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CHARLES B., JR. MORREY

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A CLASS OF ELLIPTIC DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

par Charles B. MORREY, Jr

1. Introduction.

We shall discuss equations of the form

$$(1.1) \quad \int_G \{\sum_{\alpha=1}^{\nu} v_{,\alpha}(\sum_{\beta=1}^{\nu} a^{\alpha\beta} u_{,\beta} + b^{\alpha} u + e^{\alpha}) + v(\sum_{\alpha=1}^{\nu} c^{\alpha} u_{,\alpha} + du + f)\} dx = 0$$

in which G is a bounded domain in ν dimensional space, the coefficients $a^{\alpha\beta}$, b^{α}, c^{α}, and d are bounded and measurable, e and f are in $L_2(G)$, $u \in H_2^1(D)$ for each domain D for which $\bar{D}$ (the closure of D) $\subset G$, and the equation is supposed to hold for all Lipschitz functions v with compact support. A function $u \in H_p^1(D)$ if and only if u and its "distribution derivatives", which we denote by $u_{,\alpha}$, $\alpha = 1, \ldots, \nu$, $\in L_p(D)$; that is to say $u \in L_p(D)$ and there are functions p_{α}, $\alpha = 1, \ldots, \nu$, also in $L_p(D)$, such that

$$(1.2) \qquad \int_D u(x)\ g_{,\alpha}(x)\ dx = - \int_D p_{\alpha}(x)\ g(x)\ dx$$

for every function g of class C^{∞} on D and having compact support ; here, of course, $g_{,\alpha}$ denotes the ordinary partial derivative. These spaces are well known but are discussed rather completely in [7] (see the bibliography at the end).

In case the function u is of class $C^2(G)$, the coefficients $a^{\alpha\beta}$, b^{α}, and $e^{\alpha} \in C^1(G)$, and c^{α}, d and $f \in C^0(G)$, then one sees that u satisfies (1.1) for all the v mentioned above if and only if u satisfies the differential equation

$$(1.3) \qquad \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^{\alpha}}(\sum_{\beta=1}^{\nu} a^{\alpha\beta} u_{,\beta} + b^{\alpha} u) - (\sum_{\alpha=1}^{\nu} c^{\alpha} u_{,\alpha} + du) = f - \sum_{\alpha=1}^{\nu} e^{\alpha}_{,\alpha} \quad .$$

However, if the coefficients are not smooth, examples show that there may not be any solution u of the equation (1.1) which is in $C^1(G)$, let alone $C^2(G)$; in such cases, of course, it is not legitimate to write the differential equation (1.3).

Equations of the form (1.1) with "rough" coefficients arise in attempting to prove the differentiability of the solutions of variational problems. For example, suppose that a function z minimizes an integral of the form

$$I(z, G) = \int_G f[x, z(x), \nabla z(x)]\, dx \qquad (x = (x^1, \dots, x^\nu))$$
$$\nabla z(x) = \operatorname{grad} z(x) = \{z_{,1}(x), \dots, z_{,\nu}(x)\} \tag{1.4}$$

among all admitted functions having the same boundary values (in a generalized sense). Then, if the function $f(x^1, \dots, x^\nu, z, p_1, \dots, p_\nu)$ is of class C^2 in its arguments and satisfies a set of inequalities, too long to write here (but see [7]), one can proceed as follows : first, if ζ is any Lipschitz function with compact support, then $z + \lambda\zeta$ has the same boundary values as z for any λ and so the function

$$\varphi(\lambda) = \int_G f[x, z(x) + \lambda\zeta(x), \nabla z(x) + \lambda \nabla\zeta(x)]\, dx \tag{1.5}$$

has a minimum for $\lambda = 0$ and the first step in the derivation of the Euler equation can be carried through to yield

$$\varphi'(0) = 0 = \int_G \Big(\sum_{\alpha=1}^{\nu} \zeta_{,\alpha} f_{p_\alpha} + \zeta f_z\Big)\, dx \tag{1.6}$$

for any Lipschitz function ζ with compact support ; here

$$f_{p_\alpha} = f_{p_\alpha}[x, z(x), \nabla z(x)], \text{ etc.}, \qquad f_{p_\alpha} = \partial f/\partial p_\alpha, \text{ etc.}$$

At this point, all we know about z from the existence theory (see the notes [7] referred to above) is that it belongs to some space $H^1_p(G)$. To obtain more information about z, we next apply a difference-quotient procedure to the equation (1.6) as follows : let ζ be any Lipschitz function with compact support in G. Then there is an $h_0 > 0$ and a D' with $\bar{D}' \subset G$ such that the support of the function $\zeta(x - he_\gamma) \subset D'$ for all h with $0 < |h| < h_0$ and each γ, $1 \leq \gamma \leq \nu$, e_γ being the unit vector in the x^γ direction. For a fixed γ and h, $0 < |h| < h_0$, let us define

$$\zeta_h(x) = h^{-1}[\zeta(x - he_\gamma) - \zeta(x)] , \qquad z_h(x) = h^{-1}[z(x + he_\gamma) - z(x)] \quad .$$

If ζ_h is inserted in (1.6), if next the integral is written as a sum of two integrals one for each term in ζ_h, if then the obvious change of variables is made in the integral involving $\zeta(x - he_\gamma)$, and if finally the integrals are recombined, one obtains

$$\int_G [\sum_{\alpha=1}^{\nu} \zeta_{,\alpha}(x) \, A_h^\alpha(x) + \zeta(x) \, B_h(x)] \, dx = 0 \qquad ,$$

where

$$A_h^\alpha(x) = f_{p_\alpha}[x + he_\gamma , z(x + he_\gamma) , p(x + he_\gamma)] - f_{p_\alpha}[x , z(x) , p(x)] , \tag{1.7}$$

$$p(x) = \{p_1(x) , \dots , p_\nu(x)\} , \qquad p_\alpha(x) = z_{,\alpha}(x) \qquad ,$$

