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DIMENSIONAL GRADATIONS AND COGRADATIONS OF OPERATOR IDEALS.

THE WEAK DISTANCE BETWEEN BANACH-SPACES

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In this seminar we present some new notions in the operator ideals theory. These are dimensional gradations and cogradations of operator ideals which give some approximations of a given operator ideal by auxiliary sequences of ideal norms. We recall known important examples of the natural dimensional gradations of the ideals of (q,r) - summing operators and of type p and co-type q operators. We also investigate other interesting examples of dimensional cogradations of the ideals of (t,s) - nuclear operators and L_p - factorable operators.

Our considerations on the dimensional co-gradation of the ideal of L_p-factorable operators lead to the definition of the weak distance $\hat{d}(E,F)$ between Banach spaces. The weak distance has the property $\hat{d}(E,F) \leqslant d(E,F)$, where d(E,F) is the classical Banach-Mazur distance from E to F. Still, it separates spaces with respect to ideal norms : we have $\alpha(id_E) \leqslant \hat{d}(E,F)\ \alpha(id_F)$ for every operator ideal $(\mathfrak{A},\alpha)$.

The definition of the dimensional gradation has been precised in many discussions between Prof. A. Pełczynski and the author. It can be also found in the Pełczynski Notes [15] . All the material of this seminar will be contained in notes prepared by the author for the lecture notes series of Springer Verlag [20] .

1 - GENERAL DEFINITIONS

Throughout this text E,F denote real or complex Banach spaces, E^*, F^* their duals ; the action of a functional $e^* \in E^*$ on an element $e \in E$ is denoted by (e, e^*). By $\mathbb{L}(E,F)$ we denote the space of all linear bounded opeartors from E to F and by $\|\cdot\|$ - the usual operator norm on $\mathbb{L}(E,F)$. By $\mathfrak{N}(E,F)$ we denote the space of all nuclear operators from E to F and by $\nu(.)$ - the nuclear norm on $\mathfrak{N}(E,F)$. For $e^* \in E^*$ and $f \in F$ the operator $e^* \otimes f \in \mathbb{L}(E,F)$ is defined by $(e^* \otimes f)(e) = (e,e^*)\, f$ for $e \in E$. For $u \in \mathbb{L}(E,F)$, by $u^* \in \mathbb{L}(F^*,E^*)$ we denote the adjoint operator. If $u = \sum_{k=1}^{m} e_u^* \otimes e_k^{**} \in \mathbb{L}(E,E^{**})$, we put $\operatorname{trace} u = \sum_{k=1}^{m} (e_k^*, e_k^{**})$; it is well-known that trace u does not depend on a particular representation of u.

For Banach spaces E,F, by d(E,F) we denote the Banach-Mazur distance from E to F, i.e. we put $d(E,F) = \inf \left\{ \|T\| \|T^{-1}\| \,\middle|\, T \text{ an isomorfirm from E onto F} \right\}$, if E and F are isomorphic, and $d(E,F) = \infty$ otherwise. Finally, for $1 < p < \infty$ we put $p^* = p/(p-1)$.

Let us recall some fundamental definitions from the operator ideals theory. A normed operator ideal $(\mathfrak{A},\alpha)$ is a class $\mathfrak{A}$ of linear bounded operators between Banach spaces and a function α from $\mathfrak{A}$ to $\mathbb{R}_+$ which satisfy the following conditions for all Banach spaces E_o, E, F_o, F :

(0) if $u \in L(E,F)$, rank $u = 1$, then $u \in \mathfrak{A}(E,F)$ and $\alpha(u) = \|u\|$;
(1) $(\mathfrak{A}(E,F), \alpha)$ is a Banach space ;
(2) if $v \in L(E_o,E), u \in \mathfrak{A}(E,F), w \in L(F,F_o)$, then $wuv \in \mathfrak{A}(E_o,F_o)$ and $\alpha(wuv) \leqslant \|w\| \alpha(u) \|v\|$.

This definition is equivalent to the definition given by A. Pietsch in [16] (definitions 1.1.1 and 6.2.2). In case when we restrict our attention only to the category of finite - dimensional Banach spaces, a function α from the class $\mathcal{L}$ of all linear operators in the $\mathbb{R}_+$ satisfying the conditions (1) and (2) (for all finite - dimensional Banach spaces E_o,E,F_o,F) is called a finite - dimensional ideal norm.

We recall now the notion of duality for operator ideals. Let $(\mathfrak{A},\alpha)$ be a normed operator ideal. We say that $u \in L(E,F)$ (E,F Banach spaces) belongs to the dual operator ideal $(\mathfrak{A}^*, \alpha^*)$, if

(1) $$\alpha^*(u) = \sup |\text{trace } w\,B\,u\,A|$$

is finite, where the suprenum is taken over all finite-dimensional Banach spaces E_o,F_o and operators $A \in \mathbb{L}(E_o,E)$, $B \in \mathbb{L}(F,F_o)$, $w \in \mathbb{L}(F_o,E_o)$ with $\|A\| = \|B\| = \alpha(w) = 1$. It is easy to see taht $(\mathfrak{A}^*,\alpha^*)$ is a normed operator ideal too. If E,F are finite - dimensional Banach spaces, we obviously have for $u \in L(E,F)$

(2) $$\alpha^*(u) = \sup \left\{ |\text{trace } wu| \,\middle|\, w \in L(F,E),\ \alpha(w) \leqslant 1 \right\} .$$

In this case the symmetric formula

(3) $$\alpha(w) = \sup \left\{ |\text{trace } wu| \,\middle|\, u \in L(E,F),\ \alpha^*(u) \leqslant 1 \right\}$$

is valid for all $w \in \mathbb{L}(F,E)$. Moreover it is well-known and easy to see that the space $(\mathbb{L}(E,F), \alpha^{\star})$ can be identified with the dual space $(\mathbb{L}(F,E), \alpha)^{\star}$; this identification is given by the trace formula : for $u \in \mathbb{L}(E,F)$ and $w \in \mathbb{L}(F,E)$ we have

$$(w,u) = \text{trace } wu.$$

To emphasize the symmetry between formulas (2) and (3), we say, in the finite - dimensional setting, that finite - dimensional ideal norms α and $\alpha^{\star}$ are in trace duality.

We introduce now notions of dimensional gradations and cogradations which give some approximations of a given normed operator ideal by an auxiliary sequence of normed operator ideals.

Given a normed operator ideal $(\mathfrak{A}, \alpha)$, a sequence $\{(\mathfrak{A}_k, \alpha_k)\}$ of normed operator ideals is a dimensional gradation of $(\mathfrak{A}, \alpha)$ provided for all Banach spaces E,F the following conditions are satisfied :

(Grad 0) $\mathfrak{A}_k(E,F) = \mathbb{L}(E,F) \qquad (k = 1,2,\ldots)$;

(Grad 1) $\alpha_1(u) \leqslant \alpha_2(u) \leqslant \ldots$ for $u \in \mathbb{L}(E,F)$ and $\alpha(u) = \lim_k \alpha_k(u)$ for $u \in \mathfrak{A}(E,F)$;

(Grad 2) $\alpha_k(u) = \sup \left\{\alpha_k(u \mid \widetilde{E}) \;\middle|\; \widetilde{E} < E, \dim \widetilde{E} \leqslant k\right\}$ for $u \in \mathbb{L}(E,F)$ $(k = 1,2,\ldots)$.

Similarly, a sequence of normed operator ideals $\{(\mathfrak{L}_k \beta_k)\}$ is a dimension cogradation of a normed operator ideal $(\mathfrak{L}, \beta)$ if for all Banach spaces E,F the following conditions hold

(Cograd 0) $\mathfrak{L}_k(E,F) = \mathfrak{N}(E,F) \qquad (k = 1,2,\ldots)$

(Cograd 1) $\beta_k(u) \geqslant \beta_{k+1}(u)$ for $u \in \mathfrak{L}_k(E,F)$ $(k = 1,2,\ldots)$ and $\beta(u) = \lim_k \beta_k(u)$ for $u \in \mathfrak{L}_{k_o}(E,F)$;

(Cograd 2) $B_{\beta_k} = \text{conv } \mathcal{T}_{\beta_k}$,
where B_{β_k} is the unit ball in $(\mathfrak{L}_k(E,F), \beta_k)$ and
$\mathcal{T}_{\beta_k} = \{u \in \mathfrak{L}_k(E,F) \mid$ there exist $G \subset F$, $\dim G = k$ and $w \in \mathfrak{L}_k(E,G) = \mathbb{L}(E,G)$ such that $\beta_k(w) \leqslant 1$ and $ux = wx$ for $x \in E\}$.

For a large class of normed operator ideals a dimension gradation can be obtained as follows. Given a normed operator ideal $(\mathfrak{A},\alpha)$ we put for $u \in \mathbb{L}(E,F)$ (E,F Banach spaces)

(4) $$\alpha_k(u) = \sup \left\{\alpha(u|\tilde{E}) \mid \tilde{E} \subset E,\ \dim \tilde{E} \leqslant k\right\}$$

Clearly, if for every $u \in \mathfrak{A}(E,F)$ one has $\alpha(u) = \lim_k \alpha_k(u)$, we obtain in this way a dimension gradation of $(\mathfrak{A},\alpha)$.

Dimensional gradations and cogradations of finite - dimensional ideal norms are defined by the obvious modification of the previous definitions (just as sequences of ideal norms satisfying the conditions (Grad 1), (Grad 2) or (Cograd 1), (Cograd 2) respectively).

The notions of dimensional gradation and dimensional cogradation are in the natural duality. Namely , we have

Theorem 1 : Let $(\mathfrak{A},\alpha)$ be a normed operator ideal and $(\mathfrak{A}^\star,\alpha^\star)$ the dual ideal. A sequence of normed operator ideals $S\{(\mathfrak{A}_k,\alpha_k)\}$ is a dimensional cogradation of $(\mathfrak{A},\alpha)$ if and only if the sequence $\{(\mathfrak{A}^\star_k,\alpha^\star_k)\}$ of dual ideals is a dimensional gradation of $(\mathfrak{A}^\star,\alpha^\star)$.

We omit the proof of this theorem which is standard. Let us only mention that in order to prove that a norm α_k on $\mathfrak{N}(E,F)$ satisfies (Cograd 2), if a norm $\alpha^\star_k$ on $\mathbb{L}(F,E)$ satisfies (Grad 2), we show the formula.

$$\alpha^\star_k(w) = \sup \left\{|\text{trace } wu| \;\middle|\; u \in \mathcal{F}_{\alpha_k}\right\}$$

for $w \in \mathbb{L}(F,E)$. This yields (Cograd 2) by the general separation theorem argument.

