonic function on $\Omega - K$, and it can be extended to a h-harmonic function u' on Ω such that $0 < u' \leqslant 1$ [1]. It follows that hu' is proportional to h on Ω, and hence u is proportional to h on $\Omega - K$. This shows that h restricted to $\Omega - K$ is a minimal harmonic function.

Case 2 : K is such that $\Omega - K$ is connected (and K not polar). Let $\omega = \Omega - K$. We shall show that

$$\mathfrak{G}_{h'} = \{E \subset \omega : \ (R_{h'}^{\omega - E})_\omega \neq h'\}$$

is a filter on ω (in fact equal to $\mathfrak{F}_h \cap \omega$). Then it follows that h' is a minimal harmonic function on ω [3].

Consider $E \subset \omega$ and $E \in \mathfrak{F}_h$. Then $F = K \cup \complement E$ is such that $\complement F$ belongs to $\mathfrak{F}_h$. Hence, there exists a $v \in S^+$, $v \geqslant h$ on $K \cup \complement E$, and such that $v(y) < h(y)$ at some $y \in E$, and $v \leqslant h$ on Ω. Let $p = \hat{R}_h^K = \hat{R}_v^K$. Consider $v - p$ on ω. The function $v - p > 0$, superharmonic on ω, and $v - p \geqslant h - p = h'$ on $\omega - E$. But $(v - p)(y) < h'(y)$, and hence $E \in \mathfrak{G}_{h'}$.

To prove the opposite inclusion, let us consider a relatively compact open neighbourhood δ of K. Let $q = R_h^\delta$; this function is a potential on Ω. Using the minimum principle [5], it is easy to see that the greatest harmonic minorant in ω of the function q is $R_h^K = R_q^K$. Hence, $q - R_h^K$ is a potential on ω, and $q - R_h^K = h - R_h^K$ on $\delta \cap \omega$. Now, if $E \in \mathfrak{G}_{h'}$, then, $E \cap (\omega - \delta)$ also belongs to $\mathfrak{G}_{h'}$. For,

$$(\widehat{R_{h'}^A})_\omega \leqslant (R_{h'}^{\widehat{\omega - E}})_\omega + (R_{h'}^{\omega \cap \delta})_\omega \quad ,$$

where $A = (\omega - E) \cup (\omega \cap \delta)$; and it is clear that $E \cap (\omega - \delta)$ also belongs to $\mathfrak{G}_{h'}$.

Now, there exists a superharmonic function $v \geqslant 0$ on ω such that $v \geqslant h'$ on $(\omega - E) \cup (\omega \cap \delta)$, $v \leqslant h'$ and $v'(y) < h'(y)$ for some $y \in E$. It is clear that the function $w = v + \hat{R}_h^K$ in ω and $w = h$ on K belongs to S^+. Further, $w \leqslant h$ majorises h on $(\omega - E) \cup \delta$, and $w \neq h$ on Ω. Hence $\Omega - E$ belongs to $\mathfrak{F}_h$. This shows that $\mathfrak{G}_{h'} = \mathfrak{F}_h \cap \omega$, completing the proof of the theorem in this case.

General case : As in the case 2, we can show that $\mathfrak{F}_h \cap \omega = \mathfrak{G}_{h'}$ is a filter (where $\omega = \Omega - K$). But as in [3], we can see easily that there exists a unique connected component ω_m of ω (note that the connected components of ω are at most countable) such that $\omega_m \in \mathfrak{F}_h$ (and hence $V = \bigcup_{n \neq m} \omega_n$ is thin at h). Now,

$$h' = h - R_h^K = h - R_h^{V \cup K} \quad \text{in } \omega_m$$

is minimal harmonic on ω_m , and the fine filter corresponding to h' on ω_m is precisely $\mathfrak{F}_h \cap \omega_m$.

The proof is complete.

Let us now take a point P in Ω such that $\omega = \Omega - \{P\}$ is connected. Let $\bar{\Omega}$ be the Alexandroff compactification of Ω with A the point at infinity. $\bar{\Omega}$ is also a compactification of ω .

LEMMA 6. - A minimal harmonic function $u > 0$ on ω has a pole at A if, and only if, there is a $h \in \Delta_1$ such that u is proportional to $h - R_h^{\{P\}}$ on ω .

Proof. - The proof is trivial if $\{P\}$ is a polar set. Let us assume that $\{P\}$ is not polar. The last theorem shows that the condition is sufficient. It remains to prove the necessity of the condition. By the proposition 2, we know that there is a sequence $\{x_n\}$ of elements of ω , converging to A such that there is a potential q_n on ω with support at x_n (and belonging to a compact base of the cone S_ω^+) with $q_n(x) \to \alpha u(x)$ for all $x \in \omega$. But every such q_n is a constant multiple of $p_n - R_{p_n}^{\{P\}}$, where $p_n \in \Lambda$, p_n supported by x_n . Suppose

$$q_n = \beta_n (p_n - R_{p_n}^{\{A\}}) \quad .$$

But, by the compactness of Λ , we can find a subsequence of $\{p_n\}$ which converges to a harmonic function h on Ω (since the supports x_n of p_n converges to infinity). From this subsequence, we can choose yet another subsequence such that $R_{p_n}^{\{P\}}$ converges to a potential p with support at P . It is seen easily that u is equal to $V(h - p)$. Now, using the minimum principle, it can be shown that $p = \hat{R}_h^{\{P\}}$. Hence, u is a constant multiple of $h - R_h^{\{P\}}$ in ω . If h is not minimal, suppose $h = h_1 + h_2$ with h_1 and h_2 not proportional, then

$$u = C[h_1 - R_{h_1}^{\{P\}} + h_2 - R_{h_2}^{\{P\}}] \quad ,$$

and this is impossible. The proof is complete.

COROLLARY. - Every minimal (positive) harmonic function on ω has a unique pole (either at P or A).

Remark. - The proof shows that, even if $\Omega - \{P\}$ is not connected, a minimal harmonic function on any connected component with a pole at the Alexandroff point of Ω comes necessarily from an element of Δ_1 . The same proof applies to minimal harmonic functions on the connected components of $\Omega - K$, where K is a compact set of Ω .

THEOREM 10. - Let $\{P\}$ be a polar set of Ω . Then the set of extreme potentials on Ω with support at P have the same cardinality as the set of positive minimal harmonic functions on $\Omega - \{P\}$ with pole at P .

Proof. - Since $\{P\}$ is polar, there is a one-one correspondence between the minimal harmonic functions on ω $(= \Omega - \{P\})$ and the extreme potentials on Ω with support at P . The proof is easily completed.

THEOREM 11. - Let $P \in \Omega$ be such that $\{P\}$ is not polar, and $\Omega - \{P\}$ connected. If the set of positive minimal harmonic functions on $\Omega - \{P\}$ with a pole at P consists of a single element, then the potentials on Ω with support at P are proportional to each other.

Proof. - Let u be the unique positive minimal harmonic function on $\Omega - \{P\}$ having a pole at P . Let p_1 and p_2 be potentials on Ω with support at P . The canonical measure of the harmonic function p_1 on $\Omega - \{P\}$ does not charge the set of elements of the form $h - R_h^{\{P\}}$ where $h \in \Delta_1$. Hence, both p_1 and p_2 are constant multiples of u on $\Omega - \{P\}$. Hence, p_1 and p_2 are proportional to each other on the whole of Ω . The proof is complete.

Remark. - An example of N. BOBOC and A. CORNEA shows that the converse of this theorem is not true.

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