B_h being the corresponding difference of f_z. Using the integral form of the theorem of the mean, we may write

$$\begin{aligned} A_h^\alpha(x) &= \sum_{\beta=1}^{\nu} a_h^{\alpha\beta}(x) \, z_{h,\beta}(x) + b_h^\alpha(x) \, z_h(x) + e_h^\alpha(x) \qquad , \\ B_h(x) &= \sum_{\alpha=1}^{\nu} c_h^\alpha(x) \, z_{h,\alpha}(x) + d_h(x) \, z_h(x) + f_h(x) \qquad , \\ a_h^{\alpha\beta}(x) &= \int_0^1 f_{p_\alpha p_\beta}[x + the_\gamma , z(x) + t\Delta z , p(x) + t\Delta p] \, dt \qquad , \\ \Delta z &= z(x + he_\gamma) - z(x) , \qquad \Delta p_\varepsilon = p_\varepsilon(x + he_\gamma) - p_\varepsilon(x) \end{aligned} \tag{1.8}$$

for almost every x; of course the other coefficients are given by corresponding formulas. Since we have the solution, we may regard the coefficients as known and we see that z_h satisfies an equation of the form (1.1) ; but of course the coefficients are known only to be measurable and, in the general cases considered in the Notes [7], are not even known to be bounded. However, in case f "has degree $2k$ at infinity", i. e. satisfies

$$(1.9)\quad mV^k - K \leq f(x, z, p) \leq MV^k, \quad V = 1 + z^2 + \sum_\alpha p_\alpha^2 \quad 0 < m \leq M, \quad k \geq 1$$

and the other inequalities in equation (3.1) of the Notes [7], it is possible, by using interior boundedness properties something like those proved in § 3, to show that we may let $h \to 0$ and conclude that the derivatives $p_\gamma \equiv z_{,\gamma}$ and the function $U = V^{k/2}$ (see (1.9)) $\in H_2^1(D)$ for each D with $\overline{D} \subset G$ and that the derivatives p_γ satisfy the differentiated equations

$$(1.10)\quad \begin{gathered} \int_D V^{k-1}\{\sum_\alpha \zeta_{,\alpha}(\sum_\beta a^{\alpha\beta} p_{\gamma,\beta} + b^\alpha p_\gamma + V^{1/2} e^{\alpha\gamma}) \\ + \zeta(\sum_\alpha c^\alpha p_{\gamma,\alpha} + dp_\gamma + V^{1/2} f^\gamma)\ dx = 0, \quad \gamma = 1, \ldots, \nu, \\ V^{k-1} a^{\alpha\beta} = f_{p_\alpha p_\beta}; \quad V^{k-1} b^\alpha = V^{k-1} c^\alpha = f_{p_\alpha z}; \\ V^{k-1} d = f_{zz}; \quad V^{k-1/2} e^{\alpha\gamma} = f_{p_\alpha x^\gamma}; \quad V^{k-1/2} f^\gamma = f_{zx^\gamma}; \end{gathered}$$

and the coefficients $a^{\alpha\beta}$, b^α, c^α, d, $e^{\alpha\gamma}$, and f^γ are all bounded and measurable and satisfy

$$(1.11)\quad \begin{gathered} m \sum_{\alpha=1}^{\nu} \lambda_\alpha^2 \leq \sum_{\alpha,\beta=1}^{\nu} a^{\alpha\beta}(x)\, \lambda_\alpha \lambda_\beta, \qquad m > 0, \\ \sum [(a^{\alpha\beta})^2 + (b^\alpha)^2 + (c^\alpha)^2 + d^2] \leq M^2 . \end{gathered}$$

By setting $\zeta = V^{-\varepsilon}\psi p_\gamma$ (ψ Lipschitz with compact support in D) in equations (1.10) (this ζ is not Lipschitz but technical lemmas allow its use) and summing with respect to γ, it can be shown that the function $U = V^{k/2}$ mentioned above has the following property :

<u>There is a number</u> λ, $1 \leq \lambda < 2$, <u>such that the function</u> $W = U^\lambda$ <u>satisfies the differential inequality</u>

$$(1.12)\quad \int_G [\sum_\alpha \psi_{,\alpha}(\sum_\beta a^{\alpha\beta} W_{,\beta} + b^\alpha W) + \psi(\sum_\alpha c^\alpha W_{,\alpha} + dW)]\, dx \leq 0$$

for all $\psi \in Lip_c(G)$ with $\psi(x) \geqslant 0$, $Lip_c(G)$ denoting the set of Lipschitz functions having compact support in G.

This inequality is interesting, because if all functions are smooth, it is equivalent to the inequality

$$\int_G \psi[\sum \frac{\partial}{\partial x^\alpha}(\sum a^{\alpha\beta} W_{,\beta} + b^\alpha W) - (\sum c^\alpha W_{,\alpha} + dW)]\, dx \geqslant 0$$

for all $\psi \geqslant 0$ from which one concludes that

$$(1.13) \qquad \sum \frac{\partial}{\partial x^\alpha}(\sum a^{\alpha\beta} W_{,\beta} + b^\alpha W) - (\sum c^\alpha W_{,\alpha} + dW) \geqslant 0 \qquad .$$

In case $a^{\alpha\beta} = \delta^{\alpha\beta}$ (the Kronecker delta) and $b^\alpha = c^\alpha = d = 0$, (1.13) implies that W is sub-harmonic. In § 4, it is shown (not quite in all detail) using the method of MOSER [8] that U is bounded on each domain D with $\bar{D} \subset G$. This implies that z is Lipschitz and that all the p_γ are bounded on such D. Then since V is bounded, we see that the equations (1.10) assume the form (1.1) ; in fact, we may absorb the terms $b^\alpha p_\gamma$ and dp_γ into e^α and f, repectively. For such equations, we show in § 5 (not incomplete detail) that the solutions p_γ are Hölder continuous on interior domains. Then, if the second derivatives of f are Hölder continuous, the coefficients in (1.10) are Hölder continuous and this implies that the derivatives $p_{\gamma,\beta}$ are Hölder continuous, so that the second derivatives of z are Hölder continuous (in the case $\nu = 2$, this result has been known a long time (see [1]). Higher differentiability can be deduced by repeating the difference-quotient procedure and using the other theorems.

Some of the techniques used in studying the equations (1.1) are useful even the coefficients are smooth. For example one of the simplest ways of proving the existence of the solutions of the equation (1.3) is to show first the existence of the solutions of the corresponding equations (1.1), using the result of § 2, and then applying the difference-quotient procedure illustrated above together with the interior boundedness theorem of § 3 to show that the second derivatives of u are in $L_2(D)$. Then the Hölder continuity theorems of § 5 show that the first derivatives of u are Hölder continuous and the classical results show that the second derivatives are.

We are presenting these results before this seminar with the hope that the solutions will satisfy some or all of the axioms of abstract potential theory. We believe, however, that the theorems which are necessary for this purpose have not all been proved.

2. The existence theory.

We shall consider the equations (1.1) in which the coefficients $a^{\alpha\beta}$, b^{α}, c^{α}, and d satisfy (1.11) but we shall allow e^{α} and f to be in L_2; the domain G will always be bounded. We shall not assume that $c^{\alpha} = b^{\alpha}$ or that $a^{\beta\alpha} = a^{\alpha\beta}$; this makes no difference in the proofs and is useful in studing certain non-linear equations of the form

$$\sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^{\alpha}} A^{\alpha}(x, z, \nabla z) = B(x, z, \nabla z) \qquad ,$$

which have the same form as the Euler equations for the integral (1.4) except that we do not assume that $A^{\alpha} = f_{p_\alpha}$ and hence don't assume that $\partial A^{\alpha}/\partial p_{\beta} = \partial A^{\beta}/\partial p_{\alpha}$.