2 - BASIC EXAMPLES

As the first example we introduce the natural dimensional gradation of the ideal of (q,r) - summing operators. Let $1 \leqslant r \leqslant q < \infty$. Let E,F be

Banach spaces and let $u \in \mathbb{L}(E,F)$. For a positive integer k we put

$$\pi_{q,r}^{(k)}(u) = \sup \left(\sum_{j=1}^{k} \| ux_j \|^2\right)^{1/2},$$

where the supremum is taken over all $x_1,\dots,x_k \in E$ such that $\sum |(x_j,x^\star)|^r \leq \|x^\star\|^r$ for all $x^\star \in E^\star$. We say that $u \in L(E,F)$ is <u>(q,r) - summing</u>, in symbols $u \in \Pi_{q,r}(E,F)$, if $\sup_k \pi_{q,r}^{(k)}(u) < \infty$. For $u \in \Pi_{q,r}(E,F)$ we put $\pi_{q,r}(u) = \sup_k \pi_{q,r}^{(k)}(u)$. It is well-known and easy to see that $(\Pi_{q,r}, \pi_{q,r})$ is a normed operator ideal. A sequence $\left\{(\Pi_{q,r}^{(k)}, \pi_{q,r}^{(k)})\right\}$ (where we simply put $\Pi_{q,r}^{(k)}(E,F) = \mathbb{L}(E,F)$) is a dimensional gradation of $(\Pi_{q,r}, \pi_{q,r})$. It can be easily checked that this gradation has a submultiplicative property

$$\pi_{q,r}^{(nk)}(uw) \leq \pi_{q,r}^{(n)}(w)\ \pi_{q,r}^{(k)}(u), \tag{5}$$

for $w \in \mathbb{L}(E,F)$, $u \in \mathbb{L}(F,G)$ (E,F,G Banach spaces) and positive integers n, k.

In the last few years several theorems about a behaviour of a sequence $\left\{\pi_{q,r}^{(k)}(u)\right\}$ for finite rank operators were proven. They can be found e.g. in [18], [7], [8], [10], the detailed proofs of some of them were presented also in [15] (lectures 17-23). Most of results concern the case $q = r = 2$ or $2 = r < q < \infty$, which seem to be the most important.

The first result says that for a rank n operator u the sequence $\left\{\pi_2^{(k)}(u)\right\}$ stabilizes at the N - the place, with $N = \frac{1}{2} n(n+1)$ in the real case and $N = n^2$ in the complex case.

<u>Proposition 1</u> : Let $u \in \mathbb{L}(E,F)$ be a rank n operator (E,F Banach spaces). Then $\pi_2(u) = \pi_2^{(N)}(u)$, with N defined above.

This result is essentially due to T. Figiel ; its proof is a modification of the argument used in [3] (Lemma 6.1).

The next theorem is due to the author.

Theorem 2 : Let $u \in \mathbb{L}(E,F)$ be a rank n operator (E,F Banach spaces). Then $\pi_2(u) \leqslant 2\pi_2^{(n)}(u)$.

The proof of this theorem can be found in [18] and we omit it.

Finally, let us recall the result of H. König ([7], cf. also [8] and [10]) which says

Theorem 3 : Let $2 < q < \infty$. There is a constant c_q such that for every rank n operator $u \in \mathbb{L}(E,F)$ (E,F Banach spaces), $\pi_{q,2}(u) \leqslant c_q\ \pi_{q,2}^{(n)}(u)$.

Let us formulate two problems related to theorem 3 and Proposition 1.

Problems 1 : (a) Let $2 < q < \infty$. Does there exist a function $N = N(n)$ such that for every positive integer n and every rank n operator u one has $\pi_{q,2}(u) = \pi_{q,2}^{(N)}(u)$?

(b) Does there exist a constant c such that for every $2 < q < \infty$, every positive integer n and every rank n operator u one has $\pi_{q,2}(u) \leqslant c\pi_{q,2}^{(n)}(u)$?

Theorems 2 and 3 provide a useful tool in investigations of finite - rank operators. Many of its important applications will be shown in our further considerations. As a first application of Theorem 2 we show a sharper version of the inequality between different (q,2) - summing norms of a finite-rank operator, which was originally obtained in [9] (cf. also [17], Théoreme 3.1 and Remarques 3.1, 3.2 and 3.3).

Corollary 1 : Let $2 \leqslant q < \infty$, let u be a rank n operator. Then

$$\pi_2(u) \leqslant 2\ n^{1/2-1/q}\ \pi_{q,2}^{(n)}(u).$$

Moreover, for every $2 \leqslant r \leqslant q < \infty$ we have

$$\pi_{r,2}(u) \leqslant 2^{\Theta}\ n^{1/2-1/q}\ \pi_{q,2}(u),$$

where $0 \leqslant \Theta \leqslant 1$ is chosen so that $1/r = \Theta/2 + (1-\Theta)/q$.

Proof : The first inequality is obvious. To get the second one we interpolate between two extreme cases : r = 2, established in the first inequality and r = q, when the respective inequality is trivial.

We pass now to the duality theory for the dimensional gradation $\left\{\pi_{q,r}^{(k)}\right\}$, which is fully analogous to that for the ideal of (q,r) - summing operators. We begin with the definition. Let $1 \leqslant t \leqslant s < \infty$ and let k be a positive integer. For Banach spaces E,F and $u \in \mathbb{L}(E,F)$ with rank $u \leqslant k$ we put

(6) $$\nu_{t,s}^{(k)}(u) = \inf \left\{ \left(\sum_{j=1}^{k} \|x_j^\star\|^t\right)^{1/t} \sup_{\substack{y^\star \in F^\star \\ \|y^\star\| \leqslant 1}} \left(\sum_{j=1}^{k} |(y_j,y^\star)|^{s^\star}\right)^{1/s^\star} \right\},$$

where the infimum extends over all representations $u = \sum_{j=1}^{k} x_j^\star \circ y_j$ ($x_j^\star \in E^\star$, $y_j \in F$, $j = 1,\dots,k$). We use the notation $\nu_{t,\infty}^{(k)}$ (for $1 \leqslant t < \infty$) and $\nu_{\infty,\infty}^{(k)}$ for the usual modifications of (5). Next, for $u \in \mathfrak{N}(E,F)$ we put

(7) $$\hat{\nu}_{t,s}^{(k)}(u) = \inf \sum_{m=1}^{\infty} \nu_{t,s}^{(k)}(u_m),$$

where the infimum is taken over all representations $u = \sum u_m$ with $u_m \in L(E,F)$, rank $u_m \leqslant k$ $(m = 1,2,\dots)$. Obviously, for $u \in \mathfrak{N}_{t,s}(E,F)$ we have $\hat{\nu}_{t,s}^{(k)}(u) \leqslant \nu_1(u) < \infty$.

The definition (6) should be compared with the well-known definition of (t,s)-nuclear operators. We say that $u \in \mathbb{L}(E,F)$ is <u>(t,s)-nuclear</u>, in symbols $u \in \mathfrak{N}_{t,s}(E,F)$, if u admits a representation
$u = \sum_{j=1}^{\infty} x_j^\star \otimes y_j$ with $x_j^\star \in E^\star$, $y_j \in F$ (j=1,2,...) such that $\sum_{j=1}^{\infty} \|x_j^\star\|^t < \infty$ and $\sup\{ \sum_{j=1}^{\infty} |(y_j,y^\star)|^{s^\star} \mid y^\star \in F^\star,\ \|y^\star\| \leqslant 1 \} < \infty$. For $u \in \mathfrak{N}_{t,s}(E,F)$ we put

(8) $$\hat{\nu}_{t,s}(u) = \inf \sum_{m=1}^{\infty} \left(\sum_{j=1}^{\infty} \|x_{jm}^\star\|^t\right)^{1/t} \sup_{\substack{y^\star \in F^\star \\ \|y^\star\| \leqslant 1}} \left(\sum_{j=1}^{\infty} |(y_{jm},y^\star)|^{s^\star}\right)^{1/s^\star},$$

where the infimum is taken over all representations $u = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} x_{jm}^\star \otimes y_{jm}$ with $x_{jm}^\star \in E$, $y_{jm} \in F$ ($j, m = 1,2,\dots$).

It is well-known (cf. [16] , Theorems 18.2.5, 18.4.5), that for $1 < t \leqslant s \leqslant \infty$, $(\mathfrak{N}_{t,s}, \hat{\nu}_{t,s})^\star = (\Pi_{t^\star,s^\star}, \pi_{t^\star,s^\star})$. The fillowing result is a graded version of this duality. (In the sequel we denote $\mathfrak{N}^{(k)}_{t,s}(E,F) = \mathfrak{N}(E,F)$ and $\pi^{(k)}_{t^*,s^*}(E,F) = \mathbb{L}(E,F)$, for E,F - Banach spaces).

Theorem 4 : Let $1 < t \leq s \leq \infty$ and let k be a positive integer. Then $(\mathfrak{N}^{(k)}_{t,s}, \hat{\nu}^{(k)}_{t,s})^\star = (\Pi^{(k)}_{t^*,s^*}, \pi^{(k)}_{t^*,s^*})$. Moreover, $\hat{\nu}^{(k)}_{t,s}$ and $\pi^{(k)}_{t^*,s^*}$ considered as finite - dimensional ideal norms are in trace duality.

Proof : The second assertion is just a reformulation of the first one in the finite - dimensional case. The proof of the first assertion is a slight modification of the usual proof of the duality $(\mathfrak{N}_{t,s})^* = \Pi_{t^*,s^*}$. We give here only a sketch of it.

Let E,F be Banach spaces, let $u \in \mathbb{L}(E,F)$. It is easy to see that for all finite - dimensional Banach spaces E_o, F_o and all operators $A \in \mathbb{L}(E_o,E)$, $B \in \mathbb{L}(F,F_o)$, $w \in \mathbb{L}(F_o,E_o)$,

$$|\text{trace } w\,B\,u\,A| \leq \hat{\nu}^{(k)}_{t,s}(w)\;\|B\|\;\pi^{(k)}_{t^*,s^*}(u)\,\|A\|\;.$$

Therefore, $(\hat{\nu}^{(k)}_{t,s})^\star(u) \leq \pi^{(k)}_{t^*,s^*}(u)$.

On the other hand, given $\varepsilon > 0$ pick a sequence $x_1,\dots,x_k \in E$ such that $\sum_{j=1}^{k} |(x_j,x^\star)|^{s^*} \leq \|x^\star\|^{s^*}$ for all $x^\star \in E^\star$ and $(\sum_{j=1}^{k} \|ux_j\|^{t^\star})^{1/t^\star} \leq \pi^{(k)}_{t^*,s^*}(u) + \varepsilon$. For $j = 1, \dots, k$ pick $y_j^\star \in F^\star$ with $\|y_j^\star\| = 1$ such that $(u\,x_j, y_j^\star) = \|u\,x_j\|$. Put $E_o = \text{span}(x_j)_{j=1}^{k} \subset E$ and let $J : E_o \to E$ be the canonical embedding. Put $F_1 = (\text{span}(y_j^\star)_{j=1}^{k})^{\perp} = \{f \in F \mid (f,g_j^\star) = 0 \text{ for } j=1,\dots k\} \subset F$ and $F_o = F/F_1$, let $Q : F \longrightarrow F/F_1$ be the quotient map. Finally define $w \in \mathbb{L}(F_o,E_o)$ by $w = \sum_{j=1}^{k} \xi_j y_j^\star \otimes x_j$ where $\xi_j > 0$ $(j = 1,\dots,k)$ such that $\sum_{j=1}^{k} \xi_j^{\,t} = 1$ and $\sum_{j=1}^{k} \|ux_j\|\;\xi_j = (\sum_{j=1}^{k} \|ux_j\|^{t^*})^{1/t^*}$. Then

$$\text{trace } w\,Q\,u\,J = (\sum_{j=1}^{k} \|ux_j\|^{t})^{1/t} \leq \pi^{(k)}_{t^*,s^*}(u) + \varepsilon,$$

while

$$\hat{\nu}^{(k)}_{t,s}(w) \int \leq (\sum_{j=1}^{k} (\xi_j \|y_j^\star\|)^{t})^{1/t} \sup\left\{(\sum_{j=1}^{k} |(x_j,x^\star)|^{s^*})^{1/s^*} \,\middle|\, \begin{array}{l} x^\star \in E^\star \\ \|x^\star\| \leq 1 \end{array}\right\}$$

$$\leq 1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\pi^{(k)}_{t^*,s^*}(u) \leq (\nu^{(k)}_{t^*,s^*})^\star(u)$. This completes the proof.

As an immediate consequence of Theorems 1 and 4 we get

Corollary 2 : The sequence $\left\{(\mathcal{N}^{(k)}_{t,s}, \hat{\nu}^{(k)}_{t,s})\right\}$ is a dimensional cogradation of the ideal $(\mathcal{N}_{t,s}, \hat{\nu}_{t,s})$ of (t,s) - nuclear operators.

The next corollary is a formal consequence of Theorem 4 and the fact that π_2, as a finite-dimensional ideal norm, is "trace self-dual".

Corollary 3 : Let E,F be finite-dimensional Banach spaces, let $c \geqslant 1$ and let k be a positive interger. The following conditions are equivalent

(i) $\pi_2(u) \leqslant c\, \pi_2^{(k)}(u)$ for all $u \in L(F,E)$

(ii) if $v \in \mathbb{L}(E,F)$ with $\pi_2(v) \leqslant 1$, then $v \in \text{conv}\ (c\, W_k)$, where W_k is a set of all operators $w \in \mathbb{L}(E,F)$ which admit a factorization

$$E \xrightarrow[A]{} \ell_\infty^k \xrightarrow[\Delta]{} \ell_2^k \xrightarrow[B]{} F\ ,$$

where $A \in \mathbb{L}(E,\ell_\infty^k)$, $\Delta \in \mathbb{L}(\ell_\infty^k, \ell_2^k)$ is diagonal operator, $B \in \mathbb{L}(\ell_2^k,F)$ and $\|A\|\ \|\Delta\|\ \|B\| \leqslant 1$.

Proof : Observe that for every $v \in L(E,F)$ the condition $v \in \text{conv}\ (\ c\, W_k)$ is equivalent to $\hat{\nu}^{(k)}_{2,2}(v) \leqslant c$. Next recall that on the (finite dimensional) space $\mathbb{L}(E,F)$ the norms π_2 and $\hat{\nu}_{2,2}$ (usually denoted by ν_2) coincide, hence $\pi_2 = \pi_2^{\star}$. Moreover, $\pi_2^{(k)} = (\nu^{(k)}_{2,2})^{\star}$, by Theorem 4. Therefore (i) $\Rightarrow$ (ii) easily follows from the formula (3) ; similarly, (ii) $\Rightarrow$ (i) follows from the formula (2).

Combining Corollary 3, Theorem 2 and Proposition 1 we get

Proposition 2 : Let E,F be Banach spaces, $v \in \mathbb{L}(E,F)$ has rank n and $\pi_2(v) \leqslant 1$. Then

$$v \in c\ell(\text{conv}\ W_N) \cap 2\ c\ell(\text{conv}\ W_n),$$

where $N = \frac{1}{2}n(n+1)$ in the real case and $N = n^2$ in the complex case ; "$c\ell$" denotes the "norm closure of", if E and F are finite-dimensional, the symbols "$c\ell$" in (9) can be omitted.

Proposition 3 : Let E be an n-dimensional Banach space. Then for the identity operator, id_E, we have

(i) $id_E \in \sqrt{n}\, W_N$,

(ii) $id_E = \sum_{j=1}^{k} w_j \Delta_j v_j$, where

$$E \xrightarrow{v_j} \ell_\infty^n \xrightarrow{\Delta_j} \ell_2^n \xrightarrow{w_j} E,$$

$v_j \in \mathbb{L}(E,\ell_\infty^n)$, $\Delta_j \in \mathbb{L}(\ell_\infty^n,\ell_2^n)$ is a diagonal operator, $w_j \in \mathbb{L}(\ell_2^n,E)$ $(j=1,\dots,k)$, $k \leq n^2+1$ and $\sum_{j=1}^{k} \|v_j\|\,\|\Delta_j\|\,\|w_j\| \leq 2\sqrt{n}$.

In the proof of this proposition we shall use the well-known theorem of F. John ([6]) . Let us mention that the original John's argument works only in the real case. Another proof and interesting and fruitful generalization of this theorem is due to Lewis ([12]) , whose argument can be extended also to the complex case (cf. also [15] , Lectures 15, 16).

Theorem (F. John) : Let E be an n-dimensional (real or complex) Banach space. There exist an inner product $[\cdot,\cdot]$ on E, vectors $x_1,\dots,x_N$ in E and positive scalars $\lambda_1,\dots,\lambda_N$ (where N is the same as in Proposition 10) such that

1°. $\|x\|^2 \leq [x,x]$ for $x \in E$

2°. $\|x_j\| = [x_j,x_j] = \sup\left\{|[x,x_j]| \;\middle|\; \|x\| \leq 1\right\} = 1$ $(j=1,\dots,N)$

3°. $x = \sum_{j=1}^{N} \lambda_j [x,x_j]\, x_j$ for $x \in E$, $\sum_{j=1}^{N} \lambda_j = n$.

Let $\|\cdot\|_2$ denote the euclidean norm on E induced by the inner product $[\cdot,\cdot]$, let $E_2 = (E,\|\cdot\|_2)$ and let $I : E_2 \to E$ denote the formal identy operator. From 1° it follows that $\|I\| = 1$. Another important property of the operator I is

<u>Lemma 1</u> : The operator $I^{-1} : E \to E_2$ admits a factorization

$$E \xrightarrow{\ w_1\ } \ell_\infty^N \xrightarrow{\ \Delta\ } \ell_2^N \xrightarrow{\ w_2\ } E_2,$$

where $w_1 \in \mathbb{L}(E,\ell_\infty^N)$, $\Delta \in \mathbb{L}(\ell_\infty^N,\ell_2^N)$ is a diagonal operator, $w_2 \in \mathbb{L}(\ell_2^N,E_2)$ and $\|w_1\|\,\|\Delta\|\,\|w_2\| = \sqrt{n}$. Therefore, $\pi_2(I^{-1}) = \sqrt{n}$.

<u>Proof</u> : Define $w_1 \in \mathbb{L}(E,\ell_\infty^N)$ by $w_1 x = \left\{[x,x_j]\right\}_{j=1}^N$ for $x \in E$ and $\Delta \in \mathbb{L}(\ell_\infty^N,\ell_2^N)$ by $\Delta(\left\{t_j\right\}_{j=1}^N) = \left\{\sqrt{\lambda_j}\ t_j\right\}_{j=1}^N$ for $\left\{t_j\right\}_{j=1}^N \in \ell_\infty^N$. Put $X = \Delta w_1(E) \subset \ell_2^N$ and let $P : \ell_2^N \to X$ be the orthogonal projection onto X. Define $w_3 \in \mathbb{L}(X,E_2)$ by $w_3(\left\{\sqrt{\lambda_j}[x,x_j]\right\}_{j=1}^N) = x$ for $\left\{\sqrt{\lambda_j}[x,x_j]\right\}_{j=1}^N \in X$. From 2° and 3° it follows that $\|w_1\| = \max\limits_{1 \leq j \leq N} \sup\left\{|[x,x_j]| \;\middle|\; \|x\| \leq 1\right\} =$

$$\|\Delta\| = (\sum_{j=1}^N (\sqrt{\lambda_j})^2)^{1/2} = \sqrt{n},$$

$$\|w_3(\left\{\sqrt{\lambda_j}[x,x_j]\right\}_{j=1}^N)\|_2 = \|\left\{\sqrt{\lambda_j}[x,x_j]\right\}_{j=1}^N\|_{\ell_2^N}.$$

Put $w_2 = w_3 P \in \mathbb{L}(\ell_2^N,E_2)$, then $\|w_2\| \leq \|w_3\|\,\|P\| = 1$ and the condition 3° yields that $I^{-1} = w_2\,\Delta w_1$.

Before we pass to the proof of Proposition 3 let us formulate an immediate consequence of Lemma 1 and the "trace self-duality" of the norm π_2.

<u>Corollary 4</u> : For every n-dimensional Banach space E, $\pi_2(id_E) = \sqrt{n}$.