We are looking for a solution u of (1.1) which "has given boundary values". This has, originally, to be interpreted (since we have abandoned continuity in using the spaces H_2^1) to mean that $u - u^* \in H_{20}^1(G)$, u^* being a given function $H_2^1(G)$ and $H_{20}^1(G)$ denoting the closure in $H_2^1(G)$ of the set Lipschitz functions with compact support in G. This implies that $u \in H_2^1(G)$ and so has finite Dirichlet integral. Smoothness on the interior and at the boundary is considered later, but we shall not present any such results here. But solutions of (1.1) will be shown in § 3 and § 4 to have boundedness properties on interior domains even if they are not in H_2^1 over the whole domain G.

It is clear that our problem may be reduced, by setting $U = u - u^*$ (u^* given), to that where our desired solution $H_{20}^1(G)$; the resulting terms involving u^* can be absorbed into the non-homogeneous terms e^{α} and f. Moreover, for well-known reasons, we shall modify (1.1) by allowing the functions u, v, e^{α}, and f to be complex-valued (keeping the others real), replacing v by its conjugate $\bar{v}$, and adding the term $\lambda u\bar{v}$ in the integral (λ complex). This last has the effect, in the smooth case, of replacing the equation (1.3) by $Lu - \lambda u = \varphi$, where Lu and φ denote the left and right sides of that equation.

We now define

$$\text{(2.1)}\qquad \begin{aligned} &B(u, v) = B_1(u, v) + B_2(u, v) ; \qquad C(u, v) = \int_G u\bar{v}\, dx \\ &B_1(u, v) = \int_G \sum_{\alpha,\beta} \bar{v}_{,\alpha}\, a^{\alpha\beta}\, u_{,\beta}\, dx ; \\ &B_2(u, v) = \int_G [\sum_\alpha (b^\alpha \bar{v}_{,\alpha}\, u - c^\alpha\, u_{,\alpha}\, \bar{v}) + du\bar{v}]\, dx ; \\ &L(V) = - \int_G (\sum_\alpha e^\alpha\, \bar{v}_{,\alpha} - f\bar{v})\, dx . \end{aligned}$$

We shall assume that u and $v \in H^1_{20}(G)$ and shall use as inner product, the expression

$$\text{(2.2)}\qquad ((u , v)) = \int_G \sum_\alpha u_{,\alpha}\, \bar{v}_{,\alpha}\, dx .$$

This is legitimate since we have (see the Notes [7]).

LEMMA 2.1 (Poincaré's inequality). – _If_ $G \subset B(x_0 , R)$, _then_

$$\text{(2.3)}\qquad \int_G |u|^2\, dx \leq (R^2/2) \int_G \sum_\alpha |u_{,\alpha}|^2\, dx , \qquad u \in H^1_{20}(G) ;$$

here $B(x_0 , R)$ _denotes the ball with center_ x_0 _and radius_ R .

In terms of this notation our altered equations (1.1) become

$$\text{(2.4)}\qquad B(u , v) + \lambda C(u , v) = L(v) , \qquad v \in H^1_{20}(G) ;$$

that (2.4) holds for all v in $H^1_{20}(G)$ if it does for all $v \in Lip_c(G)$ is evident from the fact that $Lip_c(G)$ is dense in $H^1_{20}(G)$.

We first prove :

THEOREM 2.1. – _There is a real number_ λ_0 _such that_

$$m\|u\|_1^2 \leq \operatorname{Re}B_1(u , u) , \qquad |B_1(u , v)| \leq M\|u\|_1 \cdot \|v\|_1$$

(2.5) $$|B_2(u , v)| \leq M(\|v\|_1 \cdot \|u\|_0 + \|u\|_1 \cdot \|v\|_0 + \|u\|_0 \cdot \|v\|_0$$

$$|B_2(u , u)| \leq (m/2) \|u\|_1^2 + \lambda_0 C(u , u)$$

where $\|u\|_1$ _and_ $\|u\|_0$ _denote the norms in_ $H^1_{20}(G)$ _and_ $L_2(G)$, _respectively_ ; _we may take_ $\lambda_0 = M(1 + 2M/m)$.

Proof. - If we write $u = u_1 + iu_2$, the first inequality follows from (1.11), since

$$\operatorname{Re}B_1(u , u) = \int_G \sum_{\alpha,\beta} a^{\alpha\beta}(u_{1,\alpha} u_{1,\beta} + u_{2,\alpha} u_{2,\beta}) \, dx \qquad .$$

The second and third inequalities are immediate consequences of (1.11) and the Schwarz inequality (and the fact that $|\sum_\alpha b^\alpha \bar{v}_{,\alpha}| \leq (\sum_\alpha |b^\alpha|^2)^{1/2} (\sum_\alpha |v_{,\alpha}|^2)^{1/2}$, etc.). The fourth follows by setting $v = u$ in the third inequality and using the Cauchy inequality

$$2\|u\|_1 \cdot \|u\|_0 \leq \varepsilon\|u\|_1^2 + \varepsilon^{-1} \cdot \|u\|_0^2 , \qquad \varepsilon = m/2M \qquad .$$

THEOREM 2.2 (Lemma of Lax and Milgram) [2]. - _Suppose, in a Hilbert space_ $\mathcal{H}$, $B_0(u , v)$ _is linear in_ u _for each_ v _and conjugate linear in_ v _for each_ u _and suppose_

(2.6) (i) $$|B_0(u , v)| \leq M_1 \|u\| \cdot \|v\|$$

(ii) $$|B_0(u , u)| \geq m_1 \|u\|^2 , \qquad m_1 > 0 \qquad .$$

Suppose the transformation T_0 _is defined by the condition_

(2.7) $$B_0(u , v) = (T_0 u , v) \qquad .$$

Then T_0 and T_0^{-1} are operators with bounds M_1 and m_1^{-1} , respectively.

Proof. - It is clear that T_0 is a linear operator with bound M_1 . From (2.6) (ii) and (2.7), we see that

$$m_1 \|u\|^2 \leq |B_0(u , u)| = |(T_0 u , u)| \leq \|u\| \cdot \|T_0 u\|$$

so that

$$\|T_0 u\| \geq m_1 \|u\| \quad .$$

It follows easily that the range of T_0 is closed. If the range were not the whole space, there would be a v such that $B_0(u , v) = (T_0 u , v) = 0$ for every u . But, by setting $u = v$, it follows from (ii) that $v = 0$. Thus T_0^{-1} is a bounded operator with norm $\leq m_1^{-1}$.