<u>Proof</u> : Since $\pi_2 = \pi_2^{\star}$, as a finite-dimensional ideal norm, it follows that $n = \text{trace } id_E \leq \pi_2(id_E)^2$. Hence $\pi_2(id_E) \geq \sqrt{n}$. On the other hand, by Lemma 1, $\pi_2(id_E) \leq \pi_2(I^{-1})\,\|I\| = \sqrt{n}$.

<u>Proof of Proposition 3</u> : A factorization of id_E, required in (i), is given by $id_E = w_4\,\Delta w_1$, where $w_1 \in \mathbb{L}(E,\ell_\infty^N)$, $\Delta \in \mathbb{L}(\ell_\infty^N,\ell_2^N)$ are the same as in Lemma 1 and $w_4 = I\,w_2 \in \mathbb{L}(\ell_2^N,E)$.

By Corollary 4 one has $\pi_2(id_E) = \sqrt{n}$. Therefore, by Proposition 2, $id_E \in 2\sqrt{n}\ \mathrm{conv}\ W_n$. This shows the existence of a representation required in (ii). The estimate for the number K follows from the Caratheodory's theorem.

Remark : It would be interesting to give a proof of Proposition 11 (ii) which does not use the duality argument.

At the end of the discussion of the dimensional gradation $\left\{\pi_{p,q}^{(k)}\right\}$ let us stress the fact that there are (p,q)-summing norms for which any analogue of Theorems 2 and 3 is false. The following theorem is due to Figiel and Pełczynski ([15] , Lecture 21).

Theorem 5 : (i) Let u be a rank n operator.
Then

$$\pi_1(u) \leqslant 3\ \pi_1^{(4^n)}(u).$$

(ii) Let $R_n \in \mathbb{L}(\ell_\infty^{2^n}, \ell_2^n)$ be the Rademacher projection defined by

$$R_n(x) = \left\{(x, r_i)\right\}_{i=1}^{n} = \left\{2^{-n} \sum_{j=1}^{2^n} x(j) r_i(j)\right\}_{k=1}^{n} \in \ell_2^n$$

for $x = \left\{x(j)\right\}_{j=1}^{2^n} \in \ell_\infty^{2^n}$, where $r_i(j)$ is the value of the i-th Rademacher function on the interval $((j-1)2^{-n}, j2^{-n})$ $(j=1,\dots,2^n)$. Then R_n is a rank n operator such that

$$\pi_1^{(k)}(R_n) \leqslant c\sqrt{\frac{\log k}{n}}\ \pi_1(R_n) \qquad \text{for } k = 2,3,\dots,$$

where c is a universal constant.

Another important examples of dimensional gradations are natural dimensional gradations of the type p and cotype q norms of operators $(1 < p \leqslant 2 \leqslant q < \infty)$. We recall the definitions. Let $g_1, g_2, \dots$ be a sequence of (real or complex) independent standard Gaussian random variables on a probability space (Ω, μ). Let $1 < p \leqslant 2 \leqslant q < \infty$ and let k be a positive integer. For Banach spaces E,F and $u \in \mathbb{L}(E,F)$ we define $\alpha_{p,k}(u)$ (resp. $\beta_{q,k}(u)$) as the smallest number c (resp. c_1) such that

$$\left(\int_\Omega \left\| \sum_{i=1}^{k} g_i(\omega)\, u x_i \right\|^2 d\mu(\omega)\right)^{1/2} \leqslant c \left(\sum_{i=1}^{k} \|x_i\|^p\right)^{1/p},$$

$$\text{resp.} \left(\sum_{i=1}^{k} \| ux_i \|^q\right)^{1/q} \leqslant c_1 \left(\int_\Omega \| \sum_{i=1}^{k} g_i(\omega)x_i \|^2 \, d\mu(\omega)\right)^{1/2}$$

holds for arbitrary vectors $x_1,\dots,x_k \in E$. We say that u is of Gaussian type p, in xymbols $u \in \mathcal{T}_p(E,F)$, (resp. Gaussian cotype q, in symbols $u \in \mathcal{C}_q(E,F)$,) if $\alpha_p(u) = \sup_k \alpha_{p,k}(u) < \infty$ (resp $\beta_q(u) = \sup_k \beta_{q,k}(u) < \infty$).

It is well-known that $(\mathcal{T}_p,\alpha_p)$ and $(\mathcal{C}_q,\beta_q)$ are normed operator ideals. It can be easily shown that the sequences $\{(\mathcal{T}_{p,k},\alpha_{p,k})\}$ and $\{(\mathcal{T}_{q,k},\beta_{q,k})\}$ (where we put $\mathcal{T}_{p,k}(E,F) = \mathbb{L}(E,F) = \mathcal{C}_{q,k}(E,F)$) are dimensional gradations of $(\mathcal{T}_p,\alpha_p)$ and $(\mathcal{C}_q,\beta_q)$ respectively.

In a similar way, using the Rademacher functions instead of Gaussian random variables, we can define the dimensional gradations of the ideals of Rademacher type p and Rademacher cotype q operators. These gradations, which were introduced by Maurey and Pisier in [14] , and their relations to the gradations of Gaussian norms were investigated by König and Tzafriri ([10] and [8]). Let us also mention that the gradations of Rademacher norms are equivalent to the gradations which have the submultiplicative property, analogous to (5) (cf. [14]).

We pass now to an investigation of behaviour of the natural gradations of Gaussian type p and cotype q norms for finite-rank operators. The obtained results answer questions raised by many authors and already have many applications in the Banach spaces theory.

Our considerations are based upon the relations between the gradations of Gaussian type p and cotype q norms and the cogradation (resp. gradation) of the ideal of (p,2)-nuclear (resp.(q,2)-summing) operators.

Proposition 4 : Let $1 < p \leqslant 2 \leqslant q < \infty$ and let k be a positive integer. Let E,F be Banach spaces and let $u \in L(E,F)$. Then

$$\alpha_{p,k}(u) = \sup\left\{ \left(\int_\Omega \| \sum_{j=1}^{k} g_j(\omega)uv(e_j) \|^2 d\mu(\omega)\right)^{1/2} \,\middle|\, \begin{array}{l} v \in \mathbb{L}(\ell_2^k,E) \\ \nu_{p,2}^{(k)}(v) \leqslant 1 \end{array} \right\},$$

$$\beta_{q,k}(u) = \sup\left\{ \pi_{q,2}^{(k)}(uw) \,\middle|\, w \in \mathbb{L}(\ell_2^k,E),\ \int_\Omega \| \sum_{j=1}^{k} g_j(\omega)w(e_j) \|^2 d\mu(\omega) \leqslant 1 \right\},$$

where $(e_j)_{j=1}^{k}$ is the unit vector basis in ℓ_2^k.

The proof of this proposition can be found in [18] (for the case $p = 2 = q$, the generalization to the case $1 < p < 2 < q < \infty$ is easy) (cf. also [15] or [17] , Propositions 2.2 and 2.4).

Combining Proposition 4 with Proposition 1 and Theorems 2 and 3 we get

Theorem 6 : Let us be an operator of rank n. Then

$$\alpha_2(u) = \alpha_{2,N}(u) \leq 2\, \alpha_{2,n}(u),$$

$$\beta_2(u) = \beta_{2,N}(u) \leq 2\, \beta_{2,n}(u),$$

where $N = \frac{1}{2} n(n+1)$ in the real case and $N = n^2$ in the complex case. If $1 < p < 2 < q < \infty$, then

$$\alpha_p(u) \leq c_p\, \alpha_{p,n}(u),$$

$$\beta_q(u) \leq c_q\, \beta_{q,n}(u),$$

where c_p and c_q are constants depending only on p and q respectively.

Let us formulate two typical applications of the obtained results, which strengthen earlier results from [9] and [4] (cf. also [17] , Proposition 3.2 and Théorème 4.1). The second result is a generalization of the Mourey's extension theorem ([13]).

Corollary 5 : Let $1 < r \leq p \leq 2 \leq q \leq s < \infty$. Let u be an operator of rank n. Then

$$\alpha_p(u) \leq 2^{\Theta}\, n^{1/r-1/p}\, \alpha_r(u),$$

$$\beta_q(u) \leq 2^{\Theta'} n^{1/q-1/s}\, \beta_s(u),$$

where $0 \leq \Theta \leq 1$ and $0 \leq \Theta' \leq 1$ are chosen to satisfy $1/p = \Theta/r + (1-\Theta)/2$ and $1/q = \Theta'/s + (1-\Theta')/2$.

This corollary easily follows from Proposition 4 and Corollary 1.

Corollary 6 : Let $1 < p \leq 2 \leq q < \infty$. Let E,F,G be Banach spaces and let $F_o \subset F$ be a subspace. Let $u \in \mathbb{L}(E,F)$, $w \in \mathbb{L}(F_o,G)$, rank $w = n$. Let $E_o = u^{-1}(F_o)$ and let $u_o = u|E_o$ be the restriction of u. Then there exists an extension $T \in \mathbb{L}(E,G)$ of the operator $w\, u_o$ such that

$$(10) \qquad \gamma_2(T) \leq 4\ \alpha_{p,n}(u)\ \beta_{q,n}(w)\ n^{1/p-1/q},$$

where $\gamma_2(.)$ is the norm of factorization through L_2-space. In particular, if E is a Banach space with type p and $E_o \subset E$ its n-dimensional subspace then

$$d(E_o,\ell_2^n) \leq 4\ \alpha_{p,n}(id_E)\ \beta_{q,n}(id_E)\ n^{1/p-1/q}.$$

Proof : The following lemma is a consequence of the Hahn-Banach theorem and the characterization of the ideal $\Gamma_2^\star$, dual to the ideal (Γ_2,γ_2) of L_2-factorable operators, due to Kwapieri ([11]) (cf. [4] , Lemma 10.1).

Lemma 2 : Let E,G be Banach spaces, let $E_o \subset E$ be a subspace and let $J : E_o \to E$ be the canonical embedding. Let $v_o \in \mathbb{L}(E_o,G)$ rank $v_o = n$ and let $c > 0$. The following conditions are equivalent

(a) there exists $T \in \Gamma_2(E,G)$ such that $T|E_o = v_o$ and $\gamma_2(T) \leq c$;

(b) $\pi_2(v_o\ v) \leq c\ \pi_2\ (v^\star\ J^\star)$ for every $v \in \mathbb{L}(\ell_2^n,E)$.

It follows from Lemma 2 that the existence of an extension satisfying (10) is an obvious consequence of the inequality.

$$(11) \qquad \pi_2(w\ u_o\ v) \leq 4\ \alpha_{2,n}(u)\ \beta_{2,n}(w)\ \pi_2(v^\star\ J^\star)$$

valid for all $v \in \mathbb{L}(\ell_2^n,E)$.

Let $v \in \mathbb{L}(\ell_2^n,E)$. By theorem 2 and Propositions 4 and 2 we have

$$\begin{aligned}\pi_2(w\ u_o v) &\leq \pi_2^{(n)}(w\ u_o v)\\ &\leq 2\ \beta_{2,n}(w)\Big(\int_\Omega \Big\|\sum_{j=1}^n g_j(\omega)\ u_o v(e_j)\Big\|^2 d\mu(\omega)\Big)^{1/2}\\ &\leq 2\ \beta_{2,n}(w)\Big(\int_\Omega \Big\|\sum_{j=1}^n g_j(\omega)\ uJv(e_j)\Big\|^2 d\mu(\omega)\Big)^{1/2}\end{aligned}$$

$$\leq 2\ \beta_{2,n}(w)\ \alpha_{2,n}(u)\ \nu_2^{(n)}\ (v^\star\ J^\star)$$

$$\leq 4\ \beta_{2,n}(w)\ \alpha_{2,n}(u)\ \pi_2(v^\star\ J^\star).$$

This cimpletes the proof of (11).

Remark : An alternative proof of Corollary 6 can be obtained by an easy modification of an argument from [17] , Théorème 4.1.

Finally let us introduce some dimensional gradations and cogradations of ideals Γ_p of L_p-factorable operators ($1 \leq p \leq \infty$). We begin with the definitions. Let $1 \leq p \leq \infty$. Let E,F be Banach spaces and let $u \in \mathbb{L}(E,F)$. We say that u is L_p-factorables, in symbols $u \in \Gamma_p(E,F)$, if $J_F\ u = vw$, where $w \in \mathbb{L}(E,L_p(\Omega,\mu))$, $v \in L(L_p(\Omega,\mu), F^{\star\star})$ and $J_F : F \to F^{\star\star}$ is the canonical embedding. For $u \in \Gamma_p(E,F)$ we put $\gamma_p(u) = \inf \|v\|\ \|w\|$, with the infimum taken over all factorizations $J_F\ u = vw$ as above. This motion was introduced by Kwapień in [11] , where it was also proven that (Γ_p,γ_p) is a normed operator ideal.

For $1 \leq p \leq \infty$ we define the sequence $\left\{\gamma_{p,k}\right\}$ by the general formula (4), i.e.

$$\gamma_{p,k}(u) = \sup \left\{ \gamma_p(u|\widetilde{E}) \ \middle|\ \widetilde{E} \subset E,\ \dim \widetilde{E} \leq k \right\} \quad ,$$

for $u \in \mathbb{L}(E,F)$ (E,Γ Banach spaces) and k = 1,2,.... Using the ultraproduct technique one can show that $\left\{(\Gamma_{p,k},\gamma_{p,k})\right\}$ is actually a dimensional gradation of (Γ_p,γ_p). This gradation has very clear geometric interpretation ; for p = 2 and u-the identity operator on a Banach space E, the condition $\gamma_{2,k}(id_E) \leq c$ is equivalent to $d(\widetilde{E},\ell_2^k) \leq c$ for all k-dimensional subspaces $\widetilde{E} \subset E$.

The next proposition gives a solution for an operator version of a problem discussed by Figiel, Lindenstrauss and Milman in [3] , Section 6. As a particular case we get an essential improvement of Theorem 6.2 in [3] (for $k \sim n/2$).

<u>Proposition 5</u> : Let E,F be Banach spaces, let $u \in \mathbb{L}(E,F)$. Let k,m be positive integers. Then

$$\gamma_{2,km}(u) \leq 2\sqrt{m}\,\gamma_{2,k}(u).$$

In particular, if dim E = n and $d(\tilde{E},\ell_2^k) \leq c$ for all k-dimensional subspaces $\tilde{E} \subset E$, with $k = \left[\frac{n+1}{2}\right]$, then $d(E,\ell_2^n) \leq 2\sqrt{2.c}$.

<u>Proof</u> : Let $\tilde{E} \subset E$ be a km-dimensional subspace and let $u_o = u|\tilde{E}$. From Lemma 2 it follows that the inequality

$$\gamma_2(u_o) \leq 2\sqrt{m}\,\gamma_{2,k}(u) \tag{12}$$

is equivalent to

$$\pi_2(u_o\,v) \leq 2\sqrt{m}\,\gamma_{2,k}(u)\,\pi_2(v^\star) \tag{13}$$

for all $v \in \mathbb{L}(\ell_2^{km},\tilde{E})$.

To show (13) fix $v \in \mathbb{L}(\ell_2^{km},\tilde{E})$ with $\pi_2(v^\star) \leq 1$. Since rank $v \leq km$, then, by Proposition 2, $v \in \text{conv}\, W_{km}^\star$, where $W_{km}^\star$ consists of all operators $w \in \mathbb{L}(\ell_2^{km},\tilde{E})$ which admit a factorization

$$\ell_2^{km} \xrightarrow[w']{} \ell_2^{km} \xrightarrow[\Delta]{} \ell_1^{km} \xrightarrow[w'']{} \tilde{E} \tag{14}$$

where $w' \in L(\ell_2^{km},\ell_2^{km})$, $\Delta \in L(\ell_2^{km},\ell_1^{km})$ is a diagonal operator, $w'' \in L(\ell_1^{km},\tilde{E})$ and $\|w'\|\,\|\Delta\|\,\|w''\| \leq 2$. Since $\pi_2(u_o\,v)$ is a convex function of v, one may assume, without loss of generality, that v itself admits a factorization (14), say $v = w''\Delta w'$. Define subspaces $X_j \subset \ell_2^{km}$ by $X_j = \text{span}\,(e_{(j-1)k+1},\ldots, e_{jk})$, for $j = 1,\ldots,m$, where $(e_i)_{i=1}^{km}$ is the unit vector basis in ℓ_2^{km}. Let $P_j \in \mathbb{L}(\ell_2^{km},\ell_2^{km})$ be the orthogonal projection onto X_j ; put $Y_j = \Delta(X_j) \subset \ell_1^{km}$ $\Delta_j \in \mathbb{L}(X_j,Y_j)$ be the restriction of Δ ; put $E_j = w''(Y_j) \subset \tilde{E}$ and let $w''_j \in \mathbb{L}(Y_j,E_j)$ be the restriction of w'' $(j=1,\ldots,m)$. Since dim $E_j \leq k$, then

$$\begin{aligned}\pi_2(u_o\,w''\,\Delta\,P_j) &= \pi_2(\,(u_o|E_j)\,w''_j\,\Delta j)\\ &= \gamma_2(u_o|E_j)\,\pi_2(\Delta_j^\star\,w_j''^\star)\\ &= \gamma_{2,k}(u_o)\,\|\Delta_j^\star\|\,\|w_j''^\star\|\\ &= \gamma_{2,k}(u_o)\,\|\Delta_j\|\,\|w''\|\,,\end{aligned}$$

for all $j=1,\dots,m$ (in the first inequality we use the easy estimate $\pi_2(TS) \leqslant \gamma_2(T)\, \pi_2(S^\star)$, valid for all $S : F_1 \to F_2$ such that $S^\star \in \Pi_2(F_2^\star, F_1^\star)$ and $T \in \Gamma_2(F_2,F_3)$ (F_1,F_2,F_3 Banach spaces)). Therefore,

$$\begin{aligned}
\pi_2(u_o\, v) &= \pi_2(u_o\, w'' \wedge w') \leqslant \pi_2(u_o\, w''\, \Delta)\ \| w' \| \\
&\leqslant \pi_2\Big(\sum_{j=1}^{m} u_o\, w'' \wedge P_j\Big) \| w' \| \\
&\leqslant \gamma_{2,k}(u_o) \sum_{j=1}^{m} \| \Delta_j \|\ \| w'' \|\ \| w' \| \\
&\leqslant \gamma_{2,k}(u_o)\ \sqrt{m} \Big(\sum_{j=1}^{m} \| \Delta_j \|^2\Big)^{1/2} \| w'' \|\ \| w' \| .
\end{aligned}$$

Now it is enough to observe that $\Big(\sum_{j=1}^{m} \| \Delta_j \|^2\Big)^{1/2} = \| \Delta \|$ to get

$$\begin{aligned}
\pi_2(u_o\, v) &\leqslant \gamma_{2,k}(u_o)\ \sqrt{m}\ \| \Delta \|\ \| w'' \|\ \| w' \| \\
&\leqslant \gamma_{2,k}(u)\ 2\ \sqrt{m}.
\end{aligned}$$

This completes the proof of (16). Since the inequality (12) holds for all km-dimensional subspaces $\widetilde{E} \subset E$ we infer that

$$\gamma_{2,km}(u) \leqslant 2\ \sqrt{m}\ \gamma_{2,k}(u),$$

completing the proof.

Proposition 5 suggests the following problems

Problems 2 : (a) Does the gradation $\{\gamma_{2,n}\}$ have the submultiplicative property, analogous to (6) and (12) ?

(b) In particular, is it true that $\gamma_{2,km}(id_E) \leqslant \gamma_{2,k}(id_E)\, \gamma_{2,m}(id_E)$ for positive integers k, m and any Banach space E ?

Remark : The affirmative answer to Problem 2(b) would imply that for any Banach space E of type $p > 1$ there is an $\alpha < 1/2$ such that $d(\widetilde{E}, \ell_2^n) \leqslant n^\alpha$ for all n-dimensional subspaces $\widetilde{E} \subset E$ and $n = 1,2,\dots$. This would give the affirmative answer to the problem raised by many specialists (cf. eg. [17] , Problème (i)).

At the end of this section, we define, as a particularly interesting example, a sequence $\left\{\hat{\gamma}_p^{(k)}\right\}$ of ideal norms ($1 \leqslant p \leqslant \infty$), which forms a dimensional cogradation of γ_p, considered as a finite-dimensional ideal norm. Our definition is analogous to the definition of the cogradation $\left\{\hat{\nu}_{t,s}^{(k)}\right\}$ (cf. (6) and (7)).