THEOREM 2.3. - Suppose the transformation U is defined on $H^1_{20}(G)$ by the condition that

$$C(u , v) = ((Uu , v))^1_{20} , \quad v \in H^1_{20}(G) \quad . \tag{2.8}$$

Then U is a completely continuous operator.

Proof. - That U is an operator follows from Poincaré's inequality (lemma 2.1) since

$$\|Uu\| = \sup (Uu , v) = \sup \int_G u\bar{v}\, dx \leq 2^{-1} R^2 \|u\| \quad \text{if} \quad \|v\| = 1 \quad .$$

Next, suppose $u_n \to u$ (locale convergence) in $H^1_{20}(G)$. Then $u_n \to u$ (strongly) in L_2 ([7], theorem 1.10 (d)) and

$$\|U(u_n - u)\| = \sup_v \int_G (u_n - u)\, \bar{v}\, dx \leq 2^{-1/2} R\|v\| \cdot \|u_n - u\|_0 \to 0$$

so that U is compact.

THEOREM 2.4. - If λ is not in a set $\mathcal{C}$, which has no limit points in the plane, the equation (2.4) has a unique solution u in $H^1_{20}(G)$ for each given e and f in $L_2(G)$. If $\lambda \in \mathcal{C}$, there are solutions of (2.4) in which $u \neq 0$ and $e = f = 0$, but the manifold of these is finite dimensional. If λ_0 is defined as in theorem 2.1, then no real number $\lambda_1 > \lambda_0$ is in $\mathcal{C}$.

Proof. - Let us define λ_0 as in theorem 2.1 and B_0 by

$$B_0(u , v) = B(u , v) + \lambda_0 C(u , v) \tag{2.9}$$

and define T_0 by (2.7). Then, equation (2.4) is equivalent to

$$T_0 u + (\lambda - \lambda_0) Uu = w , \quad \text{where} \quad ((w , v)) = L(v) \tag{2.10}$$

L being a linear functional. Moreover, from theorem 2.1, it follows that B_0 satisfies the conditions of the lemma of Lax and Milgram with $m_1 = m/2$. Accordingly T_0 has a bounded inverse so (2.10) is equivalent to

$$u + (\lambda - \lambda_0) T_0^{-1} Uu T_0^{-1} w \quad .$$

Since $T_0^{-1} U$ is compact, the theorem follows from the Riesz theory of linear operators.

As an immediate consequence of the theorems of this section and Poincaré's inequality we obtain the following theorem :

THEOREM 2.5. (Local existence and uniqueness theorem). - There is an $R_0 > 0$, depending only on m and M such that if $0 < R \leqslant R_0$ and $G \subset B(x_0 , R)$, then $\lambda = 0$ is not in the set $\mathcal{C}$ of theorem 2.4 and, in fact if u is the solution of (2.4), then

$$\|u\|_1 \leqslant 4m_1^{-1}(\|e\|_0 + R\|f\|_0) \tag{2.11}$$

3. First interior boundedness and approximation theorems.

In this section, we continue to assume that the coefficients $a^{\alpha\beta}$, b^α, c^α, and d satisfy (1.11) and that G is bounded and $G \subset B(x_0 , R)$.

THEOREM 3.1 (Interior boundedness in H_2^1). - Suppose that u, e and $f \in L_2(G)$ and that $u \in H_2^1(D)$ for each D with $\bar{D} \subset G$ and satisfies (1.1) for each $v \in H_{20}^1(G)$ which vanishes in $G - \bar{D}$ for some $\bar{D} \subset G$. Then there is a constant C depending only on m and M (and R) such that

$$(3.1) \qquad \|\nabla u\|_{0,D} \leq C[\delta^{-1} \|u\|_0 + \|e\|_0 + \delta\|f\|_0], \qquad \delta \leq 1 ,$$

being the distance from D to δG (the boundary of G).

Notations. - $|\nabla u| \geq 0$, $|\nabla u|^2 = \sum_\alpha |u_{,\alpha}|^2$; $\|\nabla u\|_{0,D}$ denotes the L_2 norm of $|\nabla u|$ over D.

Proof. - Let us define $\eta(x) = 1$ on D, $\eta(x) = 1 - 2\delta^{-1} d(x, D)$ for $0 \leq d(x, D)$ (the distance of x from D) $\leq \delta/2$, and $\eta(x) = 0$ otherwise ; and let us define

$$v = \eta U , \qquad U = \eta u$$

and substitute that v in our equation (1.1) as altered to (2.4) with $\lambda = 0$ and take the real part. Then $U \in H_{20}^1(G)$ and we have

$$\bar{v}_{,\alpha} = \eta(\bar{U}_{,\alpha} + \eta_{,\alpha} \bar{u}) , \qquad \eta u_{,\beta} = U_{,\beta} - \eta_{,\beta} u$$

and our equation becomes

$$(3.2) \qquad 0 = \mathrm{Re}B(U, U) + \mathrm{Re} \int_G \Sigma[a^{\alpha\beta}(\eta_{,\alpha} \bar{u}U_{,\beta} - \bar{U}_{,\alpha} \eta_{,\beta} u) - a^{\alpha\beta} \eta_{,\alpha} \eta_{,\beta} u\bar{u}$$

$$+ \eta e^\alpha \bar{U}_{,\alpha} + \eta_{,\alpha} \bar{u} b^\alpha U + \eta_{,\alpha} \eta \bar{u} e^\alpha - c^\alpha \eta_{,\alpha} \bar{U} u + \eta \bar{U} f_\gamma] \, dx$$

Using theorem 2.1 for U and relations like

$$|\Sigma a^{\alpha\beta} \varphi_\alpha \bar{\psi}_\beta| \leq M(\Sigma|\varphi_\alpha|^2)^{1/2} (\Sigma|\psi_\alpha|^2)^{1/2}$$

and then the Cauchy inequality to eliminate the terms involving the $U_{,\alpha}$ in the remaining integral, we see that

$$0 \geqslant (m/4)\,\|U\|_1^2 - (\lambda_0 + C_1\,\delta^{-2})\,\|u\|_0^2 - C_2\,\|e\|_0^2 - C_3\,\delta^{-1}\,\|u\|_0 \;.\; \delta\|f\|_0$$

from which the theorem follows easily.

THEOREM 3.2 (Approximation theorem). - Suppose that the coefficients $a_n^{\alpha\beta}$, b_n^{α}, c_n^{α}, and d_n satisfy (1.11) for each n on G and converge almost everywhere on G to $a^{\alpha\beta}$, b^{α}, c^{α}, and d, respectively, and suppose that $e_n \rightharpoonup e$ and $f_n \rightharpoonup f$ in $L_2(G)$. Suppose that $u_n \rightharpoonup u$ in $H_2^1(G)$ and that u_n is a solution of $(1.1)_n$ for each n. Then u is a solution of (1.1).