Let $1 \leqslant p \leqslant \infty$ and let k be a positive integer. For Banach spaces E,F and $u \in \mathfrak{N}(E,F)$ we put

$$(15) \qquad \hat{\gamma}_p^{(k)}(u) = \inf \sum_{m=1}^{\infty} \| v_m \| \, \| w_m \| ,$$

where the infimum is taken over all representations $u = \sum_{m=1}^{\infty} v_m w_m$ with $w_m \in \mathbb{L}(E,\ell_p^k)$, $v_m \in \mathbb{L}(\ell_p^k,F)$ ($m = 1,2,\ldots$). Clearly, for $u \in \mathfrak{N}(E,F)$ we have $\hat{\gamma}_p^{(k)}(u) \leqslant \nu_1(u) < \infty$.

Obviously, $\hat{\gamma}_p^{(k)}$ is a finite-dimensional ideal norm ($k = 1,2,\ldots$) ; it is not difficult to show that the sequence $\left\{\hat{\gamma}_p^{(k)}\right\}$ is a dimensional cogradation of the ideal norm γ_p. Let us notice that the sequence $\left\{(\mathfrak{N},\hat{\gamma}_p^{(k)})\right\}$ of normed operator ideals is not a cogradation of the normed operator ideal (Γ_p,γ_p). It is easy to see, however, that this sequence forms a dimensional cogradation of the normed operator ideal (κ_p,κ_p) of, so-called, p-compact operators. The ideal (κ_p,κ_p) is defined by Pietsch ([16] , Section 18.3) in the following way : for Banach spaces E,F, $\kappa_p(E,F)$ is a linear space of all operator form E to F which admit a factorization through ℓ_p, endowed with the norm defined for $u \in \kappa_p(E,F)$ by $\kappa_p(u) = \inf \| v \| \, \| w \|$, where the infimum is taken over all factorizations $u = vw$ with $w \in \mathbb{L}(E,\ell_p)$, $v \in \mathbb{L}(\ell_p,F)$.

For further convenience we also introduce the following notation : for $1 \leqslant p \leqslant \infty$, a positive integer k and $u \in \mathbb{L}(E,F)$ (E,F Banach spaces) with rank $u \leqslant k$, we put

$$\gamma_p^{(k)}(u) = \inf \left\{ \| v \| \, \| w \| \;\middle|\; w \in \mathbb{L}(E,\ell_p^k),\ v \in \mathbb{L}(\ell_p^k,F),\ u = vw \right\} .$$

obviously, for $1 \leqslant p \leqslant \infty$, we have $\gamma_p^{(k)}(u) \geqslant \hat{\gamma}_p^{(k)}(u)$ for every finite-rank operator u and $k \geqslant$ rank u. Also, $\gamma_p(u) = \lim_k \gamma_p^{(k)}(u)$ for every finite-rank operator u.

Particularly interesting is behaviour of the sequences $\left\{\hat{\gamma}_p^{(k)}(id_E)\right\}$ and $\left\{\gamma_p^{(k)}(id_E)\right\}$ for E-a finite-dimensional Banach space. It is due to the obvious fact that $\gamma_p^{(n)}(id_E) = d(E,\ell_p^n)$ for n = dim E. Unfortunately, relatively little is known on this topic.

Let us recall first the well-known fact that for $1 \leqslant p \leqslant \infty$ we have $\gamma_p(id_E) \leqslant \sqrt{n}$ for every n-dimensional Banach-space E. Actually, $\gamma_p^{(N)}(id_E) \leqslant \sqrt{n}$, where $N = \frac{1}{2}\, n\,(n+1)$ in the real case and $N = n^2$ in the complex case. For $2 \leqslant p \leqslant \infty$ this follows directly from Proposition 3(i) (see the argument below). For $1 \leqslant p < 2$ we just observe that $\gamma_p^{(N)}(id_E) = \gamma_{p^*}^{(N)}(id_{E^*})$.

In a similar way, as an application of Proposition 3(ii), we have

<u>Proposition 6</u> : Let $1 \leqslant p \leqslant \infty$, $p \neq 2$. Then $\hat{\gamma}_p^{(n)}(id_E) \leqslant 2\sqrt{n}$ for every n-dimensional Banach space.

<u>Proof</u> : Without loss generality we may assume that $2 < p \leqslant \infty$. By Proposition 3(ii), $id_E = \sum_{j=1}^{k} w_j\, \Delta_j\, v_j$ with $f_j \in \mathbb{L}(E,\ell_\infty^n)$, $\Delta_j \in \mathbb{L}(\ell_\infty^n,\ell_2^n)$ a diagonal opeartor, $w_j \in \mathbb{L}(\ell_2^n,E)$ $(j=1,\dots,k)$ and $\sum_{j=1}^{k} \|w_j\|\,\|\Delta_j\|\,\|v_j\| \leqslant 2\sqrt{n}$. Without loss of generality we may assume that $\|\Delta_j\| = 1$. For $j=1,\dots,k$ let $\left\{d_j(i)\right\}_{i=1}^{n}$ be a sequence of scalars corresponding to Δ_j, i.e. $\Delta_j\, e_i = d_j(i)\, e_i$ for $i=1,\dots,n$ (here $(e_i)_{i=1}^n$ is the unit vector basis). Put $d_j'(i) = |d_j(i)|^{2/p-1} d_j(i)$ and $d_j''(i) = |d_j(i)|^{1-2/p}$ $(i=1,\dots,n,$ $j=1,\dots,k)$. Let $\Delta_j' \in \mathbb{L}(\ell_\infty^n,\ell_p^n)$ be the diagonal operator corresponding to $\left\{d_j'(i)\right\}$ and $\Delta_j'' \in \mathbb{L}(\ell_p^n,\ell_2^n)$ be the diagonal operator corresponding to $\left\{d_j''(i)\right\}$ $(j=1,\dots,k)$. Then $\Delta_j = \Delta_j''\,\Delta_j'$ and $\|\Delta_j'\| = \|\Delta_j''\| = \|\Delta_j\| = 1$ $(j=1,\dots,k)$. Put $v_j' = \Delta_j\, v_j \in \mathbb{L}(E,\ell_p^n)$ and $w_j' = w_j\,\Delta_j'' \in \mathbb{L}(\ell_p^n,E)$ $(j=1,\dots,k)$. Therefore $id_E = \sum_{j=1}^{k} w_j'\, v_j'$ is the representation of id_E such that $\sum_{j=1}^{k} \|w_j'\|\,\|v_j'\| \leqslant 2\sqrt{n}$. This completes the proof.

Obviously, $\gamma_2(u) = \gamma_2^{(n)}(u) = \hat{\gamma}_2^{(n)}(u)$ for any operator u with rank u = n. However, for $1 < p < \infty$, $p \neq 2$, $\hat{\gamma}_p^{(n)}(id_E)$ (dim E = n) no longer has to be equivalent, up to a constant factor, to $\gamma_p(id_E)$. This can be seen e.g. for $E = \ell_2^n$.

<u>Proposition 7</u> : Let $2 < p < \infty$ and let n be a positive integer. Let $I_n = id_{\ell_2^n}$. Then

(i) $\gamma_{p*}(I_n) = \gamma_p(I_n) = \gamma_p^{(m_p)}(I_n) \leq c_p$, where $m_p = \left[n^{p/2} \right]$ and c_p is a constant depending only on p ;

(ii) for every $k \geq n$, $\hat{\gamma}_{p*}^{(k)}(I_n) = \hat{\gamma}_p^{(k)}(I_n) \geq ck^{-1/p}\, n^{1/2}$ where $c > 0$ is an absolute constant.

<u>Proof</u> : By the result of Bennet, Dor, Goodman, Johnson and Newman ([1]) there exists a constant c_p', depending only on p and $T \in \mathbb{L}(\ell_2^n, \ell_p^{m_p})$ such that $\| T\| \, \| (T_1)^{-1} \| \leq c_p'$, where $T_1 \in \mathbb{L}(\ell_2^n, T(\ell_2^n))$ is the restriction of T. By the Maurey's extension theorem ([13]) there exists $S \in \mathbb{L}(\ell_p^{m_p}, \ell_2^n)$ such that $S|T(\ell_2^n) = (T_1)^{-1}$ and $\| S\| \leq \gamma_2(S) \leq \alpha_2(id_{\ell_p^{m_p}}) \, \| (T_1)^{-1} \|$. Therefore, $I_n = ST$ is the required factorization of I_n.

The proof of (ii) is based upon the well-known lemma.

<u>Lemma 3</u> : Let $2 \leq p \leq \infty$ and let n be a positive integer. Then

$$(16) \qquad ck^{1/p} \leq \gamma_\infty(id_{\ell_p^k}) \leq \gamma_\infty^{(k)}(id_{\ell_p^k}) \leq k^{1/p},$$

where $c > 0$ is an absolute constant.

Assuming the truth of Lemma 3 we complete the proof of (ii) as follows. It is easy to see that

$$\gamma_\infty(I_n) \leq \hat{\gamma}_p^{(k)}(I_n) \, \gamma_\infty(id_{\ell_p^k}).$$

Therefore, by Lemma 3,

$$\hat{\gamma}_p^{(k)}(I_n) \geq \gamma_\infty(I_n)/\gamma_\infty(id_{\ell_p^k}) \geq ck^{-1/p}\, n^{1/2},$$

what concludes (ii).

It remains to prove Lemma 3. First write $id_{\ell_p^k} = I_{p,\infty}^{-1} I_{p,\infty}$, where $I_{p,\infty} \in \mathbb{L}(\ell_p^k, \ell_\infty^k)$ denotes the formal identity operator. Therefore,

$$\gamma_\infty^{(k)}(id_{\ell_p^k}) \leqslant \| I_{p,\infty}^{-1}\| \; \| I_{p,\infty}\| = k^{1/p}.$$

To show the left-hand side inequality in (16) fix a L_∞-factorization $id_{\ell_p^k} = vw$ with $w \in \mathbb{L}(\ell_p^k, L_\infty)$ $v \in \mathbb{L}(L_\infty, \ell_p^k)$ and by $I_{p,2} \in \mathbb{L}(\ell_p^k, \ell_2^k)$ denote the formal intensity operator. Recall that $\pi_2(id_E) = \sqrt{k}$ for every k-dimensional space E (Corollary 4), and that, by the Grothendieck theorem, $\pi_2(u) \leqslant c' \| u\|$ for every $u \in \mathbb{L}(L_\infty, \ell_2)$ (c' is a universal constant). Therefore,

$$\begin{aligned} k^{1/2} = \pi_2(id_{\ell_p^k}) &\leqslant \| I_{p,2}^{-1}\| \; \pi_2(I_{p,2}v) \; \| w\| \\ &\leqslant \| I_{p,2}^{-1}\| \; c' \; \| I_{p,2}\| \; \| v\| \; \| w\| \\ &\leqslant c' \; K^{1/2-1/p} \; \| v\| \; \| w\|. \end{aligned}$$

Thus, $\| v\| \, \| w\| \geqslant c\, k^{1/p}$. Since this estimate is valid for all L_∞-factorizations of $id_{\ell_p^k}$, it follows that $\gamma_\infty(id_{\ell_p^k}) \geqslant c\, k^{1/p}$. This completes the proof.

Let us state some problems related to Propositions 6 and 7.

Problems 3 : (a) Let $1 \leqslant p \leqslant \infty$, $p \neq 2$. Does here exist a constant c_p such that $\gamma_p^{(n)}(u) \leqslant c_p \hat{\gamma}_p^{(n)}(u)$ for every operator u of rank n ?

(b) Does here exist a constant c such that $\gamma_\infty^{(n)}(u) \leqslant c\, \gamma_\infty(u)$ for every $u \in \mathbb{L}(E,F)$, where E,F are Banach spaces, dim E = n ?. In particular, does here exist a constant c such that $\gamma_\infty^{(n)}(id_E) \leqslant c\, \gamma_\infty(id_E)$ for any n-dimensional Banach space E ?

(c) Let $1 \leqslant p \leqslant \infty$, $p \neq 2$. What is the order of growth as $n \to \infty$, of the sequence

$$\delta_n(p) = \sup \left\{ \gamma_p^{(n)}(id_E) \;\middle|\; \dim E = n \right\}$$

$$= \sup \left\{ d(E, \ell_p^n) \;\middle|\; \dim E = n \right\} \; ?$$

In particular, does there exist a constant c_p such that $\delta_n(p) \leqslant c_p \sqrt{n}$?

Remarks : (1) The second part of Problem 3(b) and Problem 3(c) are just reformulations of well-known classical questions.

(2) In connection with Problem 3(b) let us mention an example given by W.B. Johnson (unpublished) which shows that the analogous inequality is not true for all operators of rank n. Namely, for the Rademacher projection $R_n \in \mathbb{L}(\ell_\infty^{2^n}, \ell_2^n)$, considered in Theorem 5(ii), we have

$$\gamma_\infty^{(k)}(R_n) \geqslant c \sqrt{\frac{n}{\log k}} \qquad \text{for } k = 2,3,\ldots,$$

where $c > 0$ is an absolute constant, while obviously $\gamma_\infty(R_n) = 1$.

3 - E-FACTORABLE OPERATORS AND THE WEAK DISTANCE BETWEEN BANACH SPACES

The definitions of L_p-factorable operators and of the norms $\gamma_p^{(n)}$ $(1 \leqslant p \leqslant \infty)$ suggest the following uniform approach. Let E be a fixed Banach space. Let X,Y be Banach spaces and let $u \in \mathbb{L}(X,Y)$. We say that u is E-factorable, in symbols $u \in \Gamma_E(X,Y)$, if $u = \sum_k w_k v_k$ with $v_k \in \mathbb{L}(X,E)$, $w_k \in \mathbb{L}(E,Y)$ $(k=1,2,\ldots)$ and $\sum_k \|w_k\| \, \|v_k\| < \infty$. For $u \in \Gamma_E(X,Y)$ we define $\hat{\gamma}_E(u) = \inf \sum_k \|w_k\| \, \|v_k\|$, where the infimum is taken over all representations of u as above. For further convenience we also introduce the notation $\gamma_E(u) = \inf \|w\| \, \|v\|$, where the infimum is taken over all factorizations $u = wv$ with $v \in L(X,E)$, $w \in L(E,Y)$ (we put $\gamma_E(u) = \infty$, if such factorization does not exist).

Obviously, $(\Gamma_E, \hat{\gamma}_E)$ is a normed operator ideal. For $E = L_p$ we have $(\Gamma_E, \hat{\gamma}_E) = (\Gamma_p, \gamma_p)$, for $E = \ell_p$ we have $(\Gamma_E, \hat{\gamma}_E) = (\kappa_p, \kappa_p)$ - the ideal of p-compact operators and, finally, for $E = \ell_p^n$ we have $\hat{\gamma}_E = \hat{\gamma}_p^{(n)}$

($1 \leqslant p \leqslant \infty$, $n = 1,2,\ldots$). If X,Y are finite-dimensional then $\Gamma_E(X,Y) = \mathbb{L}(X,Y)$ and $\left\{ u \in \mathbb{L}(X,Y) \;\middle|\; \hat{\gamma}_E(u) \leqslant 1 \right\} = \operatorname{conv} \left\{ u \in \mathbb{L}(E,F) \;\middle|\; \gamma_E(u) \leqslant 1 \right\}$. If E is also finite-dimensional, the Caratheodory's theorem and the standard compactness argument show that given $u \in \mathbb{L}(X,Y)$ there exist $v_k \in \mathbb{L}(X,E)$, $w_k \in \mathbb{L}(E,Y)$ $(k=1,\ldots,N)$ such taht $u = \sum_{k=1}^{N} w_k v_k$ and $\hat{\gamma}_E(u) = \sum_{k=1}^{N} \|w_k\| \, \|v_k\|$ (here $N \leqslant \frac{1}{2} m(m+1)$ for real spaces and $N \leqslant m^2$ for complex spaces, where $m = \max(\dim X, \dim Y) \dim E$).

For a Banach space E the function $\gamma_E(.)$ is directly related to the notion of the Banach-Mazur distance d(E,F) (F any Banach space). We have $d(E,F) = \gamma_E(id_F)$ $(= \gamma_F(id_E))$. This observation suggest the following definition of the weak distance d(E,F). For Banach spaces E,F we put

$$(17) \qquad \hat{d}(E,F) = \max (\hat{\gamma}_E(id_F), \hat{\gamma}_F(id_E)),$$

if $id_F \in \Gamma_E(F,F)$ and $id_E \in \Gamma_F(E,E)$, and $\hat{d}(E,F) = \infty$ otherwise.

The definition of $\hat{d}(E,F)$ is very natural in the context of the theory of normed operator ideals. The value of $\hat{d}(E,F)$ shows how well one can distinguish the spaces E and F by means of normed operator ideals. More precisely, we have

<u>Proposition 8</u> : Let E,F be Banach spaces. Then

$$(18) \quad \hat{d}(E,F) = \sup \left\{ \max \left(\frac{\alpha(id_E)}{\alpha(id_F)}, \frac{\alpha(id_F)}{\alpha(id_E)} \right) \;\middle|\; (\mathfrak{A},\alpha) \text{ a normed operator ideal} \right\}.$$

(In (18) we put $\alpha(u) = \infty$ for any operator $u \in \mathbb{L}(X,Y)$, $u \notin \mathfrak{A}(X,Y)$. (X,Y - Banach spaces)).

<u>Proof</u> : It is eady to see that (18) holds if $\hat{d}(E,F) = \infty$. Assume now that $d(E,F) < \infty$. Let $(\mathfrak{A},\alpha)$ be a normed operator ideal and let $id_E = \sum_k w_k v_k$ with $v_k \in \mathbb{L}(E,F)$, $w_k \in \mathbb{L}(F.E)$ $(k=1,2,\ldots)$. Then

$$\alpha(id_E) \leqslant \sum_k \alpha(w_k v_k) \leqslant \sum_k \|w_k\| \, \alpha(id_F) \, \|v_k\|$$

$$\leqslant \alpha(id_F) \sum_k \|w_k\| \, \|v_k\| .$$

Since the estimate holds for all representations of id_E as above, we infer that

$$\frac{\alpha(id_E)}{\alpha(id_F)} \leq \hat{\gamma}_F(id_E).$$

The similar argument shows that

$$\frac{\alpha(id_F)}{\alpha(id_E)} \leq \hat{\gamma}_E(id_F).$$

It follows that

$$\hat{d}(E,F) \geq \sup \left\{ \max\left(\frac{\alpha(id_E)}{\alpha(id_F)}, \frac{\alpha(id_F)}{(\alpha(id_E)}\right) \middle| (\mathfrak{A},\alpha \text{ a normed operator ideal} \right\}.$$

The converse inequality can be obtained by evaluating the right-hand side expression on the ideals $(\Gamma_E,\hat{\gamma}_E)$ and $(\Gamma_F,\hat{\gamma}_F)$.

For Banach spaces E,F,G we obviously have $\hat{d}(E,G) \leq \hat{d}(E,F)\ \hat{d}(F,G)$. If $\dim E = \dim F = n < \infty$ and $\hat{d}(E,F) = 1$, then it is easy to show, using the observation that the norm $\gamma_F(id_E)$ is attained that there exist $v \in \mathbb{L}(E,F)$, $w \in \mathbb{L}(F,E)$ such that $\|wv\| = \|w\|\ \|v\| = 1$ and $|\text{trace } wv| = n$. This yields (cf. eg. [21] ,[22]) that $wv = a\ id_E$ for some scalar a with $|a| = 1$. Thus $\gamma_F(id_E) = \|w\|\ \|v\| = 1$, hence E = F. Therefore, for every positive integer n, the function $\log \hat{d}(.,.)$ is a metric on the space $\mathcal{F}_n$ of all n-dimensional Banach spaces. Obviously, the induced topology on $\mathcal{F}_n$ is weaker than the classical topology defined by the Banach-Mazur distance. Since the latter topology is compact, it follows that on $\mathcal{F}_n$ both topologies are equivalent. This suggest the following problems.

<u>Problems 4</u> : (a) What is the order of growth, as $n \to \infty$, of the sequence

$$\hat{\delta}_n = \sup \left\{ \hat{d}(E,F) \;\middle|\; E,F \in \mathcal{F}_n \right\} ?$$

In particular, is it true that $\hat{\delta}_n \geq c\,n$ for some $c > 0$?

(b) Does thre exist a constant c' such that $d(E,F) \leq c'\ \hat{d}(E,F)$ for all Banach spaces E,F ? In particular, is it true that $d(E,\ell_p^n) \leq c'\ \hat{d}(E,\ell_p^n)$ for all $E \in \mathcal{F}_n$, $1 \leq p \leq \infty$, $p \neq 2$?

Remarks : (1) E.D. Gluskin recently has shown in [5] that for every positive integer n there exist subspaces $E_n, F_n \subset \ell_\infty^{3n}$ with $\dim E_n = \dim F_n = n$ such that $d(E,F) \geq c_1 n$, where $c_1 > 0$ is a universal constant. It seems to be quite likely that for these spaces we also have $\hat{d}(E_n,F_n) \geq cn$.