Notation. - $\rightharpoonup$ denotes weak convergence.

Proof. - For each $v \in H_{20}^1(G)$, we see that

$$a_n^{\alpha\beta}\,\bar{v}_{,\alpha} \to a^{\alpha\beta}\,v_{,\alpha}\,, \quad b_n^{\alpha}\,\bar{v}_{,\alpha} \to b^{\alpha}\,v_{,\alpha}\,, \text{ etc.} \qquad ,$$

in $L_2(G)$ ($\to$ denotes strong or ordinary convergence) so that

$$B(u_n\,,\,v) \to B(u\,,\,v)\,, \quad C(u_n\,,\,v) \to C(u\,,\,v)\,, \text{ and } \quad L_n(v) \to L(v) \qquad .$$

4. Interior boundedness.

Suppose that a function $u \in H_2^1[B(x_0\,,\,R)]$. Then there is a lemma of SOBOLEV ([9], [7]) which states that $u \in L_{2s}[B(x_0\,,\,R)]$ and that

$$(4.1) \qquad \{\int_{B(x_0,R)} |u(x)|^{2s}\,dx\}^{1/s} \leqslant C_0 \int_{B(x_0,R)} [|\nabla u|^2 - R^{-2}\,|u|^2]\,dx \qquad ,$$

$$1 \leqslant s \leqslant \nu/(\nu - 2) \text{ if } \nu > 2\,, \quad s \geqslant 1 \text{ if } \nu = 2 \qquad ;$$

in the case $\nu = 2$, u still need not be bounded. The function U, mentioned in § 1, $\in L_2(G)$ and to $H_2^1(D)$ for each D with $\bar{D} \subset G$. If U also satisfies the conditions near (1.12), then $U^s \in L_2(D)$ and it turns out that we can conclude that $U^s \in H_2^1(\Delta)$ for each Δ with $\bar{\Delta} \subset D$. Indeed, it is possible to prove the following lemma :

LEMMA 4.1. - Suppose that U is real, $U(x) \geq 1$, and satisfies the underlined conditions near and including (1.12) and, in addition that

$$w = U^{\tau} \in L_2[B(x_0 , R + a)]$$

for some $\tau \geq 1$ where we assume that $B(x_0 , R + a) \subset G$ and $0 < a \leq R$. Then $w \in H_2^1[B(x_0 , R)]$ and

$$\int_{B(x_0,R)} |\nabla w|^2 \, dx \leq C_1 \tau^2 a^{-2} \int_{B(x_0,R-a)} w^2 \, dx , \qquad C_1 \geq 1 \tag{4.2}$$

where C_1 depends only on ν, m, M, and λ.

Proof. - A technical lemma allows us to substitute

$$\psi = \eta^2 U^{2-\lambda} U_L^{2\tau-2}$$

in (1.12), U_L being the truncated function, defined by

$$U_L(x) = U(x) \text{ if } U(x) \leq L , \qquad U_L(x) = L \text{ if } U(x) \geq L \tag{4.3}$$

and η being defined by $\eta(x) = 1$ on $B(x_0 , R)$, equal to $a^{-1}(|x - x_0| - R)$ for $R \leq |x - x_0| \leq R + a$ and 0 otherwise. Since $U_{L,\alpha} = 0$ almost everywhere on E_L, the set where $U(x) \geq L$, we see that

$$\psi_{,\alpha} = \eta^2 U^{1-\lambda} U_L^{2\tau-2} [(2 - \lambda) U_{,\alpha} + (2\tau - 2) U_{L,\alpha}] + 2\eta\eta_{,\alpha} U^{2-\lambda} U_L^{2\tau-2}$$

the inequality (1.12) becomes (again using $\nabla U_L = 0$ on E_L) :

$$\int_G \{\eta^2 U_L^{2\tau-2} [\lambda(2 - \lambda) \nabla U.a.\nabla U + (2 - \lambda) Ub.\nabla U - \lambda Uc.\nabla U] + dU^2 + (2\tau - 2)(\lambda \nabla U_L.a.\nabla U_L + U_L b.\nabla U_L) + 2\eta U U_L^{2\tau-2} \nabla\eta(\lambda a.\nabla U + bU)\} \, dx = 0 \tag{4.4}$$

where we have abbreviated $\sum U_{,\alpha} a^{\alpha\beta} U_{,\beta}$ to $\nabla U.a.\nabla U$, $b^{\alpha} U_{,\alpha}$ to $b.\nabla U$, etc.

Using the bounds for the coefficients and the inequalities of Cauchy and Schwarz as usual, we conclude that

$$(4.5) \qquad \int_G \eta^2 U_L^{2\tau-2} [|\nabla U|^2 + (\tau - 1)\ |\nabla U_L|^2]\ dx$$

$$\leq C_1 \int_G [\tau\eta^2 U_L^{2\tau-2} U^2 + |\nabla\eta|^2 U^2 U_L^{2\tau-2}]\ dx \quad .$$

If we now set $w_L = \eta U U_L^{\tau-1}$, we find (again using $\nabla U_L = 0$ on E_L) that

$$(4.6) \qquad \nabla w_L = U U_L^{\tau-1} \nabla\eta + \eta U_L^{\tau-1}[\nabla U + (\tau - 1)\ \nabla U_L] \quad .$$

It follows from (4.5) and (4.6) that

$$(4.7) \quad \int_{B(x_0,R+a)} |\nabla w_L|^2\ dx \leq C_2\ \tau^2 \int_{B(x_0,R+a)} (\eta^2 + |\nabla\eta|^2)\ U^2\ U_L^{2\tau-2}\ dx \quad .$$

Since $U^\tau \in L_2[B(x_0\ R + a)]$, we may let $L \to \infty$ to obtain our result.

THEOREM 4.1. - Suppose U satisfies the hypotheses of lemma 4.1 with $\tau = 1$. Then U is bounded on each domain $\overline{D} \subset G$ and

$$(4.8) \qquad |U(x)|^2 \leq C a^{-\nu} \int_{B(x_0,R+a)} |U(y)|^2\ dy\ , \qquad x \in B(x_0\ , R) \quad ,$$

$$0 < a \leq R\ , \quad B(x_0\ , R + a) \subset G\ , \quad \nu > 2 \quad ,$$

where C depends only on ν , m , M , and λ .

Remark. - If $\nu = 2$, U is still bounded on interior domains but the inequality in (4.8) must be replaced by

$$(4.8') \qquad |U(x)|^2 \leq C(\varepsilon)\ a^{-\nu-\varepsilon} \int_{B(x_0,R+a)} |U(y)|^2\ dy\ , \qquad \nu = 2 \quad .$$

This result is not good enough to obtain the results in the next section. However, the writer proved the results in the next section in the case $\nu = 2$ for more general systems of equations many years ago [3], [4]. A simplified version of this old work appears in [6], chapter 4. So in the next section, we assume $\nu > 2$.

Proof. - Let us define

$$s = \nu/(\nu - 2) \ , \quad w_0 = U \ , \quad w_n = U^{s^n} \ , \quad B_n = B(x_0 \ , R + 2^{-n} a) \ , \quad W_n = \int_{B_n} w_n^2 \, dx$$

Using the lemma we conclude in turn that $w_1 = w_0^s \in L_2(B_1)$, $w_1 \in H_2^1(B_2)$, $w_2 = w_1^s \in L_2(B_2)$, $w_2 \in H_2^1(B_3)$, etc. Then, using the inequalities (4.1) and (4.2) with $\tau = s^{n-1}$ and a replaced by 2^{-n} a , we obtain the recurrence relation

$$W_n^{1/s} = \{\int_{B_n} w_{n-1}^{2s} \, dx\}^{1/s} \leq C_0 \int_{B_n} (|\nabla w_{n-1}|^2 + R^{-2} w_{n-1}^2) \, dx$$

$$\leq 2C_0 \, C_1 \, s^{2n-2} \, 4^n \, a^{-2} \int_{B_{n-1}} w_{n-1}^2 \, dx = K_0 \, K_1^n \, W_{n-1} \quad ,$$

$$K_0 = 2C_0 \, C_1 \, s^{-2} \, a^{-2} \ , \quad K_1 = 4s^2 \quad .$$

From this recurrence relation for each n , we conclude that

$$W^{1/s^n} \leq K_0^\alpha \, K_1^\beta \, W_0 = Ca^{-\nu} \, W_0 \ , \qquad \alpha = (1 - s^{-1})^{-1} = \nu/2 \ , \quad \beta = \alpha^2 \quad .$$

The theorem follows by letting $n \to \infty$.

5. Hölder continuity of the solutions.

In this section we shall assume that $\nu > 2$ (see the remark after theorem 4.1) and shall restrict ourselves to the special equations

$$(5.1) \quad \int_G \sum(\zeta_{,\alpha} \, a^{\alpha\beta} \, u_{,\beta} - \zeta c^\alpha \, u_{,\alpha}) \, dx \equiv \int_G (\nabla\zeta \cdot a \cdot \nabla u - \zeta c \cdot \nabla u) \, dx = 0 \quad ,$$

$$(5.2) \qquad \int_G [\nabla\zeta(a \cdot \nabla u + e) + \zeta(c \cdot \nabla u + f)] \, dx = 0 \quad .$$

It was pointed out in the introduction that the type (5.2) with e and f bounded is sufficient for the application to the calculus of variations. The general equations (1.1) have been treated in [5] by a somewhat longer method.

We need the following two generalizations of Poincaré's inequality :

LEMMA 5.1. - <u>There are constants</u> $C_1(\nu)$ <u>and</u> $C_2(\nu\ ,\ c)$ <u>such that</u>

$$\int_{B(x_0,R)} |u|^2\, dx \leq C_1 R^2 \int_{B(x_0,R)} |\nabla u|^2\, dx \quad \underline{\text{if}} \quad \int_{B(x_0,R)} u\, dx = 0$$

$$\int_{B(x_0,R)} |u|^2\, dx \leq C_2 R^2 \int_{B(x_0,R)} |\nabla u|^2\, dx \quad \underline{\text{if}} \quad |S| \geq c|B(x_0\ ,\ R)|\ ,\quad c > 0\ ,$$

<u>for all</u> $u \in H_2^1[B(x_0\ ,\ R)]$; <u>here</u> S <u>is the set of</u> x <u>where</u> $u(x) = 0$ <u>and</u> $|S|$ <u>is its measure.</u>

<u>Proof.</u> - It is sufficient to prove these for $R = 1$ and $x_0 = 0$. We prove the second, the first is proved similarly. Suppose the second is false. Then there exists a sequence $\{u_n\}$ with $\|u_n\|_1$ (the <u>full</u> norm in H_2^1) $= 1$ such that $|S_n| \geq c|B(0\ ,\ 1)|$ and

$$\int_{B(0,1)} |u_n^2|\, dx > n \int_{B(0,1)} |\nabla u_n|^2\, dx \qquad (5.3)$$

We may assume that $u_n \rightharpoonup u$ in H_2^1 ([7], theorem 1.10 b) so that $u_n \to$ in L_2 ([7], theorem 1.10 d). From (5.3) we conclude that $\nabla u_n \to 0$ in L_2 , so that $u_n \to u$ in H_2^1 . Then u must be a constant d ([7], theorem 1.1) $\neq 0$ since $\|u\|_1 = 1$. But then

$$0 = \lim_{n\to\infty} \int_{B(0,1)} |u_n - u|^2\, dx \geq \lim_{n\to\infty} \int_{S_n} |u_n - u|^2\, dx \geq \lim d^2\, |S_n|$$

which is a contradiction.

<u>Definition.</u> - A function $v \in H_2^1(D)$ for each D with $\overline{D} \subset G$ is a <u>sub-solution</u> of (5.1) if and only if

$$\int_G (\nabla\zeta.a.\nabla v + \zeta c.\nabla v)\, dx \leqslant 0 \quad \text{for each} \quad \zeta \in \mathrm{Lip}_c(G) \;,\quad \zeta(x) \geqslant 0 \quad .$$

Remarks. - This condition is formally equivalent to the condition

$$\frac{\partial}{\partial x^\alpha} a^{\alpha\beta} v_{,\beta} - c^\alpha v_{,\alpha} \geqslant 0 \qquad .$$

LEMMA 5.2. - Suppose that

(i) F is non-negative and convex on the interval $[0, \infty)$,

(ii) $H = - e^{-F}$ is convex on that interval,

(iii) u is a non-negative solution of (5.1) on G ,

(iv) $v(x) = F[u(x)]$, and

(v) $v \in L_2(G)$.

Then v is a sub-solution of (5.1) on G and

$$\int_D |\nabla v|^2\, dx \leqslant Ca^{-2}\, |G| \quad \text{if} \quad D \subset G_a \;,\quad G \subset B(x_1 , R)$$

where C depends only on ν , m , M , and R and G_a is the set of x in G such that $B(x , a) \subset G$.