(2) Let us observe that, since $\hat{\gamma}_E(id_{\ell_2^n}) \leq d(E,\ell_2^n)$, then $\hat{d}(E,\ell_2^n) = d(E,\ell_2^n)$ for all $E \in \mathcal{F}_n$.

The next theorem should be compared with Problems 3(c) and 4(b).

Theorem 7 : Let $1 \leq p \leq \infty$, $p \neq 2$. Then

$$\hat{d}(E,\ell_p^n) \leq a\sqrt{n} \qquad \text{for all } E \in \mathcal{F}_n,$$

where $a = 20(1-3.10^{-2})^{-1}$.

Proof : The estimate $\hat{\gamma}_{\ell_p^n}(id_E) = \hat{\gamma}_p^{(n)}(id_E) \leq 2\sqrt{n}$ has been shown, for $p \neq 2$, in Proposition 6. For $p = 2$, we obviously have $\hat{\gamma}_2(id_E) = \gamma_2(id_E) \leq \pi_2(id_E) = \sqrt{n}$, by Corollary 4.

Since $\hat{d}(F_1,F_2) = \hat{d}(F_1^*,F_2^*)$ for all Banach spaces F_1,F_2, then without loss of generality we may assume that $2 \leq p \leq \infty$. To estimate $\hat{\gamma}_E(id_{\ell_p^n})$, we need the proposition, which is a slight modification of Theorem 3 in [2].

Proposition 9 : Let E be an n-dimensional Banach space and let $2 \leq p \leq \infty$. Then there exist $X \subset \ell_p^n$ with $\dim X \geq n/10$ and $v_1 \in \mathbb{L}(\ell_p^n,\ell_2^n)$, $v_2 \in \mathbb{L}(\ell_2^n,E)$, $w \in \mathbb{L}(E,\ell_p^n)$ such that $wv_2v_1|X = id_X$, $\|w\|\,\|v_2\|\,\|v_1\| \leq 2(1-3.10^{-2})^{-1}\sqrt{n}$ and $v_1wv_2 \geq 0$, as an operator acting in the Hilbert space ℓ_2^n.

Assuming the truth of Proposition 9 we complete the proof of Theorem 7 as follows. Let G be the symmetry group of ℓ_p^n and let $d\mu$ be the Haar measure on G. It is well-known and easy to see that ℓ_p^n has enough symmetries, i.e. for any $u \in \mathbb{L}(\ell_p^n,\ell_p^n)$, if $ug = gu$ for all $g \in G$ then $u = \lambda id_{\ell_p^n}$, for some $\lambda \in C$.

Let $X \subset \ell_p^n$ and v_1, v_2, w be as in Proposition 9. Put $u = \int_G g^{-1} w v_2 v_1 g \, d\mu(g)$. Then, by the translation invariance of the Haar measure, ug = g u for all $g \in G$. Therefore,

$$\int_G g^{-1} w v_2 v_1 g \, d\mu(g) = \lambda \, id_{\ell_p^n},$$

with some $\lambda \in C$. It follows that

$$id_{\ell_p^n} = \frac{1}{\lambda} \int_G g^{-1} w v_2 v_1 g \, d\mu(g).$$

Moreover,

$$\frac{1}{|\lambda|} \int \| g^{-1} w\| \, \| v_2 v_1 g\| \, d\mu(g) =$$

$$= \frac{1}{|\lambda|} \|w\| \, \|v_2 v_1\| \leqslant \frac{1}{|\lambda|} \, 2(1-3.10^{-2})^{-1} \sqrt{n}.$$

Thus,

$$\hat{\gamma}_E(id_{\ell_p^n}) \leqslant \frac{1}{|\lambda|} \, 2(1-3.10^{-2})^{-1} \sqrt{n}.$$

To estimate $|\lambda|$ observe that, because $v_1 w v_2 \geqslant 0$.

$$\begin{aligned} \text{trace } g^{-1} w v_2 v_1 g &= \text{trace } v_1 w v_2 \\ &\geqslant \text{trace } v_1 w v_2 | v_1(X) \\ &= \text{trace } id_{v_1(X)} = \dim X \geqslant n/10. \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda n &= \text{trace } \lambda \, id_{\ell_p^n} = \text{trace } u \\ &= \text{trace} \int g^{-1} w v_2 v_1 g \, d\mu(g) \geqslant n/10. \end{aligned}$$

It follows that, $\frac{1}{|\lambda|} = \frac{1}{\lambda} \leqslant 10$. Thus $\hat{\gamma}_E(id_{\ell_p^n}) \leqslant a \sqrt{n}$, with $a = 20(1-3.10^{-2})^{-1}$, as derived.

It remains to prove Proposition 9.

Proof of Proposition 9 : Let $[\,.,.\,]$ be the inner product on E defined by F. John's theorem. Let $\|\,.\,\|_2$ be the induced euclidean norm on E, let $E_2 = (E, \|\,.\,\|_2)$ and let $I : E_2 \to E$ be the formal identity operator. By Lemma 1 we have $\pi_2(I^{-1}) = \sqrt{n}$. Therefore, by Proposition 2 and Theorem 2, it follows that I^{-1} admits a representation $I^{-1} = \sum_{j=1}^{k} c_j B_j \Delta_j A_j$,

$$E \xrightarrow[A_j]{} \ell_\infty^n \xrightarrow[\Delta_j]{} \ell_2^n \xrightarrow[B_j]{} E_2,$$

with $A_j \in \mathbb{L}(E, \ell_\infty^n)$, $\Delta_j \in \mathbb{L}(\ell_\infty^n, \ell_2^n)$ a diagonal operator, $B_j \in \mathbb{L}(\ell_2^n, E_2)$, $\|A_j\|\,\|\Delta_j\|\,\|B_j\| \leqslant 2\sqrt{n}$, $c_j \geqslant 0$ $(j=1,\dots,k)$ and $\sum_{j=1}^{k} c_j = 1$.

It is easy to see that there is a j_o, $1 \leqslant j_o \leqslant k$, such that $|\text{trace}\ \Delta_{jo} A_{jo} I B_{jo}| = |\text{trace}\ B_{jo} \Delta_{jo} A_{jo} I| \geqslant n$. Let $U \in \mathbb{L}(\ell_2^n, \ell_2^n)$ be an unitary operator such that $U\Delta_{jo} A_{jo} I B_{jo} \geqslant 0$. Let $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0$ be the sequence of eigenvalues of $U \Delta_{jo} A_{jo} I B_{jo}$ and let $(\varphi_i)_{i=1}^n$ be the orthonormal basis in ℓ_2^n of corresponding eigenvectors. Then

$$(19) \quad \sum_{i=1}^{n} \lambda_i = \text{trace}\ U \Delta_{jo} A_{jo} I B_{jo} \geqslant |\text{trace}\ \Delta_{jo} A_{jo} I B_{jo}| \geqslant n.$$

On the other hand

$$(20) \quad \left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2} = \pi_2(U\Delta_{jo} A_{jo} I B_{jo}) \leqslant \|U\|\,\|\Delta_{jo}\|\,\|A_{jo}\|\,\|I\|\,\|B_{jo}\| \leqslant 2\sqrt{n}.$$

Let $Jo = \{ i \mid \lambda_i \leqslant 1-3.10^{-2} \}$. It follows from (19) and (20) that card $Jo \geqslant n/10$. Put $Y = \text{span}(\varphi_i)_{i \in Jo} \subset \ell_2^n$ and define $T \in \mathbb{L}(\ell_2^n, \ell_2^n)$ by $T\varphi_i = \frac{1}{\lambda_i}\varphi_i$ if $i \in Jo$ and $T\varphi_i = \varphi_i$ if $i \notin Jo$. Then $\|T\| \leqslant (1-3.10^{-2})^{-1}$, $T U \Delta_{jo} A_{jo} I B_{jo} | Y = id_Y$ and $T U \Delta_{jo} A_{jo} I B_{jo} \geqslant 0$.

Let $\Delta_{jo} = \Delta''\Delta''$ be a diagonal factorization of Δ_{jo} such that $\Delta' \in \mathbb{L}(\ell_\infty^n, \ell_p^n)$, $\Delta'' \in \mathbb{L}(\ell_p^n, \ell_2^n)$ and $\|\Delta_{jo}\| = \|\Delta''\|\,\|\Delta'\|$ (cf. the proof of Proposition 6). Put $v_1 = TU\Delta'' \in \mathbb{L}(\ell_p^n, \ell_2^n)$, $v_2 = IB_{jo} \in \mathbb{L}(\ell_2^n, E)$, $w = \Delta' A_{jo} \in \mathbb{L}(E, \ell_p^n)$ and $X = w\, v_2(Y) \subset \ell_p^n$. Obviously, $v_1\, w\, v_2 \geqslant 0$. Since $v_1\, w\, v_2 | Y = id_Y$ it follows that $\dim X = \dim Y \geqslant n/10$ and $w\, v_2\, v_1 | X = id_X$. Finally, $\|w\|\,\|v_2\|\,\|v_1\| \leqslant \|\Delta'\|\,\|A_{jo}\|\,\|I\|\,\|B_{jo}\|\,\|T\|\,\|U\|\,\|\Delta''\| \leqslant \|T\|\,\|A_{jo}\|\,\|B_{jo}\|\,\|\Delta'\|\,\|\Delta''\| \leqslant (1-3.10^{-2})^{-1}\, 2\sqrt{n}$. This completes the proof of Proposition 9.

Remark : It is easy to observe that the conclusion of Proposition 9 holds if we replace the space ℓ_p^n by any n-dimensional Banach space F which has the property :

(Δ) every diagonal operator $\Delta \in \mathbb{L}(\ell_\infty^n, \ell_2^n)$, admits a factorization

$$\Delta = \Delta'' \Delta' \text{ with } \Delta' \in \mathbb{L}(\ell_\infty^n, F),\ \Delta'' \in \mathbb{L}(F, \ell_2^n) \text{ and } \|\Delta\| = \|\Delta''\|\,\|\Delta'\| .$$

Therefore, $\hat{d}(E,F) \leqslant a\sqrt{n}$ for every n-dimensional Banach spaces E,F such that F has enough symmetries and F or $F^\star$ have (Δ). It is known (cf. eg. [19] , or [15] , Proposition 3.2) that every symmetric space which is 2-convex or 2-concave (with corresponding constants equal to 1) may be taken as a space F above.

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