Proof. - First, we assume that

$$H \in C^2[0 , \infty) \;,\quad - 1 \leqslant H(u) \leqslant - \varepsilon \qquad (\varepsilon > 0)$$

and that H'' is bounded on $[0 , \infty)$. Then $F \in C^2[0 , \infty)$, and F , F' , and F'' are bounded there with $F''(u) \geqslant [F'(u)]^2$. Let us set $\zeta = \eta^2 F'(u)$ in equation (5.1), where η is defined as usual. It follows that

$$0 = \int_G [2\eta\nabla\eta.a.\nabla v + \eta^2 F''(u)\, \nabla u.a.\nabla u + \eta^2\, c.\nabla v]dx$$

$$\geqslant \int_G [\eta^2 (\nabla v.a.\nabla v + c.\nabla v) + 2\eta\nabla\eta.a.\nabla v]\, dx \qquad ,$$

since $F'' \geqslant (F')^2$. Finally

$$(5.4) \qquad \int_G \eta^2 \, |\nabla v|^2 \, dx \leq C \int_G (\eta^2 c^2 + |\nabla \eta|^2) \, dx$$

from which the inequality follows easily.

In the general case, H is convex with $- \leq H(u) < 0$ on $(0, \infty)$. It is easy to see that H can be approximated from below by functions H_n having the properties in the preceding paragraph. It follows that the functions $v_n(x) \to v(x)$ from below and hence strongly in $L_2(G)$. Clearly, also, $V_n \to v$ in $H_2^1(D)$ for each D with $\bar{D} \subset G$, on account of the inequality (5.4) which holds for each n. The inequality holds in the limit by lower-semicontinuity.

THEOREM 5.1 (Harnack type). - Suppose that

(i) u is a non-negative solution of (5.1) on $B_{2R} \equiv B(x_0, 2R)$ and

(ii) the set S where $u(x) \geq 1$ has measure $\geq c_1 |B_{2R}|$, $c_1 > 0$. Then

$$u(x) \geq c_2 > 0 \quad \text{for} \quad x \in B_R$$

where c_2 depends only on ν, m, M, and c_1.

Proof. - There is a k, $1 < k < 2$, such that $|B_{2R} - B_{kR}| = (1/2)c_1 \, |B_{2R}|$. Then $|S \cap B_{kR}| \geq (1/2)c_1 \, |B_{kR}|$. Let us define $F(u) = \max[- \log(u + \varepsilon), 0]$, where $0 < \varepsilon < 1$. It is easy to see that F satisfies the hypotheses of lemma 5.1. Consequently

$$\int_{B_{kR}} |\nabla v|^2 \, dx \leq C_1 \, R^{\nu-2} \quad \text{where} \quad v(x) = F[u(x)]. \quad .$$

Since $v(x) = 0$ on S and $|S \cap B_{kR}| \geq (c_1/2)|B_{kR}|$, it follows from lemma 5.1 that

$$\int_{B_{kR}} v^2 \, dx \leq C_2 \, R^{\nu} \qquad .$$

The theorem follows from this and theorem 4.1.

Notation. - $u \in C_\mu^0(\bar{G})$ if and only if u satisfies a uniform Hölder condition with exponent μ on $\bar{G}$; $u \in C_\mu^0(G)$ if and only if $u \in C_\mu^0(\bar{D})$ for each D with $\bar{D} \subset G$.

THEOREM 5.2. - Suppose u is a solution of (5.1) on G. Then $u \in C^0_{\mu_0}(G)$ where $0 < \mu_0 < 1$ and μ_0 depends only on ν, m, and M.

More precisely

$$|u(x) - u(x_0)| \leqslant CL\delta^{-\tau}(|x - x_0|/R)^{\mu_0}, \qquad x \in B(x_0, R),$$

where

$$L = \|u\|^0_{2,R+\delta}, \quad B(x_0, R + \delta) \subset G, \quad \tau = \nu/2, \quad \delta \leqslant R,$$

and C depends only on ν, m, and M.

Proof. - It is sufficient to prove the inequality. It follows from theorem 4.1 that

$$|u(x)| \leqslant C_0 \, L\delta^{-\tau}, \qquad x \in B_R \equiv B(x_0, R).$$

Let us define m^* and M^* as the essential inf and sup of $u(x)$ on B_R and let us choose $\overline{m}$ (unique) so that $|S^{\pm}| \leqslant |B_R|/2$, S^+ and S^- being the sets of points $x \in B_R$ for which $u(x) > \overline{m}$ and $u(x) < \overline{m}$, respectively.

If $m^* < \overline{m} < M^*$, the functions $[M^* - u(x)]/(M^* - \overline{m})$ and $[u(x) - m^*]/(\overline{m} - m^*)$ satisfy the hypotheses of theorem 5.1 on B_R with $c_1 = 1/2$. It follows that $m_1 \leqslant u(x) \leqslant M_1$ for $x \in B_{R/2}$, where

$$m_1 = \overline{m} - h(\overline{m} - m^*), \qquad M_1 = \overline{m} + h(M^* - \overline{m}), \qquad h = 1 - c_2 < 1,$$

c_2 being the constant of theorem 5.1 with $c_1 = 1/2$. The same results hold if $\overline{m} = m^*$ or $\overline{m} = M^*$ or both.

Now, let us define

$$\varphi(r) = [\text{ess sup } u(x)] - [\text{ess inf } u(x)] \quad \text{for} \quad x \in B_r, \quad r \leqslant R.$$

We conclude from the preceding paragraph that

$$\varphi(2^{-n} R) \leqslant h^n S , \quad S = 2C_1 L\delta^{-\tau} , \quad n = 1 , 2 , \ldots$$

Thus

$$\log \varphi(r) \leqslant \log S - \log h + (n + 1) \log h < \log(S/h) - (\log h)/(\log 2) \log(R/r) ,$$

if $n \log 2 \leqslant \log(R/r) < (n + 1) \log 2$.

From this it follows that

$$\varphi(r) \leqslant h^{-1} S(r/R)^{\mu_0} , \quad \mu_0 \leqslant - \log h/\log 2 . \quad .$$

THEOREM 5.3. - <u>There are constants</u> $R_1 > 0$ <u>and</u> C <u>which depend only on</u> ν , m , <u>and</u> M , <u>such that</u>

$$\|\nabla u\|^0_{2,r} \leqslant CL(r/R)^{\tau-1+\mu_0} , \quad 0 \leqslant r \leqslant R , \quad L = \|\nabla u\|^0_{2,R} , \quad \tau = \nu/2 ,$$

<u>for each</u> R , $0 < R \leqslant R_1$, <u>and each solution of</u> (5.1) <u>with</u> $\|\nabla u\|^0_{2,R} < + \infty$.

<u>Proof</u>. - Evidently we may suppose that the average value of $u = 0$. From lemma 5.1, we conclude that

$$\|u\|^0_{2,R} \leqslant CLR \quad .$$

From theorem 5.2, we then obtain

$$(5.5) \quad |u(x) - u(x_0)| \leqslant Z_1 . \|u\|^0_{2,R}(R/2)^{-\tau} (|x - x_0|/R)^{\mu_0}$$

$$\leqslant Z_2 \, LR^{1-\tau-\mu_0}|x - x_0|^{\mu_0} , \quad |x - x_0| \leqslant R/2 \quad .$$

We define η as usual with a , G , and D replaced by r , $B(x_0 , 2r)$ and $B(x_0 , r)$, respectively, and put

$$(5.6) \quad \zeta(x) = \eta^2 [u(x) - u(x_0)] , \quad x \in B(x_0 , 2r) , \quad 0 < r \leqslant R/4 \quad ,$$

in (5.1). We obtain

$$0 = \int_{B_{2r}} \eta^2 [\nabla u . a . \nabla u + c(u - u_0) . \nabla u + 2\eta(u - u_0) \nabla\eta . a . \nabla u] \, dx \quad .$$

The theorem follows easily by using (5.5) and the inequalities of Cauchy and Schwarz.

THEOREM 5.4. - Suppose that $u \in H_2^1(G)$ and is a solution of (5.2) there, where f is bounded and $e \in L_2(G)$ and satisfies

$$(5.7) \qquad \int_{B(x_0,r)} |e|^2 \, dx \leqslant L^2 (r/R)^{\nu-2+2\mu} ,$$

$$0 < \mu < \mu_0 , \quad 0 \leqslant r \leqslant R \leqslant R_0 \quad \text{for every} \quad B(x_0 , R) \subset G$$

R_0 being the number in theorem 2.5. Then $u \in C_\mu^0(G)$ and, in fact, satisfies a condition of the form

$$(5.8) \qquad \int_{B(x_0,r)} |\nabla u|^2 \, dx \leqslant K^2 (r/R)^{\nu-2+2\mu}$$

for all x_0 , r and R as above.

Proof. - Let V be the potential of f . It is well known that V is of class C^1 with $|\nabla V(x)| \leqslant Cp \max|f(x)|$ with $|B(0 , p)| = |G|$. Also

$$\int vf \, dx = \pm \int \Sigma \, v_{,\alpha} \, V_{,\alpha} \, dx$$

so equation (5.2) is equivalent to another such with $f \equiv 0$ and e replaced by $e \pm \nabla V$, which satisfies a condition (5.7) with a different L . Moreover, by vertue of a old theorem of the writer ([7], theorem 1.12) it is sufficient to prove (5.8) for some K .

Since we have assumed that $R \leqslant R_0$, we conclude from theorem 2.5 that $u = U + H$ on $B_R \equiv B(x_0 , R)$, where U is the solution of (5.2) which $\in H_{20}^1(B_R)$ and H is the solution of (5.1) such that $H - u \in H_{20}^1(B_R)$, and we also conclude that

$$(5.9) \qquad \|\nabla U\|_R \leqslant Z_1\|e\|_R \leqslant Z_1\|e\|_G \;, \qquad \|\nabla H\|_R \leqslant Z_2\|\nabla u\|_R \leqslant Z_2\|\nabla u\|_G \;,$$

where we have chosen a fixed ball $B(x_0 , R) \subset G$ and will denote the L_2 norm of ψ on $B_r \equiv B(x_0 , r)$ by $\|\psi\|_r$. Then it follows from theorem 5.3 that

$$\|\nabla H\|_r \leqslant C_3\|\nabla u\|_G \; (r/R)^{\tau-1+\mu_0} \quad .$$

Now, let us define $\varphi(s) = L^{-1} \sup \|\nabla U\|_{S_s}$ for all e which satisfy (5.7) with L_1 replaced by L, R replaced by $S \leqslant R$, U being the solution of (5.2) $\in H^1_{20}(B_S)$. Next, choose an arbitrary e which satisfies (5.7) (L_1 replaced by L). We may write $U = U_S + H_S$ on B_S where U_S is the solution of (5.2) $\in H^1_{20}(B_S)$. Obviously e satisfies

$$\int_{B_r} |e|^2 \, dx \leqslant [L^2(S/R)^{\nu-2+2\mu}].(r/S)^{\nu-2+2\mu} \;, \quad 0 \leqslant r \leqslant S \quad .$$

Thus, using the ideas of (5.9) and the definition of φ, we conclude that

$$\|\nabla U_S\|_S \leqslant Z_1 \; L(S/R)^{\tau-1+\mu} \;, \qquad \|\nabla H_S\|_S \leqslant Z_2\|\nabla U\|_S \leqslant Z_2 \; L\varphi(S/R) \quad .$$

Now, suppose that $0 < r < S < R$. Then

$$\|\nabla U\|_r \leqslant \|\nabla U_S\|_r + \|\nabla H_S\|_r \leqslant L(S/R)^{\tau-1+\mu} \, \varphi(r/S) + Z_3 \; L\varphi(S/R) \; (r/S)^{\tau-1+\mu_0} \quad .$$

Since e is arbitrary, we conclude (setting $s = r/R$, $t = S/R$) that

$$(5.10) \qquad \varphi(s) \leqslant t^{\tau-1+\mu} \, \varphi(s/t) + Z_3 \; \varphi(t)(s/t)^{\tau-1+\mu_0} \quad .$$

Obviously φ is monotone and $\varphi(1) \leqslant Z_1$. So let us choose o, $0 < o < 1$. Then, obviously

$$\varphi(s) \leqslant S_0 \; s^{\tau-1+\mu} \;, \quad o \leqslant s \leqslant 1 \;, \quad S_0 \leqslant Z_1 \; o^{-\tau+1-\mu} \quad .$$

Using (5.10) with $o^2 \leqslant s \leqslant o$ and $t = o^{-1} s$, we obtain

$$(5.11) \qquad \varphi(s) \leq S_1 \; s^{\tau-1+\mu} \text{ , where } S_1 = S_0(1 + Z_3 \, w) \text{ , } \; w = o^{\mu_0 - \mu} \quad .$$

Since $S_1 \geq S_0$, (5.11) holds for $o^2 \leq s \leq 1$. Using (5.10) with $o^4 \leq s \leq o^2$ and $t = o^{-2} s$, we conclude that

$$\varphi(s) \leq S_2 \; s^{\tau-1+\mu} \text{ , } \; o^4 \leq s \leq 1 \text{ , } \; S_2 = S_0(1 + Z_3 \, w)(1 + Z_3 \, w^2) \quad .$$

By repeating the argument, we obtain

$$\varphi(s) \leq S s^{\tau-1+\mu} \text{ , } \; 0 \leq s \leq 1 \text{ , } \; S = S_0(1 + Z_3 \, w)(1 + Z_3 \, w^2)(1 + Z_3 \, w^4)\ldots$$

from which the theorem follows immediately.

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